1. Let $m$ be a countably additive measure defined for all sets in a $\sigma$-algebra $\mathcal{M}$.

If $A$ and $B$ are two sets in $M$ with $A \subset B$, then $mA \leq mB$. This property is called \textit{monotonicity}.

\textbf{Proof:} $B = A \cup (A - B)$. $A$ and $A - B$ are disjoint. Since $m$ is a countably additive measure, $mB = mA + m(A - B)$. Note that $m$ is nonnegative, and $m(A - B) \geq 0$. Hence $mA \leq mB$.

2. Let $< E_n >$ be any sequence of sets in $\mathcal{M}$. Then $m(\bigcup E_n) \leq \sum mE_n$. [Hint: Use Proposition 1.2.] This property of a measure is called \textit{countable subadditivity}.

\textbf{Proof:} By Proposition 1.2 on page 17, since $\mathcal{M}$ is a $\sigma$-algebra, there is a sequence $< B_n >$ of sets in $\mathcal{M}$ such that $B_n \cap B_m = \phi$ for $n \neq m$ and

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} E_i.$$  

Since $m$ is a countably additive measure and $B_i \subset E_i$ for all $i$, by Problem 3.1 I have that

$$m(\bigcup E_n) = m(\bigcup B_n) = \sum mB_n \leq \sum mE_n.$$  

3. If there is a set $A$ in $\mathcal{M}$ such that $mA < \infty$, then $m\phi = 0$.

\textbf{Proof:} Note that $A = A \cup \phi$ and $A$ and $\phi$ are disjoint, and thus

$$mA = mA + m\phi.$$  

Since $mA < \infty$, $m\phi = 0$ precisely.
4.

5. Let $A$ be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering $A$. Then $\sum l(I_n) \geq 1$.

**Proof 1:** (due to Meng-Gen Tsai) Since $0 \in A$, there is an open interval $J_1$ in $\{I_n\}$ such that $0 \in J_1$. Let $J_1 = (a_1, b_1)$. Note that $a_1 < 0$ and $b_1 > 0$. If $b_1 \geq 1$, then

$$\sum l(I_n) \geq l(J_1) = b_1 - a_1 \geq 1.$$ 

Suppose not. If $a_1$ is rational, then I can find an open interval $J_2 \in \{I_n\}$ such that $a_1 \in J_2$. If $a_1$ is irrational, I consider the following cases.

**Case 1.** There is an open interval $J_2$ such that $a_1 \in J_2$.

**Case 2.** There is an open interval $J_2$ such that $a_1$ is the right endpoint of $J_2$.

**Case 3.** Otherwise.

I claim that Case 3 is impossible. Consider the following subcollection

$$K_1, ..., K_m$$

where $K_i \in \{I \in I_n|a_1 < x \text{ for all } x \in I\}$. Thus I select a open interval $K_0 = (x, y)$ nearest $a_1$. Hence $(a_1, x) \cap Q$ cannot be covered by any elements of $\{I_n\}$, a contradiction.

Hence we can find $J_2$ in **Case 1** and **Case 2**. Continue the process to find out $J_3, ....$ Since $\{I_n\}$ is a finite covering of $A$, this process can be done in finite steps. Hence

$$\sum l(I_n) \geq \sum J_n = \sum (a_{n+1} - a_n) = a_m - a_1.$$ 

Note that $a_m \geq 1$ and $a_n \leq 0$. Hence $\sum l(I_n) \geq 1$. 

2
**Proof 2:** (due to Shin-Yi Lee) \( A \) is contained in \( \bigcup I_k \), then \( \overline{A} \) is contained in \( \bigcup \overline{I_k} = \bigcup \overline{I_k} \). So, \( |A| = 1 \leq | \bigcup I_k | \leq \sum | I_k | \).

Where \(| . |\) in Zygmund’s book is the same as \( m(.)\) in Royden’s book if I do not misunderstand.

**Proof 3:** (due to ljl) (1) We can suppose that \( I_n \) are disjoint; otherwise, if \( I_m \cap I_n \) is not empty, then we can use \( I = I_m \cap I_n \) to replace \( I_m \) and \( I_n \). \( I \) is also an interval, and also covers \( A \), and will make \( \sum l(I_n) \) smaller. Hence if the sum is still \( \geq 1 \) after our adjustment, the original one is \( \geq 1 \) surely.

(2) Now \( I_n \) are disjoint, so we can suppose that \( I_n = (a_n, b_n) \), \( n = 1, ..., N \) with \( a_i < b_i \leq a_{i+1} < b_{i+1} \) for \( i = 1, ..., N - 1 \).

(3) It is easy to show that \( b_i = a_{i+1} \) (or that collection cannot cover \( A \)). Hence \( \sum l(I_n) = b_N - a_1 \). Also, it is easy to show that \( a_1 < 0 \) and \( b_N > 1 \). Proved.

6. Given any set \( A \) and any \( \epsilon > 0 \), prove that there is an open set \( O \) such that \( A \subset O \) and \( m^*O \leq m^*A + \epsilon \). Also, prove that there is a \( G \in G_\delta \) such that \( A \subset G \) and \( m^*A = m^*G \).

**Proof:** Note that

\[
m^*A = \inf_{A \subset \bigcup I_n} \sum l(I_n).
\]

where \( I_n \) are open intervals by the definition of the outer measure. Let \( O = \bigcup I_n \). \( O \) is also open. For any \( \epsilon > 0 \), there exists \( \{I_n\} \) such that

\[
m^*A + \epsilon \geq \sum l(I_n).
\]

By Proposition 1 and Problem 2,

\[
\sum l(I_n) = \sum m^*(I_n) \geq m^*(\bigcup I_n) = m^*O.
\]
Combine them and I have $m^*O \leq m^*A + \epsilon$.

By previous conclusion, for any $n \in N$ there is an open set $O_n$ such that $A \subset O_n$ and $m^*O_n \leq m^*A + \frac{1}{n}$. Take $G = \bigcap O_n$. Hence

$$m^*G \leq m^*O_n \leq m^*A + \frac{1}{n}$$

for all $n \in N$. Therefore

$$m^*G \leq m^*A.$$ 

Note that $A \subset O_n$ for all $n$, that is, $A \subset G$, that is,

$$m^*A \leq m^*G.$$

Hence

$$m^*A = m^*G.$$ 

7. Prove that $m^*$ is translation invariant.

**Proof:** Let $E$ be a set. Consider the countable collections $\{I_n\}$ of open intervals that cover $E$. Then $\{I'_n\}$ covers $E+y$ where $I'_n = I_n + y$. Note that $I'_n$ is also an open interval, and

$$l(I'_n) = l(I_n).$$

Hence for any $\epsilon > 0$ there is $\{I_n\}$ such that $m^*E + \epsilon > \sum l(I_n)$. Thus,

$$m^*E + \epsilon > \sum l(I_n) = \sum l(I'_n) \geq m^*(E + y),$$

that is,

$$m^*E \geq m^*(E + y).$$

Similarly (regard $E$ as $(E + y) - y$),

$$m^*E \leq m^*(E + y).$$

Hence $m^*E = m^*(E + y)$, that is, $m^*$ is translation invariant.
9. Show that if $E$ is a measurable set, then each translate $E + y$ of $E$ is also measurable.

**Proof:** Since $E$ is a measurable set, for each set $A$ I have

$$m^*A = m^*(A \cap E) + m^*(A \cap E^c).$$

Suppose $x \in A \cap (E + y)$, then

$$x \in A \cap (E + y) \iff x \in A \text{ and } x \in (E + y)$$
$$\iff x \in A \text{ and } x - y \in E$$
$$\iff x - y \in (A - y) \text{ and } x - y \in E$$
$$\iff x - y \in (A - y) \cap E$$
$$\iff x \in (A - y) \cap E + y.$$

Hence $A \cap (E + y) = (A - y) \cap E + y$; thus

$$m^*(A \cap (E + y)) = m^*((A - y) \cap E + y)$$
$$= m^*((A - y) \cap E)$$

(since Problem 3.7). Similarly, $A \cap (E + y)^c = (A - y) \cap E^c + y$; thus

$$m^*(A \cap (E + y)^c) = m^*((A - y) \cap E^c + y)$$
$$= m^*((A - y) \cap E^c)$$

Hence

$$m^*(A \cap (E + y)) + m^*(A \cap (E + y)^c)$$
$$= m^*((A - y) \cap E) + m^*((A - y) \cap E^c)$$
$$= m^*(A - y)$$
$$= m^*A$$

for each set $A$ (since $E$ is measurable). Therefore, $E + y$ is also measurable.
10.

11. Show that the condition $mE_1 < \infty$ is necessary in Proposition 14 by giving a decreasing sequence $< E_n >$ of measurable set with $\phi = \bigcap E_n$ and $mE_n = \infty$ for each $n$.

**Solution:** Let $$E_n = \bigcup_{k=1}^{n} A_k$$ where $A_k = (k - \frac{1}{4}, k + \frac{1}{4})$.

12. Let $< E_n >$ be a sequence of disjoint measurable sets and $A$ any set. Then

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i).$$

**Proof:** By Lemma 9,

$$m^*(A \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Since

$$A \cap \bigcup_{i=1}^{\infty} E_i \subset A \cap \bigcup_{i=1}^{n} E_i$$

for all $n$,

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) \geq m^*(A \cap \bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

for all $n$. Hence,

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i).$$

Similarly, by using the fact that $m^*$ is nonnegative, I have

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i).$$

Therefore I got the conclusion.
13. Prove Proposition 15. [Hints: a. Show that for $m^*E < \infty$, (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (vi) (cf. Proposition 5).

b. Use (a) to show that for arbitrary sets $E$, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

c. Use (b) to show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i).]

Proposition 15: Let $E$ be a given set. Then the following five statements are equivalent:

i. $E$ is measurable.

ii. Given $\epsilon > 0$, there is an open set $O \supset E$ with $m^*(O - E) < \epsilon$.

iii. Given $\epsilon > 0$, there is a closed set $F \subset E$ with $m^*(E - F) < \epsilon$.

iv. There is a $G$ in $G_\delta$ with $E \subset G$, $m^*(G - E) = 0$.

v. There is an $F$ in $F_\sigma$ with $F \subset E$, $m^*(E - F) = 0$.

If $m^*E$ is finite, the above statements are equivalent to:

vi. Given $\epsilon > 0$, there is a finite union $U$ of open intervals such that $m^*(U \triangle E) < \epsilon$.

**Proof of (a):**

(i) $\Rightarrow$ (ii): Since $E$ is measurable, by Proposition 5 there is an open set $O$ such that $E \subset O$ and $m^*O \leq m^*E + 2\epsilon$. By Lemma 9

$$m^*O = m^*(O - E) + m^*(O \cap E) = m^*(O - E) + m^*E$$

since $E \subset O$). Since $m^*E$ is finite, $m^*(O - E) \leq 2\epsilon < \epsilon$.

(ii) $\Rightarrow$ (vi): Take $\epsilon = 1/n$ for all $n \in N$, there is an open set $O_n \subset E$ with $m^*(O - E) < 1/n$. Take $G = \bigcap O_n$; $G$ is open and $G \in G_\delta$. And

$$m^*(G - E) \leq m^*(O_n - E) < 1/n$$

for all $n \in N$. Hence $m^*(G - E) = 0$.

(vi) $\Rightarrow$ (ii): If not, there is a real $\epsilon_0 > 0$ such that

$$m^*(O - E) \geq \epsilon_0$$
for any open set $O$. Note that $G$ is the intersection of countable open set, write $G = \bigcap O_n$. Hence
\[
m^*(\bigcap_{k=1}^{n} O_k - E) \geq \epsilon_0
\]
for all $n$. Hence
\[
m^*(\bigcap_{k=1}^{\infty} O_k - E) \geq \epsilon_0,
\]
a contradiction.

14.