Sequences of Functions

Uniform convergence

9.1 Assume that $f_n \to f$ uniformly on S and that each f_n is bounded on S. Prove that $\{f_n\}$ is uniformly bounded on S.

Proof: Since $f_n \to f$ uniformly on S, then given $\varepsilon = 1$, there exists a positive integer n_0 such that as $n \geq n_0$, we have

$$|f_n(x) - f(x)| \leq 1 \text{ for all } x \in S. \tag{*}$$

Hence, $f(x)$ is bounded on S by the following

$$|f(x)| \leq |f_{n_0}(x)| + 1 \leq M(n_0) + 1 \text{ for all } x \in S. \tag{**}$$

where $|f_{n_0}(x)| \leq M(n_0)$ for all $x \in S$.

Let $|f_1(x)| \leq M(1), ..., |f_{n_0-1}(x)| \leq M(n_0 - 1)$ for all $x \in S$, then by (*) and (**),

$$|f_n(x)| \leq 1 + |f(x)| \leq M(n_0) + 2 \text{ for all } n \geq n_0.$$

So,

$$|f_n(x)| \leq M \text{ for all } x \in S \text{ and for all } n$$

where $M = \max(M(1), ..., M(n_0 - 1), M(n_0) + 2)$.

Remark: (1) In the proof, we also shows that the limit function f is bounded on S.

(2) There is another proof. We give it as a reference.

Proof: Since Since $f_n \to f$ uniformly on S, then given $\varepsilon = 1$, there exists a positive integer n_0 such that as $n \geq n_0$, we have

$$|f_n(x) - f_{n+k}(x)| \leq 1 \text{ for all } x \in S \text{ and } k = 1, 2, ...$$

So, for all $x \in S$, and $k = 1, 2, ...$

$$|f_{n_0+k}(x)| \leq 1 + |f_{n_0}(x)| \leq M(n_0) + 1 \tag{*}$$

where $|f_{n_0}(x)| \leq M(n_0)$ for all $x \in S$.

Let $|f_1(x)| \leq M(1), ..., |f_{n_0-1}(x)| \leq M(n_0 - 1)$ for all $x \in S$, then by (*),

$$|f_n(x)| \leq M \text{ for all } x \in S \text{ and for all } n$$

where $M = \max(M(1), ..., M(n_0 - 1), M(n_0) + 1)$.

9.2 Define two sequences $\{f_n\}$ and $\{g_n\}$ as follows:

$$f_n(x) = x\left(1 + \frac{1}{n}\right) \text{ if } x \in R, \ n = 1, 2, ...,$$

$$g_n(x) = \begin{cases} \frac{1}{n} \text{ if } x = 0 \text{ or if } x \text{ is irrational,} \\ b + \frac{1}{n} \text{ if } x \text{ is rational, say } x = \frac{a}{b}, \ b > 0. \end{cases}$$

Let $h_n(x) = f_n(x) g_n(x)$.

(a) Prove that both $\{f_n\}$ and $\{g_n\}$ converges uniformly on every bounded interval.

Proof: Note that it is clear that

$$\lim_{n \to \infty} f_n(x) = f(x) = x, \text{ for all } x \in R$$

and

$$\lim_{n \to \infty} g_n(x) = g(x) = \begin{cases} 0 \text{ if } x = 0 \text{ or if } x \text{ is irrational,} \\ b \text{ if } x \text{ is ratonal, say } x = \frac{a}{b}, b > 0. \end{cases}$$

In addition, in order to show that $\{f_n\}$ and $\{g_n\}$ converges uniformly on every bounded interval, it suffices to consider the case of any compact interval $[-M, M]$, $M > 0$.

Given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$\frac{M}{n} < \varepsilon \text{ and } \frac{1}{n} < \varepsilon.$$

Hence, for this ε, we have as $n \geq N$

$$|f_n(x) - f(x)| = \left|\frac{x}{n}\right| \leq \frac{M}{n} < \varepsilon \text{ for all } x \in [-M, M]$$

and

$$|g_n(x) - g(x)| \leq \frac{1}{n} < \varepsilon \text{ for all } x \in [-M, M].$$

That is, we have proved that $\{f_n\}$ and $\{g_n\}$ converges uniformly on every bounded interval.

Remark: In the proof, we use the easy result directly from definition of uniform convergence as follows. If $f_n \to f$ uniformly on S, then $f_n \to f$ uniformly on T for every subset T of S.

(b) Prove that $h_n(x)$ does not converges uniformly on any bounded interval.

Proof: Write

$$h_n(x) = \begin{cases} \frac{x}{n}\left(1+\frac{1}{n}\right) & \text{if } x=0 \text{ or } x \text{ is irrational} \\ a+\frac{a}{n}\left(1+\frac{1}{b}+\frac{1}{bn}\right) & \text{if } x \text{ is rational, say } x=\frac{a}{b} \end{cases}.$$

Then

$$\lim_{n\to\infty} h_n(x) = h(x) = \begin{cases} 0 \text{ if } x=0 \text{ or } x \text{ is irrational} \\ a \text{ if } x \text{ is rational, say } x=\frac{a}{b} \end{cases}.$$

Hence, if $h_n(x)$ converges uniformly on any bounded interval I, then $h_n(x)$ converges uniformly on $[c,d] \subseteq I$. So, given $\varepsilon = \max(|c|,|d|) > 0$, there is a positive integer N such that as $n \geq N$, we have

$$\begin{aligned} \max(|c|,|d|) &> |h_n(x) - h(x)| \\ &= \begin{cases} \left|\frac{x}{n}\left(1+\frac{1}{n}\right)\right| = \frac{|x|}{n}\left|1+\frac{1}{n}\right| & \text{if } x \in Q^c \cap [c,d] \text{ or } x=0 \\ \left|\frac{a}{n}\left(1+\frac{1}{b}+\frac{1}{bn}\right)\right| & \text{if } x \in Q \cap [c,d],\ x=\frac{a}{b} \end{cases} \end{aligned}$$

which implies that ($x \in [c,d] \cap Q^c$ or $x=0$)

$$\max(|c|,|d|) > \frac{|x|}{n}\left|1+\frac{1}{n}\right| \geq \frac{|x|}{n} \geq \frac{\max(|c|,|d|)}{n}$$

which is absurb. So, $h_n(x)$ does not converges uniformly on any bounded interval.

9.3 Assume that $f_n \to f$ uniformly on S, $g_n \to f$ uniformly on S.

(a) Prove that $f_n + g_n \to f + g$ uniformly on S.

Proof: Since $f_n \to f$ uniformly on S, and $g_n \to f$ uniformly on S, then given $\varepsilon > 0$, there is a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \text{ for all } x \in S$$

and

$$|g_n(x) - g(x)| < \frac{\varepsilon}{2} \text{ for all } x \in S.$$

Hence, for this ε, we have as $n \geq N$,

$$\begin{aligned} |f_n(x) + g_n(x) - f(x) - g(x)| &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \varepsilon \text{ for all } x \in S. \end{aligned}$$

That is, $f_n + g_n \to f + g$ uniformly on S.

Remark: There is a similar result. We write it as follows. If $f_n \to f$ uniformly on S, then $cf_n \to cf$ uniformly on S for any real c. Since the proof is easy, we omit the proof.

(b) Let $h_n(x) = f_n(x) g_n(x)$, $h(x) = f(x) g(x)$, if $x \in S$. Exercise 9.2 shows that the assertion $h_n \to h$ uniformly on S is, in general, incorrect. Prove that it is correct if each f_n and each g_n is bounded on S.

Proof: Since $f_n \to f$ uniformly on S and each f_n is bounded on S, then f is bounded on S by **Remark (1)** in the **Exercise 9.1.** In addition, since $g_n \to g$ uniformly on S and each g_n is bounded on S, then g_n is uniformly bounded on S by **Exercise 9.1.**

Say $|f(x)| \leq M_1$ for all $x \in S$, and $|g_n(x)| \leq M_2$ for all x and all n. Then given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2(M_2+1)} \text{ for all } x \in S$$

and

$$|g_n(x) - g(x)| < \frac{\varepsilon}{2(M_1+1)} \text{ for all } x \in S$$

which implies that as $n \geq N$, we have

$$\begin{aligned} |h_n(x) - h(x)| &= |f_n(x) g_n(x) - f(x) g(x)| \\ &= |[f_n(x) - f(x)][g_n(x)] + [f(x)][g_n(x) - g(x)]| \\ &\leq |f_n(x) - f(x)| |g_n(x)| + |f(x)| |g_n(x) - g(x)| \\ &< \frac{\varepsilon}{2(M_2+1)} M_2 + M_1 \frac{\varepsilon}{2(M_1+1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for all $x \in S$. So, $h_n \to h$ uniformly on S.

9.4 Assume that $f_n \to f$ uniformly on S and suppose there is a constant $M > 0$ such that $|f_n(x)| \leq M$ for all x in S and all n. Let g be continuous on the closure of the disk $B(0; M)$ and define $h_n(x) = g[f_n(x)]$, $h(x) = g[f(x)]$, if $x \in S$. Prove that $h_n \to h$ uniformly on S.

Proof: Since g is continuous on a compact disk $B(0; M)$, g is uniformly continuous on $B(0; M)$. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $|x - y| < \delta$, where $x, y \in S$, we have

$$|g(x) - g(y)| < \varepsilon. \tag{*}$$

For this $\delta > 0$, since $f_n \to f$ uniformly on S, then there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \delta \text{ for all } x \in S. \tag{**}$$

Hence, by (*) and (**), we conclude that given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|g(f_n(x)) - g(f(x))| < \varepsilon \text{ for all } x \in S.$$

Hence, $h_n \to h$ uniformly on S.

9.5 (a) Let $f_n(x) = 1/(nx + 1)$ if $0 < x < 1, n = 1, 2, ...$ Prove that $\{f_n\}$ converges pointwise but not uniformly on $(0, 1)$.

Proof: First, it is clear that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0, 1)$. Supppos that $\{f_n\}$ converges uniformly on $(0, 1)$. Then given $\varepsilon = 1/2$, there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| = \left|\frac{1}{1 + nx}\right| < 1/2 \text{ for all } x \in (0, 1).$$

So, the inequality holds for all $x \in (0, 1)$. It leads us to get a contradiction since

$$\frac{1}{1 + Nx} < \frac{1}{2} \text{ for all } x \in (0, 1) \Rightarrow \lim_{x\to 0^+} \frac{1}{1 + Nx} = 1 < 1/2.$$

That is, $\{f_n\}$ converges **NOT** uniformly on $(0, 1)$.

(b) Let $g_n(x) = x/(nx + 1)$ if $0 < x < 1, n = 1, 2, ...$ Prove that $g_n \to 0$ uniformly on $(0, 1)$.

Proof: First, it is clear that $\lim_{n\to\infty} g_n(x) = 0$ for all $x \in (0, 1)$. Given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$1/n < \varepsilon$$

which implies that

$$|g_n(x) - g| = \left|\frac{x}{1+nx}\right| = \left|\frac{1}{\frac{1}{x}+n}\right| < \frac{1}{n} < \varepsilon.$$

So, $g_n \to 0$ uniformly on $(0,1)$.

9.6 Let $f_n(x) = x^n$. The sequence $\{f_n(x)\}$ converges pointwise but not uniformly on $[0,1]$. Let g be continuous on $[0,1]$ with $g(1) = 0$. Prove that the sequence $\{g(x)x^n\}$ converges uniformly on $[0,1]$.

Proof: It is clear that $f_n(x) = x^n$ converges **NOT** uniformly on $[0,1]$ since each term of $\{f_n(x)\}$ is continuous on $[0,1]$ and its limit function

$$f = \begin{cases} 0 \text{ if } x \in [0,1) \\ 1 \text{ if } x = 1. \end{cases}$$

is not a continuous function on $[0,1]$ by **Theorem 9.2**.

In order to show $\{g(x)x^n\}$ converges uniformly on $[0,1]$, it suffices to shows that $\{g(x)x^n\}$ converges uniformly on $[0,1)$. Note that

$$\lim_{n\to\infty} g(x)x^n = 0 \text{ for all } x \in [0,1).$$

We partition the interval $[0,1)$ into two subintervals: $[0,1-\delta]$ and $(1-\delta,1)$.

As $x \in [0,1-\delta]$: Let $M = \max_{x\in[0,1]}|g(x)|$, then given $\varepsilon > 0$, there is a positive integer N such that as $n \geq N$, we have

$$M(1-\delta)^n < \varepsilon$$

which implies that for all $x \in [0,1-\delta]$,

$$|g(x)x^n - 0| \leq M|x^n| \leq M(1-\delta)^n < \varepsilon.$$

Hence, $\{g(x)x^n\}$ converges uniformly on $[0,1-\delta]$.

As $x \in (1-\delta,1)$: Since g is continuous at 1, given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $|x-1| < \delta$, where $x \in [0,1]$, we have

$$|g(x) - g(1)| = |g(x) - 0| = |g(x)| < \varepsilon$$

which implies that for all $x \in (1-\delta,1)$,

$$|g(x)x^n - 0| \leq |g(x)| < \varepsilon.$$

Hence, $\{g(x)x^n\}$ converges uniformly on $(1-\delta, 1)$.

So, from above sayings, we have proved that the sequence of functions $\{g(x)x^n\}$ converges uniformly on $[0, 1]$.

Remark: It is easy to show the followings by definition. So, we omit the proof.

(1) Suppose that for all $x \in S$, the limit function f exists. If $f_n \to f$ uniformly on $S_1 (\subseteq S)$, then $f_n \to f$ uniformly on S, where $\#(S - S_1) < +\infty$.

(2) Suppose that $f_n \to f$ uniformly on S and on T. Then $f_n \to f$ uniformly on $S \cup T$.

9.7 Assume that $f_n \to f$ uniformly on S and each f_n is continuous on S. If $x \in S$, let $\{x_n\}$ be a sequence of points in S such that $x_n \to x$. Prove that $f_n(x_n) \to f(x)$.

Proof: Since $f_n \to f$ uniformly on S and each f_n is continuous on S, by **Theorem 9.2**, the limit function f is also continuous on S. So, given $\varepsilon > 0$, there is a $\delta > 0$ such that as $|y - x| < \delta$, where $y \in S$, we have

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

For this $\delta > 0$, there exists a positive integer N_1 such that as $n \geq N_1$, we have

$$|x_n - x| < \delta.$$

Hence, as $n \geq N_1$, we have

$$|f(x_n) - f(x)| < \frac{\varepsilon}{2}. \qquad (*)$$

In addition, since $f_n \to f$ uniformly on S, given $\varepsilon > 0$, there exists a positive integer $N \geq N_1$ such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \text{ for all } x \in S$$

which implies that

$$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}. \qquad (**)$$

By (*) and (**), we obtain that given $\varepsilon > 0$, there exists a positie integer N such that as $n \geq N$, we have

$$\begin{aligned} |f_n(x_n) - f(x)| &= |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

That is, we have proved that $f_n(x_n) \to f(x)$.

9.8 Let $\{f_n\}$ be a seuqnece of continuous functions defined on a compact set S and assume that $\{f_n\}$ converges pointwise on S to a limit function f. Prove that $f_n \to f$ uniformly on S if, and only if, the following two conditions hold.:

(i) The limit function f is continuous on S.

(ii) For every $\varepsilon > 0$, there exists an $m > 0$ and a $\delta > 0$, such that $n > m$ and $|f_k(x) - f(x)| < \delta$ implies $|f_{k+n}(x) - f(x)| < \varepsilon$ for all x in S and all $k = 1, 2, ...$

Hint. To prove the sufficiency of (i) and (ii), show that for each x_0 in S there is a neighborhood of $B(x_0)$ and an integer k (depending on x_0) such that

$$|f_k(x) - f(x)| < \delta \text{ if } x \in B(x_0).$$

By compactness, a finite set of integers, say $A = \{k_1, ..., k_r\}$, has the property that, for each x in S, some k in A satisfies $|f_k(x) - f(x)| < \delta$. Uniform convergence is an easy consequences of this fact.

Proof: ($\Rightarrow$) Suppose that $f_n \to f$ uniformly on S, then by **Theorem 9.2**, the limit function f is continuous on S. In addition, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in S$$

Let $m = N$, and $\delta = \varepsilon$, then (ii) holds.

($\Leftarrow$) Suppose that (i) and (ii) holds. We prove $f_k \to f$ uniformly on S as follows. By (ii), given $\varepsilon > 0$, there exists an $m > 0$ and a $\delta > 0$, such that $n > m$ and $|f_k(x) - f(x)| < \delta$ implies $|f_{k+n}(x) - f(x)| < \varepsilon$ for all x in S and all $k = 1, 2, ...$

Consider $\left|f_{k(x_0)}(x_0) - f(x_0)\right| < \delta$, then there exists a $B(x_0)$ such that as $x \in B(x_0) \cap S$, we have

$$\left|f_{k(x_0)}(x) - f(x)\right| < \delta$$

by continuity of $f_{k(x_0)}(x) - f(x)$. Hence, by (ii) as $n > m$

$$\left| f_{k(x_0)+n}(x) - f(x) \right| < \varepsilon \text{ if } x \in B(x_0) \cap S. \tag{*}$$

Note that S is compact and $S = \cup_{x \in S} (B(x) \cap S)$, then $S = \cup_{k=1}^{p} (B(x_k) \cap S)$. So, let $N = \max_{i=1}^{p} (k(x_p) + m)$, as $n > N$, we have

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in S$$

with help of (*). That is, $f_n \to f$ uniformly on S.

9.9 (a) Use Exercise 9.8 to prove the following theorem of Dini: **If $\{f_n\}$ is a sequence of real-valued continuous functions converginf pointwise to a continuous limit function f on a compact set S, and if $f_n(x) \geq f_{n+1}(x)$ for each x in S and every $n = 1, 2, ...$, then $f_n \to f$ uniformly on S.**

Proof: By **Exercise 9.8**, in order to show that $f_n \to f$ uniformly on S, it suffices to show that (ii) holds. Since $f_n(x) \to f(x)$ and $f_{n+1}(x) \leq f_n(x)$ on S, then fixed $x \in S$, and given $\varepsilon > 0$, there exists a positive integer $N(x) = N$ such that as $n \geq N$, we have

$$0 \leq f_n(x) - f(x) < \varepsilon.$$

Choose $m = 1$ and $\delta = \varepsilon$, then by $f_{n+1}(x) \leq f_n(x)$, then (ii) holds. We complete it.

Remark: (1) **Dini's Theorem** is important in Analysis; we suggest the reader to keep it in mind.

(2) There is another proof by using **Cantor Intersection Theorem**. We give it as follows.

Proof: Let $g_n = f_n - f$, then g_n is continuous on S, $g_n \to 0$ pointwise on S, and $g_n(x) \geq g_{n+1}(x)$ on S. If we can show $g_n \to 0$ uniformly on S, then we have proved that $f_n \to f$ uniformly on S.

Given $\varepsilon > 0$, and consider $S_n := \{x : g_n(x) \geq \varepsilon\}$. Since each $g_n(x)$ is continuous on a compact set S, we obtain that S_n is compact. In addition, $S_{n+1} \subseteq S_n$ since $g_n(x) \geq g_{n+1}(x)$ on S. Then

$$\cap S_n \neq \phi \tag{*}$$

if each S_n is non-empty by **Cantor Intersection Theorem**. However (*) contradicts to $g_n \to 0$ pointwise on S. Hence, we know that there exists a positive integer N such that as $n \geq N$,

$$S_n = \phi.$$

That is, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|g_n(x) - 0| < \varepsilon.$$

So, $g_n \to 0$ uniformly on S.

(b) Use he sequence in Exercise 9.5(a) to show that compactness of S is essential in Dini's Theorem.

Proof: Let $f_n(x) = \frac{1}{1+nx}$, where $x \in (0,1)$. Then it is clear that each $f_n(x)$ is continuous on $(0,1)$, the limit function $f(x) = 0$ is continuous on $(0,1)$, and $f_{n+1}(x) \leq f_n(x)$ for all $x \in (0,1)$. However, $f_n \to f$ not uniformly on $(0,1)$ by **Exercise 9.5 (a)**. Hence, compactness of S is essential in Dini's Theorem.

9.10 Let $f_n(x) = n^c x (1-x^2)^n$ for x real and $n \geq 1$. Prove that $\{f_n\}$ converges pointwsie on $[0,1]$ for every real c. Determine those c for which the convergence is uniform on $[0,1]$ and those for which term-by-term integration on $[0,1]$ leads to a correct result.

Proof: It is clear that $f_n(0) \to 0$ and $f_n(1) \to 0$. Consider $x \in (0,1)$, then $|1-x^2| := r < 1$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n^c r^n x = 0 \text{ for any real } c.$$

Hence, $f_n \to 0$ pointwise on $[0,1]$.

Consider

$$f_n'(x) = n^c \left(1-x^2\right)^{n-1} (2n-1) \left(\frac{1}{2n-1} - x^2\right),$$

then each f_n has the absolute maximum at $x_n = \frac{1}{\sqrt{2n-1}}$.

As $c < 1/2$, we obtain that

$$\begin{aligned} |f_n(x)| &\leq |f_n(x_n)| \\ &= \frac{n^c}{\sqrt{2n-1}} \left(1 - \frac{1}{2n-1}\right)^n \\ &= n^{c-\frac{1}{2}} \left[\sqrt{\frac{n}{2n-1}} \left(1 - \frac{1}{2n-1}\right)^n\right] \to 0 \text{ as } n \to \infty. \qquad (*) \end{aligned}$$

In addition, as $c \geq 1/2$, if $f_n \to 0$ uniformly on $[0,1]$, then given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x)| < \varepsilon \text{ for all } x \in [0,1]$$

which implies that as $n \geq N$,

$$|f_n(x_n)| < \varepsilon$$

which contradicts to

$$\lim_{n\to\infty} f_n(x_n) = \begin{cases} \frac{1}{\sqrt{2e}} \text{ if } c = 1/2 \\ \infty \text{ if } c > 1/2 \end{cases}. \tag{**}$$

From (*) and (**), we conclude that only as $c < 1/2$, the seqences of functions converges uniformly on $[0,1]$.

In order to determine those c for which term-by-term integration on $[0,1]$, we consider

$$\int_0^1 f_n(x)\,dx = \frac{n^c}{2(n+1)}$$

and

$$\int_0^1 f(x)\,dx = \int_0^1 0dx = 0.$$

Hence, only as $c < 1$, we can integrate it term-by-term.

9.11 Prove that $\sum x^n (1-x)$ converges pointwise but not uniformly on $[0,1]$, whereas $\sum (-1)^n x^n (1-x)$ converges uniformly on $[0,1]$. This illustrates that **uniform convergence of $\sum f_n(x)$ along with pointwise convergence of $\sum |f_n(x)|$ does not necessarily imply uniform convergence of $\sum |f_n(x)|$.**

Proof: Let $s_n(x) = \sum_{k=0}^{n} x^k (1-x) = 1 - x^{n+1}$, then

$$s_n(x) \to \begin{cases} 1 \text{ if } x \in [0,1) \\ 0 \text{ if } x = 1 \end{cases}.$$

Hence, $\sum x^n (1-x)$ converges pointwise but not uniformly on $[0,1]$ by **Theorem 9.2** since each s_n is continuous on $[0,1]$.

Let $g_n(x) = x^n (1-x)$, then it is clear that $g_n(x) \geq g_{n+1}(x)$ for all $x \in [0,1]$, and $g_n(x) \to 0$ uniformly on $[0,1]$ by **Exercise 9.6**. Hence, by **Dirichlet's Test for uniform convergence**, we have proved that $\sum (-1)^n x^n (1-x)$ converges uniformly on $[0,1]$.

9.12 Assume that $g_{n+1}(x) \leq g_n(x)$ for each x in T and each $n = 1, 2, ...,$ and suppose that $g_n \to 0$ uniformly on T. Prove that $\sum (-1)^{n+1} g_n(x)$ converges uniformly on T.

Proof: It is clear by **Dirichlet's Test for uniform convergence.**

9.13 Prove Abel's test for uniform convergence: Let $\{g_n\}$ be a sequence of real-valued functions such that $g_{n+1}(x) \leq g_n(x)$ for each x in T and for every $n = 1, 2, ...$ If $\{g_n\}$ is uniformly bounded on T and if $\sum f_n(x)$ converges uniformly on T, then $\sum f_n(x) g_n(x)$ also converges uniformly on T.

Proof: Let $F_n(x) = \sum_{k=1}^{n} f_k(x)$. Then

$$s_n(x) = \sum_{k=1}^{n} f_k(x) g_k(x) = F_n g_1(x) + \sum_{k=1}^{n} (F_n(x) - F_k(x)) (g_{k+1}(x) - g_k(x))$$

and hence if $n > m$, we can write

$$s_n(x) - s_m(x) = (F_n(x) - F_m(x)) g_{m+1}(x) + \sum_{k=m+1}^{n} (F_n(x) - F_k(x)) (g_{k+1}(x) - g_k(x))$$

Hence, if M is an uniform bound for $\{g_n\}$, we have

$$|s_n(x) - s_m(x)| \leq M |F_n(x) - F_m(x)| + 2M \sum_{k=m+1}^{n} |F_n(x) - F_k(x)|. \quad (*)$$

Since $\sum f_n(x)$ converges uniformly on T, given $\varepsilon > 0$, there exists a positive integer N such that as $n > m \geq N$, we have

$$|F_n(x) - F_m(x)| < \frac{\varepsilon}{M+1} \text{ for all } x \in T \quad (**)$$

By (*) and (**), we have proved that as $n > m \geq N$,

$$|s_n(x) - s_m(x)| < \varepsilon \text{ for all } x \in T.$$

Hence, $\sum f_n(x) g_n(x)$ also converges uniformly on T.

Remark: In the proof, we establish the lemma as follows. We write it as a reference.

(**Lemma**) If $\{a_n\}$ and $\{b_n\}$ are two sequences of complex numbers, define

$$A_n = \sum_{k=1}^{n} a_k.$$

Then we have the identity

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k \left(b_{k+1} - b_k\right) \quad \text{(i)}$$

$$= A_n b_1 + \sum_{k=1}^{n} \left(A_n - A_k\right)\left(b_{k+1} - b_k\right). \quad \text{(ii)}$$

Proof: The identity (i) comes from **Theorem 8.27**. In order to show (ii), it suffices to consider

$$b_{n+1} = b_1 + \sum_{k=1}^{n} b_{k+1} - b_k.$$

9.14 Let $f_n(x) = x/\left(1 + nx^2\right)$ if $x \in R$, $n = 1, 2, \ldots$ Find the limit function f of the sequence $\{f_n\}$ and the limit function g of the sequence $\{f_n'\}$.

(a) Prove that $f'(x)$ exists for every x but that $f'(0) \neq g(0)$. For what values of x is $f'(x) = g(x)$?

Proof: It is easy to show that the limit function $f = 0$, and by $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$, we have

$$\lim_{n\to\infty} f_n'(x) = g(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}.$$

Hence, $f'(x)$ exists for every x and $f'(0) = 0 \neq g(0) = 1$. In addition, it is clear that as $x \neq 0$, we have $f'(x) = g(x)$.

(b) In what subintervals of R does $f_n \to f$ uniformly?

Proof: Note that

$$\frac{1 + nx^2}{2} \geq \sqrt{n}\,|x|$$

by $A.P. \geq G.P.$ for all real x. Hence,

$$\left| \frac{x}{1+nx^2} \right| \leq \frac{1}{2\sqrt{n}}$$

which implies that $f_n \to f$ uniformly on R.

(c) In what subintervals of R does $f_n' \to g$ uniformly?

Proof: Since each $f_n' = \frac{1-nx^2}{(1+nx^2)^2}$ is continuous on R, and the limit function g is continuous on $R - \{0\}$, then by **Theorem 9.2**, the interval I that we consider does not contains 0. Claim that $f_n' \to g$ uniformly on such interval $I = [a, b]$ which does not contain 0 as follows.

Consider

$$\left| \frac{1-nx^2}{(1+nx^2)^2} \right| \leq \frac{1}{1+nx^2} \leq \frac{1}{na^2},$$

so we know that $f_n' \to g$ uniformly on such interval $I = [a, b]$ which does not contain 0.

9.15 Let $f_n(x) = (1/n) e^{-n^2x^2}$ if $x \in R$, $n = 1, 2, \ldots$ Prove that $f_n \to 0$ uniformly on R, that $f_n' \to 0$ pointwise on R, but that the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

Proof: It is clear that $f_n \to 0$ uniformly on R, that $f_n' \to 0$ pointwise on R. Assume that $f_n' \to 0$ uniformly on $[a, b]$ that contains 0. We will prove that it is impossible as follows.

We may assume that $0 \in (a, b)$ since other cases are similar. Given $\varepsilon = \frac{1}{e}$, then there exists a positive integer N' such that as $n \geq \max\left(N', \frac{1}{b}\right) := N$ ($\Rightarrow \frac{1}{N} \leq b$), we have

$$|f_n'(x) - 0| < \frac{1}{e} \text{ for all } x \in [a, b]$$

which implies that

$$\left| 2\frac{Nx}{e^{(Nx)^2}} \right| < \frac{1}{e} \text{ for all } x \in [a, b]$$

which implies that, let $x = \frac{1}{N}$,

$$\frac{2}{e} < \frac{1}{e}$$

which is absurb. So, the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

9.16 Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on $[0,1]$ and assume that $f_n \to f$ uniformly on $[0,1]$. Prove or disprove

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

Proof: By **Theorem 9.8**, we have

$$\lim_{n\to\infty} \int_0^{1} f_n(x)\,dx = \int_0^1 f(x)\,dx. \tag{*}$$

Note that $\{f_n\}$ is uniform bound, say $|f_n(x)| \leq M$ for all $x \in [0,1]$ and all n by **Exercise 9.1.** Hence,

$$\left| \int_{1-1/n}^{1} f_n(x)\,dx \right| \leq \frac{M}{n} \to 0. \tag{**}$$

Hence, by (*) and (**), we have

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

9.17 Mathematicinas from Slobbovia decided that the Riemann integral was too complicated so that they replaced it by **Slobbovian integral**, defined as follows: If f is a function defined on the set Q of rational numbers in $[0,1]$, the Slobbovian integral of f, denoted by $S(f)$, is defined to be the limit

$$S(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right),$$

whenever the limit exists. Let $\{f_n\}$ be a sequence of functions such that $S(f_n)$ exists for each n and such that $f_n \to f$ uniformly on Q. Prove that $\{S(f_n)\}$ converges, that $S(f)$ exists, and $S(f_n) \to S(f)$ as $n \to \infty$.

Proof: $f_n \to f$ uniformly on Q, then given $\varepsilon > 0$, there exists a positive integer N such that as $n > m \geq N$, we have

$$|f_n(x) - f(x)| < \varepsilon/3 \tag{1}$$

and

$$|f_n(x) - f_m(x)| < \varepsilon/2. \tag{2}$$

So, if $n > m \geq N$,

$$\begin{aligned}
|S(f_n) - S(f_m)| &= \left|\lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \left(f_n\left(\frac{j}{k}\right) - f_m\left(\frac{j}{k}\right)\right)\right| \\
&= \lim_{k\to\infty} \frac{1}{k} \left|\sum_{j=1}^{k} \left(f_n\left(\frac{j}{k}\right) - f_m\left(\frac{j}{k}\right)\right)\right| \\
&\leq \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \varepsilon/2 \text{ by } (2) \\
&= \varepsilon/2 \\
&< \varepsilon
\end{aligned}$$

which implies that $\{S(f_n)\}$ converges since it is a Cauchy sequence. Say its limit S.

Consider, by (1) as $n \geq N$,

$$\frac{1}{k} \sum_{j=1}^{k} \left[f_n\left(\frac{j}{k}\right) - \varepsilon/3\right] \leq \frac{1}{k} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) \leq \frac{1}{k} \sum_{j=1}^{k} \left[f_n\left(\frac{j}{k}\right) + \varepsilon/3\right]$$

which implies that

$$\left[\frac{1}{k} \sum_{j=1}^{k} f_n\left(\frac{j}{k}\right)\right] - \varepsilon/3 \leq \frac{1}{k} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) \leq \left[\frac{1}{k} \sum_{j=1}^{k} f_n\left(\frac{j}{k}\right)\right] + \varepsilon/3$$

which implies that, let $k \to \infty$

$$S(f_n) - \varepsilon/3 \leq \lim_{k\to\infty} \sup \frac{1}{k} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) \leq S(f_n) + \varepsilon/3 \tag{3}$$

and

$$S(f_n) - \varepsilon/3 \leq \lim_{k\to\infty} \inf \frac{1}{k} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) \leq S(f_n) + \varepsilon/3 \tag{4}$$

which implies that

$$
\begin{aligned}
&\left|\lim_{k\to\infty}\sup\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{j}{k}\right)-\lim_{k\to\infty}\inf\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{j}{k}\right)\right| \\
&\leq\left|\lim_{k\to\infty}\sup\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{j}{k}\right)-S\left(f_n\right)\right|+\left|\lim_{k\to\infty}\inf\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{j}{k}\right)-S\left(f_n\right)\right| \\
&\leq\frac{2\varepsilon}{3}\text{ by (3) and (4)} \\
&<\varepsilon. \qquad (5)
\end{aligned}
$$

Note that (3)-(5) imply that the existence of $S(f)$. Also, (3) or (4) implies that $S(f)=S$. So, we complete the proof.

9.18 Let $f_n(x)=1/(1+n^2x^2)$ if $0\leq x\leq 1$, $n=1,2,...$ Prove that $\{f_n\}$ converges pointwise but not uniformly on $[0,1]$. Is term-by term integration permissible?

Proof: It is clear that

$$\lim_{n\to\infty}f_n(x)=0$$

for all $x\in[0,1]$. If $\{f_n\}$ converges uniformly on $[0,1]$, then given $\varepsilon=1/3$, there exists a positive integer N such that as $n\geq N$, we have

$$|f_n(x)|<1/3\text{ for all }x\in[0,1]$$

which implies that

$$\left|f_N\left(\frac{1}{N}\right)\right|=\frac{1}{2}<\frac{1}{3}$$

which is impossible. So, $\{f_n\}$ converges pointwise but not uniformly on $[0,1]$.

Since $\{f_n(x)\}$ is clearly uniformly bounded on $[0,1]$, i.e., $|f_n(x)|\leq 1$ for all $x\in[0,1]$ and n. Hence, by **Arzela's Theorem**, we know that the sequence of functions can be integrated term by term.

9.19 Prove that $\sum_{n=1}^{\infty}x/n^{\alpha}(1+nx^2)$ converges uniformly on every finite interval in R if $\alpha>1/2$. Is the convergence uniform on R?

Proof: By $A.P.\geq G.P.$, we have

$$\left|\frac{x}{n^{\alpha}(1+nx^2)}\right|\leq\frac{1}{2n^{\alpha+\frac{1}{2}}}\text{ for all }x.$$

So, by **Weierstrass M-test,** we have proved that $\sum_{n=1}^{\infty} x/n^{\alpha}\left(1+nx^2\right)$ converges uniformly on R if $\alpha > 1/2$. Hence, $\sum_{n=1}^{\infty} x/n^{\alpha}\left(1+nx^2\right)$ converges uniformly on every finite interval in R if $\alpha > 1/2$.

9.20 Prove that the series $\sum_{n=1}^{\infty}\left(\left(-1\right)^n/\sqrt{n}\right)\sin\left(1+\left(x/n\right)\right)$ converges uniformly on every compact subset of R.

Proof: It suffices to show that the series $\sum_{n=1}^{\infty}\left(\left(-1\right)^n/\sqrt{n}\right)\sin\left(1+\left(x/n\right)\right)$ converges uniformly on $[0,a]$. Choose n large enough so that $a/n \leq 1/2$, and therefore $\sin\left(1+\left(\frac{x}{n+1}\right)\right) \leq \sin\left(1+\frac{x}{n}\right)$ for all $x \in [0,a]$. So, if we let $f_n(x) = \left(-1\right)^n/\sqrt{n}$ and $g_n(x) = \sin\left(1+\frac{x}{n}\right)$, then by **Abel's test for uniform convergence**, we have proved that the series $\sum_{n=1}^{\infty}\left(\left(-1\right)^n/\sqrt{n}\right)\sin\left(1+\left(x/n\right)\right)$ converges uniformly on $[0,a]$.

Remark: In the proof, we metion something to make the reader get more. (1) since a compact set K is a bounded set, say $K \subseteq [-a,a]$, if we can show the series converges uniformly on $[-a,a]$, then we have proved it. (2) The interval that we consider is $[0,a]$ since $[-a,0]$ is similar. (3) **Abel's test for uniform convergence** holds for $n \geq N$, where N is a fixed positive integer.

9.21 Prove that the series $\sum_{n=0}^{\infty}\left(x^{2n+1}/\left(2n+1\right)-x^{n+1}/\left(2n+2\right)\right)$ converges pointwise but not uniformly on $[0,1]$.

Proof: We show that the series converges pointwise on $[0,1]$ by considering two cases: (1) $x \in [0,1)$ and (2) $x=1$. Hence, it is trivial. Define $f(x) = \sum_{n=0}^{\infty}\left(x^{2n+1}/\left(2n+1\right)-x^{n+1}/\left(2n+2\right)\right)$, if the series converges uniformly on $[0,1]$, then by **Theorem 9.2**, $f(x)$ is continuous on $[0,1]$. However,

$$f(x) = \begin{cases} \frac{1}{2}\log\left(1+x\right) \text{ if } x \in [0,1) \\ \log 2 \text{ if } x = 1 \end{cases}.$$

Hence, the series converges not uniformly on $[0,1]$.

Remark: The function $f(x)$ is found by the following. Given $x \in [0,1)$, then both

$$\sum_{n=0}^{\infty} t^{2n} = \frac{1}{1-t^2} \text{ and } \frac{1}{2}\sum_{n=0}^{\infty} t^n = \frac{1}{2\left(1-t\right)}$$

converges uniformly on $[0,x]$ by **Theorem 9.14.** So, by **Theorem 9.8**, we have

$$\int_0^x \sum_{n=0}^{\infty} t^{2n} - \frac{1}{2}\sum_{n=0}^{\infty} t^n = \int_0^x \frac{1}{1-t^2} - \frac{1}{2(1-t)}dt$$
$$= \int_0^x \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right) - \frac{1}{2}\left(\frac{1}{1-t}\right)dt$$
$$= \frac{1}{2}\log(1+x).$$

And as $x = 1$,

$$\sum_{n=0}^{\infty}\left(x^{2n+1}/(2n+1) - x^{n+1}/(2n+2)\right)$$
$$= \sum_{n=0}^{\infty}\frac{1}{2n+1} - \frac{1}{2n}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1} \text{ by } \textbf{Theorem8.14}.$$
$$= \log 2 \text{ by } \textbf{Abel's Limit Theorem}.$$

9.22 Prove that $\sum a_n \sin nx$ and $\sum b_n \cos nx$ are uniformly convergent on R if $\sum |a_n|$ converges.

Proof: It is trivial by **Weierstrass M-test.**

9.23 Let $\{a_n\}$ be a decreasing sequence of positive terms. Prove that the series $\sum a_n \sin nx$ converges uniformly on R if, and only if, $na_n \to 0$ as $n \to \infty$.

Proof: ($\Rightarrow$) Suppose that the series $\sum a_n \sin nx$ converges uniformly on R, then given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$\left|\sum_{k=n}^{2n-1} a_k \sin kx\right| < \varepsilon. \qquad (*)$$

Choose $x = \frac{1}{2n}$, then $\sin\frac{1}{2} \leq \sin kx \leq \sin 1$. Hence, as $n \geq N$, we always have, by (*)

$$
\begin{aligned}
(\varepsilon >) \left| \sum_{k=n}^{2n-1} a_k \sin kx \right| &= \sum_{k=n}^{2n-1} a_k \sin kx \\
&\geq \sum_{k=n}^{2n-1} a_{2n} \sin \frac{1}{2} \text{ since } a_k > 0 \text{ and } a_k \searrow \\
&= \left(\frac{1}{2} \sin \frac{1}{2} \right) (2na_{2n}) .
\end{aligned}
$$

That is, we have proved that $2na_{2n} \to 0$ as $n \to \infty$. Similarly, we also have $(2n-1)\, a_{2n-1} \to 0$ as $n \to \infty$. So, we have proved that $na_n \to 0$ as $n \to \infty$.

($\Leftarrow$) Suppose that $na_n \to 0$ as $n \to \infty$, then given $\varepsilon > 0$, there exists a positive integer n_0 such that as $n \geq n_0$, we have

$$
|na_n| = na_n < \frac{\varepsilon}{2(\pi+1)}. \tag{*}
$$

In order to show the uniform convergence of $\sum_{n=1}^{\infty} a_n \sin nx$ on R, it suffices to show the uniform convergence of $\sum_{n=1}^{\infty} a_n \sin nx$ on $[0, \pi]$. So, if we can show that as $n \geq n_0$

$$
\left| \sum_{k=n+1}^{n+p} a_k \sin kx \right| < \varepsilon \text{ for all } x \in [0, \pi], \text{ and all } p \in N
$$

then we complete it. We consider two cases as follows. $(n \geq n_0)$

As $x \in \left[0, \frac{\pi}{n+p}\right]$, then

$$
\begin{aligned}
\left| \sum_{k=n+1}^{n+p} a_k \sin kx \right| &= \sum_{k=n+1}^{n+p} a_k \sin kx \\
&\leq \sum_{k=n+1}^{n+p} a_k kx \text{ by } \sin kx \leq kx \text{ if } x \geq 0 \\
&= \sum_{k=n+1}^{n+p} (ka_k)\, x \\
&\leq \frac{\varepsilon}{2(\pi+1)} \frac{p\pi}{n+p} \text{ by } (*) \\
&< \varepsilon.
\end{aligned}
$$

And as $x \in \left[\frac{\pi}{n+p}, \pi\right]$, then

$$\begin{aligned}
\left|\sum_{k=n+1}^{n+p} a_k \sin kx\right| &\leq \sum_{k=n+1}^{m} a_k \sin kx + \left|\sum_{k=m+1}^{n+p} a_k \sin kx\right|, \text{ where } m = \left[\frac{\pi}{x}\right] \\
&\leq \sum_{k=n+1}^{m} a_k kx + \frac{2a_{m+1}}{\sin\frac{x}{2}} \text{ by } \textbf{Summation by parts} \\
&\leq \frac{\varepsilon}{2(\pi+1)}(m-n)x + \frac{2a_{m+1}}{\sin\frac{x}{2}} \\
&\leq \frac{\varepsilon}{2(\pi+1)} mx + 2a_{m+1}\frac{\pi}{x} \text{ by } \frac{2x}{\pi} \leq \sin x \text{ if } x \in \left[0, \frac{\pi}{2}\right] \\
&\leq \frac{\varepsilon}{2(\pi+1)}\pi + 2a_{m+1}(m+1) \\
&< \frac{\varepsilon}{2} + 2\frac{\varepsilon}{2(\pi+1)} \\
&< \varepsilon.
\end{aligned}$$

Hence, $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on R.

Remark: (1) In the proof ($\Leftarrow$), if we can make sure that $na_n \searrow 0$, then we can use **the supplement on the convergnce of series in Ch8, (C)-(6)** to show the uniform convergence of $\sum_{n=1}^{\infty} a_n \sin nx = \sum_{n=1}^{\infty} (na_n)\left(\frac{\sin nx}{n}\right)$ by **Dirichlet's test for uniform convergence.**

(2)There are similar results; we write it as references.

(a) Suppose $a_n \searrow 0$, then for each $\alpha \in \left(0, \frac{\pi}{2}\right)$, $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on $[\alpha, 2\pi - \alpha]$.

Proof: The proof follows from **(12) and (13) in Theorem 8.30 and Dirichlet's test for uniform convergence.** So, we omit it. The reader can see the textbook, **example in pp 231.**

(b) Let $\{a_n\}$ be a decreasing sequence of positive terms. $\sum_{n=1}^{\infty} a_n \cos nx$ uniformly converges on R if and only if $\sum_{n=1}^{\infty} a_n$ converges.

Proof: ($\Rightarrow$) Suppose that $\sum_{n=1}^{\infty} a_n \cos nx$ uniformly converges on R, then let $x = 0$, then we have $\sum_{n=1}^{\infty} a_n$ converges.

($\Leftarrow$) Suppose that $\sum_{n=1}^{\infty} a_n$ converges, then by **Weierstrass M-test,** we have proved that $\sum_{n=1}^{\infty} a_n \cos nx$ uniformly converges on R.

9.24 Given a convergent series $\sum_{n=1}^{\infty} a_n$. Prove that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly on the half-infinite interval $0 \leq s < +\infty$. Use this to prove that $\lim_{s\to 0^+} \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} a_n$.

Proof: Let $f_n(s) = \sum_{k=1}^{n} a_k$ and $g_n(s) = n^{-s}$, then by **Abel's test for uniform convergence**, we have proved that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly on the half-infinite interval $0 \leq s < +\infty$. Then by **Theorem 9.2**, we know that $\lim_{s\to 0^+} \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} a_n$.

9.25 Prove that the series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges uniformly on every half-infinite interval $1 + h \leq s < +\infty$, where $h > 0$. Show that the equation

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

is valid for each $s > 1$ and obtain a similar formula for the kth derivative $\zeta^{(k)}(s)$.

Proof: Since $n^{-s} \leq n^{-(1+h)}$ for all $s \in [1+h, \infty)$, we know that $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges uniformly on every half-infinite interval $1+h \leq s < +\infty$ by **Weierstrass M-test.** Define $T_n(s) = \sum_{k=1}^{n} k^{-s}$, then it is clear that

1. For each n, $T_n(s)$ is differentiable on $[1+h, \infty)$,

2. $\lim_{n\to\infty} T_n(2) = \frac{\pi^2}{6}$.

And

3. $T_n'(s) = -\sum_{k=1}^{n} \frac{\log k}{k^s}$ converges uniformly on $[1+h, \infty)$

by **Weierstrass M-test.** Hence, we have proved that

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

by **Theorem 9.13**. By **Mathematical Induction**, we know that

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \frac{(\log n)^k}{n^s}.$$

0.1 Supplement on some results on Weierstrass M-test.

1. In the textbook, **pp 224-223,** there is a surprising result called **Space-filling curve.** In addition, note the proof is related with **Cantor set** in **exercise 7. 32** in the textbook.

2. There exists a continuous function defined on R which is nowhere differentiable. The reader can see the book, **Principles of Mathematical Analysis by Walter Rudin, pp 154.**

Remark: The first example comes from **Bolzano** in **1834**, however, he did **NOT** give a proof. In fact, he only found the function $f : D \to R$ that he constructed is not differentiable on $D' \, (\subseteq D)$ where D' is countable and dense in D. Although the function f is the example, but he did not find the fact.

In **1861**, **Riemann** gave

$$g(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 \pi x)}{n^2}$$

as an example. However, **Reimann** did **NOT** give a proof in his life until **1916**, the proof is given by **G. Hardy.**

In **1860, Weierstrass** gave

$$h(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x), \ b \text{ is odd}, \ 0 < a < 1, \ \text{and } ab > 1 + \frac{3\pi}{2},$$

until **1875,** he gave the proof. The fact surprises the world of **Math**, and produces many examples. There are many researches related with it until now **2003.**

Mean Convergence

9.26 Let $f_n(x) = n^{3/2} x e^{-n^2 x^2}$. Prove that $\{f_n\}$ converges pointwise to 0 on $[-1, 1]$ but that $l.i.m._{n \to \infty} f_n \neq 0$ on $[-1, 1]$.

Proof: It is clear that $\{f_n\}$ converges pointwise to 0 on $[-1, 1]$, so it remains to show that $l.i.m._{n\to\infty}f_n \neq 0$ on $[-1,1]$. Consider

$$\begin{aligned}
\int_{-1}^{1} f_n^2(x)\,dx &= 2\int_0^1 n^3x^2e^{-2n^2x^2}dx \text{ since } f_n^2(x) \text{ is an even function on } [-1,1] \\
&= \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}n} y^2e^{-y^2}dy \text{ by \textbf{Change of Variable}, let } y=\sqrt{2}nx \\
&= \frac{1}{-2\sqrt{2}}\int_0^{\sqrt{2}n} yd\left(e^{-y^2}\right) \\
&= \frac{1}{-2\sqrt{2}}\left[ye^{-y^2}\Big|_0^{\sqrt{2}n} - \int_0^{\sqrt{2}n} e^{-y^2}dy\right] \\
&\to \frac{\sqrt{\pi}}{4\sqrt{2}} \text{ since } \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \text{ by \textbf{Exercise 7. 19}.}
\end{aligned}$$

So, $l.i.m._{n\to\infty}f_n \neq 0$ on $[-1,1]$.

9.27 Assume that $\{f_n\}$ converges pointwise to f on $[a,b]$ and that $l.i.m._{n\to\infty}f_n = g$ on $[a,b]$. Prove that $f=g$ if both f and g are continuous on $[a,b]$.

Proof: Since $l.i.m._{n\to\infty}f_n = g$ on $[a,b]$, given $\varepsilon_k = \frac{1}{2^k}$, there exists a n_k such that

$$\int_a^b |f_{n_k}(x) - g(x)|^p\,dx \leq \frac{1}{2^k}, \text{ where } p>0$$

Define

$$h_m(x) = \sum_{k=1}^m \int_a^x |f_{n_k}(t) - g(t)|^p\,dt,$$

then

a. $h_m(x) \nearrow$ as $x \nearrow$
b. $h_m(x) \leq h_{m+1}(x)$
c. $h_m(x) \leq 1$ for all m and all x.

So, we obtain $h_m(x) \to h(x)$ as $m\to\infty$, $h(x) \nearrow$ as $x\nearrow$, and

$$h(x) - h_m(x) = \sum_{k=m+1}^\infty \int_a^x |f_{n_k}(t) - g(t)|^p\,dt \nearrow \text{ as } x \nearrow$$

which implies that

$$\frac{h(x+t)-h(x)}{t} \geq \frac{h_m(x+t)-h_m(x)}{t} \text{ for all } m. \tag{*}$$

Since h and h_m are increasing, we have h' and h'_m exists a.e. on $[a,b]$. Hence, by (*)

$$h'_m(x) = \sum_{k=1}^{m} |f_{n_k}(t) - g(t)|^p \leq h'(x) \text{ a.e. on } [a,b]$$

which implies that

$$\sum_{k=1}^{\infty} |f_{n_k}(t) - g(t)|^p \text{ exists a.e. on } [a,b].$$

So, $f_{n_k}(t) \to g(t)$ a.e. on $[a,b]$. In addition, $f_n \to f$ on $[a,b]$. Then we conclude that $f = g$ a.e. on $[a,b]$. Since f and g are continuous on $[a,b]$, we have

$$\int_a^b |f-g|\,dx = 0$$

which implies that $f = g$ on $[a,b]$. In particular, as $p = 2$, we have $f = g$.

Remark: (1) A property is said to hold **almost everywhere on a set** S (written: a.e. on S) if it holds everywhere on S except for a set of measurer zero. Also, see the textbook, **pp 254**.

(2) In this proof, we use the theorem which states: A monotonic function h defined on $[a,b]$, then h is differentiable a.e. on $[a,b]$. The reader can see the book, The reader can see the book, **Measure and Integral (An Introduction to Real Analysis) written by Richard L. Wheeden and Antoni Zygmund**, **pp 113.**

(3) There is another proof by using **Fatou's lemma**: Let $\{f_k\}$ be a measruable function defined on a measure set E. If $f_k \geq \phi$ a.e. on E and $\phi \in L(E)$, then

$$\int_E \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int_E f_k.$$

Proof: It suffices to show that $f_{n_k}(t) \to g(t)$ a.e. on $[a,b]$. Since $l.i.m._{n\to\infty} f_n = g$ on $[a,b]$, and given $\varepsilon > 0$, there exists a n_k such that

$$\int_a^b |f_{n_k} - g|^2\,dx < \frac{1}{2^k}$$

which implies that

$$\int_a^b \sum_{k=1}^m |f_{n_k} - g|^2 \, dx < \sum_{k=1}^m \frac{1}{2^k}$$

which implies that, by **Fatou's lemma,**

$$\begin{aligned}\int_a^b \liminf_{m\to\infty} \sum_{k=1}^m |f_{n_k} - g|^2 \, dx &\le \liminf_{m\to\infty} \int_a^b \sum_{k=1}^m |f_{n_k} - g|^2 \, dx \\ &= \sum_{k=1}^\infty \int_a^b |f_{n_k} - g|^2 \, dx < 1.\end{aligned}$$

That is,

$$\int_a^b \sum_{k=1}^\infty |f_{n_k} - g|^2 \, dx < 1$$

which implies that

$$\sum_{k=1}^\infty |f_{n_k} - g|^2 < \infty \text{ a.e. on } [a,b]$$

which implies that $f_{n_k} \to g$ a.e. on $[a,b]$.

Note: The reader can see the book, **Measure and Integral (An Introduction to Real Analysis) written by Richard L. Wheeden and Antoni Zygmund**, **pp 75.**

(4) There is another proof by using **Egorov's Theorem**: Let $\{f_k\}$ be a measurable functions defined on a finite measurable set E with finite limit function f. Then given $\varepsilon > 0$, there exists a closed set $F\,(\subseteq E)$, where $|E - F| < \varepsilon$ such that

$$f_k \to f \text{ uniformly on } F.$$

Proof: If $f \ne g$ on $[a,b]$, then $h := |f - g| \ne 0$ on $[a,b]$. By continuity of h, there exists a compact subinterval $[c,d]$ such that $|f - g| \ne 0$. So, there exists $m > 0$ such that $h = |f - g| \ge m > 0$ on $[c,d]$. Since

$$\int_a^b |f_n - g|^2 \, dx \to 0 \text{ as } n \to \infty,$$

we have

$$\int_c^d |f_n - g|^2 \, dx \to 0 \text{ as } n \to \infty.$$

then by **Egorov's Theorem,** given $\varepsilon > 0$, there exists a closed susbet F of $[c, d]$, where $|[c, d] - F| < \varepsilon$ such that

$$f_n \to f \text{ uniformly on } F$$

which implies that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_F |f_n - g|^2 \, dx \\ &= \int_F \lim_{n\to\infty} |f_n - g|^2 \, dx \\ &= \int_F |f - g|^2 \, dx \geq m^2 |F| \end{aligned}$$

which implies that $|F| = 0$. If we choose $\varepsilon < d - c$, then we get a contradiction. Therefore, $f = g$ on $[a, b]$.

Note: The reader can see the book, **Measure and Integral (An Introduction to Real Analysis) written by Richard L. Wheeden and Antoni Zygmund**, **pp 57.**

9.28 Let $f_n(x) = \cos^n x$ if $0 \leq x \leq \pi$.

(a) Prove that $l.i.m._{n\to\infty} f_n = 0$ on $[0, \pi]$ but that $\{f_n(\pi)\}$ does not converge.

Proof: It is clear that $\{f_n(\pi)\}$ does not converge since $f_n(\pi) = (-1)^n$. It remains to show that $l.i.m._{n\to\infty} f_n = 0$ on $[0, \pi]$. Consider $\cos^{2n} x := g_n(x)$ on $[0, \pi]$, then it is clear that $\{g_n(x)\}$ is boundedly convergent with limit function

$$g = \begin{cases} 0 \text{ if } x \in (0, \pi) \\ 1 \text{ if } x = 0 \text{ or } \pi \end{cases}.$$

Hence, by **Arzela's Theorem**,

$$\lim_{n\to\infty} \int_0^\pi \cos^{2n} x dx = \int_0^\pi g(x) \, dx = 0.$$

So, $l.i.m._{n\to\infty} f_n = 0$ on $[0, \pi]$.

(b) Prove that $\{f_n\}$ converges pointwise but not uniformly on $[0, \pi/2]$.

Proof: Note that each $f_n(x)$ is continuous on $[0, \pi/2]$, and the limit function

$$f = \begin{cases} 0 \text{ if } x \in (0, \pi/2] \\ 1 \text{ if } x = 0 \end{cases}.$$

Hence, by **Theorem9.2**, we know that $\{f_n\}$ converges pointwise but not uniformly on $[0, \pi/2]$.

9.29 Let $f_n(x) = 0$ if $0 \leq x \leq 1/n$ or $2/n \leq x \leq 1$, and let $f_n(x) = n$ if $1/n < x < 2/n$. Prove that $\{f_n\}$ converges pointwise to 0 on $[0,1]$ but that $l.i.m._{n\to\infty} f_n \neq 0$ on $[0,1]$.

Proof: It is clear that $\{f_n\}$ converges pointwise to 0 on $[0,1]$. In order to show that $l.i.m._{n\to\infty} f_n \neq 0$ on $[0,1]$, it suffices to note that

$$\int_0^1 f_n(x)\, dx = 1 \text{ for all } n.$$

Hence, $l.i.m._{n\to\infty} f_n \neq 0$ on $[0,1]$.

Power series

9.30 If r is the radius of convergence if $\sum a_n (z - z_0)^n$, where each $a_n \neq 0$, show that

$$\lim_{n\to\infty} \inf \left| \frac{a_n}{a_{n+1}} \right| \leq r \leq \lim_{n\to\infty} \sup \left| \frac{a_n}{a_{n+1}} \right|.$$

Proof: By **Exercise 8.4**, we have

$$\frac{1}{\lim_{n\to\infty} \sup \left| \frac{a_{n+1}}{a_n} \right|} \leq r = \frac{1}{\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}} \leq \frac{1}{\lim_{n\to\infty} \inf \left| \frac{a_{n+1}}{a_n} \right|}.$$

Since

$$\frac{1}{\lim_{n\to\infty} \sup \left| \frac{a_{n+1}}{a_n} \right|} = \lim_{n\to\infty} \inf \left| \frac{a_n}{a_{n+1}} \right|$$

and

$$\frac{1}{\lim_{n\to\infty} \inf \left| \frac{a_{n+1}}{a_n} \right|} = \lim_{n\to\infty} \sup \left| \frac{a_n}{a_{n+1}} \right|,$$

we complete it.

9.31 Given that two power series $\sum a_n z^n$ has radius of convergence 2. Find the radius convergence of each of the following series: In (a) and (b), k is a fixed positive integer.

(a) $\sum_{n=0}^{\infty} a_n^k z^n$

Proof: Since

$$2 = \frac{1}{\lim_{n\to\infty} \sup |a_n|^{1/n}}, \tag{*}$$

we know that the radius of $\sum_{n=0}^{\infty} a_n^k z^n$ is

$$\frac{1}{\lim_{n\to\infty} \sup |a_n^k|^{1/n}} = \frac{1}{\left(\lim_{n\to\infty} \sup |a_n|^{1/n}\right)^k} = 2^k.$$

(b) $\sum_{n=0}^{\infty} a_n z^{kn}$

Proof: Consider

$$\lim_{n\to\infty} \sup \left|a_n z^{kn}\right|^{1/n} = \lim_{n\to\infty} \sup |a_n|^{1/n} |z|^k < 1$$

which implies that

$$|z| < \left(\frac{1}{\lim_{n\to\infty} \sup |a_n|^{1/n}}\right)^{1/k} = 2^{1/k} \text{ by } (*).$$

So, the radius of $\sum_{n=0}^{\infty} a_n z^{kn}$ is $2^{1/k}$.

(c) $\sum_{n=0}^{\infty} a_n z^{n^2}$

Proof: Consider

$$\lim \sup \left|a_n z^{n^2}\right|^{1/n} = \lim_{n\to\infty} \sup |a_n|^{1/n} |z|^n$$

and claim that the radius of $\sum_{n=0}^{\infty} a_n z^{n^2}$ is 1 as follows.

If $|z| < 1$, it is clearly seen that the series converges. However, if $|z| > 1$,

$$\lim_{n\to\infty} \sup |a_n|^{1/n} \lim_{n\to\infty} \inf |z|^n \le \lim_{n\to\infty} \sup |a_n|^{1/n} |z|^n$$

which impliest that

$$\lim_{n\to\infty}\sup|a_n|^{1/n}|z|^n = +\infty.$$

so, the series diverges. From above, we have proved the claim.

9.32 Given a power series $\sum a_n x^n$ whose coefficents are related by an equation of the form

$$a_n + Aa_{n-1} + Ba_{n-2} = 0\ (n = 2, 3, ...).$$

Show that for any x for which the series converges, its sum is

$$\frac{a_0 + (a_1 + Aa_0)\,x}{1 + Ax + Bx^2}.$$

Proof: Consider

$$\sum_{n=2}^{\infty}(a_n + Aa_{n-1} + Ba_{n-2})\,x^n = 0$$

which implies that

$$\sum_{n=2}^{\infty}a_n x^n + Ax\sum_{n=2}^{\infty}a_{n-1}x^{n-1} + Bx^2\sum_{n=2}^{\infty}a_{n-2}x^{n-2} = 0$$

which implies that

$$\sum_{n=0}^{\infty}a_n x^n + Ax\sum_{n=0}^{\infty}a_n x^n + Bx^2\sum_{n=0}^{\infty}a_n x^n = a_0 + a_1 x + Aa_0 x$$

which implies that

$$\sum_{n=0}^{\infty}a_n x^n = \frac{a_0 + (a_1 + Aa_0)\,x}{1 + Ax + Bx^2}.$$

Remark: We prove that for any x for which the series converges, then $1 + Ax + Bx^2 \neq 0$ as follows.

Proof: Consider

$$\left(1 + Ax + Bx^2\right)\sum_{n=0}^{\infty}a_n x^n = a_0 + (a_1 + Aa_0)\,x,$$

if $x = \lambda\,(\neq 0)$ is a root of $1 + Ax + Bx^2$, and $\sum_{n=0}^{\infty} a_n \lambda^n$ exists, we have

$$1 + A\lambda + B\lambda^2 = 0 \text{ and } a_0 + (a_1 + Aa_0)\,\lambda = 0.$$

Note that $a_1 + Aa_0 \neq 0$, otherwise, $a_0 = 0\,(\Rightarrow a_1 = 0)$, and therefore, $a_n = 0$ for all n. Then there is nothing to prove it. So, put $\lambda = \frac{-a_0}{a_1 + Aa_0}$ into $1 + A\lambda + B\lambda^2 = 0$, we then have

$$a_1^2 = a_0 a_2.$$

Note that $a_0 \neq 0$, otherwise, $a_1 = 0$ and $a_2 = 0$. Similarly, $a_1 \neq 0$, otherwise, we will obtain a trivial thing. Hence, we may assume that all $a_n \neq 0$ for all n. So,

$$a_2^2 = a_1 a_3.$$

And it is easy to check that $a_n = a_0 \frac{1}{\lambda^n}$ for all $n \geq N$. Therefore, $\sum a_n \lambda^n = \sum a_0$ diverges. So, for any x for whcih the series converges, we have $1 + Ax + Bx^2 \neq 0$.

9.33 Let $f(x) = e^{-1/x^2}$ if $x \neq 0$, $f(0) = 0$.

(a) Show that $f^{(n)}(0)$ exists for all $n \geq 1$.

Proof: By **Exercise 5.4**, we complete it.

(b) Show that the Taylor's series about 0 generated by f converges everywhere on R but that it represents f only at the origin.

Proof: The Taylor's series about 0 generated by f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 0 x^n = 0.$$

So, it converges everywhere on R but that it represents f only at the origin.

Remark: It is an important example to tell us that even for functions $f \in C^{\infty}(R)$, the Taylor's series about c generated by f may **NOT** represent f on some open interval. Also see the textbook, **pp 241**.

9.34 Show that the binomial series $(1 + x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ exhibits the following behavior at the points $x = \pm 1$.

(a) If $x = -1$, the series converges for $\alpha \geq 0$ and diverges for $\alpha < 0$.

Proof: If $x = -1$, we consider three cases: (i) $\alpha < 0$, (ii) $\alpha = 0$, and (iii) $\alpha > 0$.

(i) As $\alpha < 0$, then

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!},$$

say $a_n = (-1)^n \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$, then $a_n \geq 0$ for all n, and

$$\frac{a_n}{1/n} = \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{(n-1)!} \geq -\alpha > 0 \text{ for all } n.$$

Hence, $\sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n$ diverges.

(ii) As $\alpha = 0$, then the series is clearly convergent.

(iii) As $\alpha > 0$, define $a_n = n(-1)^n \binom{\alpha}{n}$, then

$$\frac{a_{n+1}}{a_n} = \frac{n-\alpha}{n} \geq 1 \text{ if } n \geq [\alpha]+1. \qquad (*)$$

It means that $a_n > 0$ for all $n \geq [\alpha]+1$ or $a_n < 0$ for all $n \geq [\alpha]+1$. Without loss of generality, we consider $a_n > 0$ for all $n \geq [\alpha]+1$ as follows.

Note that (*) tells us that

$$a_n > a_{n+1} > 0 \Rightarrow \lim_{n\to\infty} a_n \text{ exists.}$$

and

$$a_n - a_{n+1} = \alpha(-1)^n \binom{\alpha}{n}.$$

So,

$$\sum_{n=[\alpha]+1}^{m} (-1)^n \binom{\alpha}{n} = \frac{1}{\alpha} \sum_{n=[\alpha]+1}^{m} (a_n - a_{n+1}).$$

By **Theorem 8.10**, we have proved the convergence of the series $\sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n$.

(b) If $x = 1$, the series diverges for $\alpha \leq -1$, converges conditionally for α in the interval $-1 < \alpha < 0$, and converges absolutely for $\alpha \geq 0$.

Proof: If $x = 1$, we consider four cases as follows: (i) $\alpha \leq -1$, (ii) $-1 < -\alpha < 0$, (iii) $\alpha = 0$, and (iv) $\alpha > 0$:

(i) As $\alpha \leq -1$, say $a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. Then

$$|a_n| = \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{n!} \geq 1 \text{ for all } n.$$

So, the series diverges.

(ii) As $-1 < \alpha < 0$, say $a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. Then $a_n = (-1)^n b_n$, where

$$b_n = \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{n!} > 0.$$

with

$$\frac{b_{n+1}}{b_n} = \frac{n-\alpha}{n} < 1 \text{ since } -1 < -\alpha < 0$$

which implies that $\{b_n\}$ is decreasing with limit L. So, if we can show $L = 0$, then $\sum a_n$ converges by **Theorem 8.16.**

Rewrite

$$b_n = \prod_{k=1}^{n}\left(1 - \frac{\alpha+1}{k}\right)$$

and since $\sum \frac{\alpha+1}{k}$ diverges, then by **Theroem 8.55**, we have proved $L = 0$.

In order to show the convergence is conditionally, it suffices to show the divergence of $\sum b_n$. The fact follows from

$$\frac{b_n}{1/n} = \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{(n-1)!} \geq -\alpha > 0.$$

(iii) As $\alpha = 0$, it is clearly that the series converges absolutely.

(iv) As $\alpha > 0$, we consider $\sum \left|\binom{\alpha}{n}\right|$ as follows. Define $a_n = \left|\binom{\alpha}{n}\right|$, then

$$\frac{a_{n+1}}{a_n} = \frac{n-\alpha}{n+1} < 1 \text{ if } n \geq [\alpha] + 1.$$

It implies that $na_n - (n+1)a_n = \alpha a_n$ and $(n+1)a_{n+1} < na_n$. So, by **Theroem 8.10**,

$$\sum a_n = \frac{1}{\alpha}\sum na_n - (n+1)a_n$$

converges since $\lim_{n\to\infty} na_n$ exists. So, we have proved that the series converges absolutely.

9.35 Show that $\sum a_n x^n$ converges uniformly on $[0,1]$ if $\sum a_n$ converges. Use this fact to give another proof of Abel's limit theorem.

Proof: Define $f_n(x) = a_n$ on $[0,1]$, then it is clear that $\sum f_n(x)$ converges uniformly on $[0,1]$. In addition, let $g_n(x) = x^n$, then $g_n(x)$ is unifomrly bouned with $g_{n+1}(x) \leq g_n(x)$. So, by Abel's test for uniform convergence,

$\sum a_n x^n$ converges uniformly on $[0,1]$. Now, we give another proof of **Abel's Limit Theorem** as follows. Note that each term of $\sum a_n x^n$ is continuous on $[0,1]$ and the convergence is uniformly on $[0,1]$, so by **Theorem 9.2,** the power series is continuous on $[0,1]$. That is, we have proved **Abel's Limit Theorem:**

$$\lim_{x\to 1^-} \sum a_n x^n = \sum a_n.$$

9.36 If each $a_n > 0$ and $\sum a_n$ diverges, show that $\sum a_n x^n \to +\infty$ as $x \to 1^-$. (Assume $\sum a_n x^n$ converges for $|x| < 1$.)

Proof: Given $M > 0$, if we can find a y near 1 from the left such that $\sum a_n y^n \geq M$, then for $y \leq x < 1$, we have

$$M \leq \sum a_n y^n \leq \sum a_n x^n.$$

That is, $\lim_{x\to 1^-} \sum a_n x^n = +\infty$.

Since $\sum a_n$ diverges, there is a positive integer p such that

$$\sum_{k=1}^{p} a_k \geq 2M > M. \tag{*}$$

Define $f_n(x) = \sum_{k=1}^{n} a_k x^k$, then by continuity of each f_n, given $0 < \varepsilon\,(< M)$, there exists a $\delta_n > 0$ such that as $x \in [\delta_n, 1)$, we have

$$\sum_{k=1}^{n} a_k - \varepsilon < \sum_{k=1}^{n} a_k x^k < \sum_{k=1}^{n} a_k + \varepsilon \tag{**}$$

By (*) and (**), we proved that as $y = \delta_p$

$$M \leq \sum_{k=1}^{p} a_k - \varepsilon < \sum_{k=1}^{p} a_k y^k.$$

Hence, we have proved it.

9.37 If each $a_n > 0$ and if $\lim_{x\to 1^-} \sum a_n x^n$ exists and equals A, prove that $\sum a_n$ converges and has the sum A. (Compare with Theorem 9.33.)

Proof: By **Exercise 9.36**, we have proved the part, $\sum a_n$ converges. In order to show $\sum a_n = A$, we apply **Abel's Limit Theorem** to complete it.

9.38 For each real t, define $f_t(x) = xe^{xt}/(e^x - 1)$ if $x \in R$, $x \neq 0$, $f_t(0) = 1$.

(a) Show that there is a disk $B(0;\delta)$ in which f_t is represented by a power series in x.

Proof: First, we note that $\frac{e^x-1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} := p(x)$, then $p(0) = 1 \neq 0$. So, by **Theorem 9. 26**, there exists a disk $B(0;\delta)$ in which the reciprocal of p has a power series exapnsion of the form

$$\frac{1}{p(x)} = \sum_{n=0}^{\infty} q_n x^n.$$

So, as $x \in B(0;\delta)$ by **Theorem 9.24.**

$$\begin{aligned} f_t(x) &= xe^{xt}/(e^x - 1) \\ &= \left(\sum_{n=0}^{\infty} \frac{(xt)^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}\right) \\ &= \sum_{n=0}^{\infty} r_n(t) x^n. \end{aligned}$$

(b) Define $P_0(t), P_1(t), P_2(t), \ldots$, by the equation

$$f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!}, \text{ if } x \in B(0;\delta),$$

and use the identity

$$\sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}$$

to prove that $P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0) t^{n-k}$.

Proof: Since

$$f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \frac{x}{e^x - 1},$$

and

$$f_0(x) = \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

So, we have the identity

$$\sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}.$$

Use the identity with $e^{tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n$, then we obtain

$$\begin{aligned} \frac{P_n(t)}{n!} &= \sum_{k=0}^{n} \frac{t^{n-k}}{(n-k)!} \frac{P_k(0)}{k!} \\ &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} P_k(0)\, t^{n-k} \end{aligned}$$

which implies that

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0)\, t^{n-k}.$$

This shows that each function P_n is a polynomial. There are the **Bernoulli polynomials**. The numbers $B_n = P_n(0)$ $(n = 0, 1, 2, ...)$ are called the **Bernoulli numbers**. Derive the following further properties:

(c) $B_0 = 1$, $B_1 = -\frac{1}{2}$, $\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$, if $n = 2, 3, ...$

Proof: Since $1 = \frac{p(x)}{p(x)}$, where $p(x) := \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$, and $\frac{1}{p(x)} := \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}$. So,

$$\begin{aligned} 1 &= p(x) \frac{1}{p(x)} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} C_n x^n \end{aligned}$$

where

$$C_n = \frac{1}{(n+1)!} \sum_{k=0}^{n} \binom{n+1}{k} P_k(0).$$

So,

$$B_0 = P_0(0) = C_0 = 1,$$

$$B_1 = P_1(0) = \frac{C_1 - P_0(0)}{2} = -\frac{1}{2}, \text{ by } C_1 = 0$$

and note that $C_n = 0$ for all $n \geq 1$, we have

$$\begin{aligned} 0 &= C_{n-1} \\ &= \frac{1}{n!} \sum_{k=0}^{n-1} \tbinom{n}{k} P_k(0) \\ &= \frac{1}{n!} \sum_{k=0}^{n-1} \tbinom{n}{k} B_k \text{ for all } n \geq 2. \end{aligned}$$

(d) $P'_n(t) = nP_{n-1}(t)$, if $n = 1, 2, ...$

Proof: Since

$$\begin{aligned} P'_n(t) &= \sum_{k=0}^{n} \tbinom{n}{k} P_k(0)(n-k)t^{n-k-1} \\ &= \sum_{k=0}^{n-1} \tbinom{n}{k} P_k(0)(n-k)t^{n-k-1} \\ &= \sum_{k=0}^{n-1} \frac{n!(n-k)}{k!(n-k)!} P_k(0)t^{(n-1)-k} \\ &= \sum_{k=0}^{n-1} n\frac{(n-1)!}{k!(n-1-k)!} P_k(0)t^{(n-1)-k} \\ &= n\sum_{k=0}^{n-1} \tbinom{n-1}{k} P_k(0)t^{(n-1)-k} \\ &= nP_{n-1}(t) \text{ if } n = 1, 2, ... \end{aligned}$$

(e) $P_n(t+1) - P_n(t) = nt^{n-1}$ if $n = 1, 2, ...$

Proof: Consider

$$\begin{aligned} f_{t+1}(x) - f_t(x) &= \sum_{n=0}^{\infty} \left[P_n(t+1) - P_n(t)\right] \frac{x^n}{n!} \text{ by (b)} \\ &= xe^{xt} \text{ by } f_t(x) = xe^{xt}/\left(e^x - 1\right) \\ &= \sum_{n=0}^{\infty} (n+1)\, t^n \frac{x^{n+1}}{(n+1)!}, \end{aligned}$$

so as $n = 1, 2, ...$, we have

$$P_n(t+1) - P_n(t) = nt^{n-1}.$$

(f) $P_n(1-t) = (-1)^n P_n(t)$

Proof: Note that

$$f_t(-x) = f_{1-t}(x),$$

so we have

$$\sum_{n=0}^{\infty} (-1)^n P_n(t) \frac{x^n}{n!} = \sum_{n=0}^{\infty} P_n(1-t) \frac{x^n}{n!}.$$

Hence, $P_n(1-t) = (-1)^n P_n(t)$.

(g) $B_{2n+1} = 0$ if $n = 1, 2, ...$

Proof: With help of (e) and (f), let $t = 0$ and $n = 2k+1$, then it is clear that $B_{2k+1} = 0$ if $k = 1, 2, ...$

(h) $1^n + 2^n + ... + (k-1)^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1}$ $(n = 2, 3, ...)$

Proof: With help of (e), we know that

$$\frac{P_{n+1}(t+1) - P_{n+1}(t)}{n+1} = t^n$$

which implies that

$$1^n + 2^n + ... + (k-1)^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1} \quad (n = 2, 3, ...)$$

Remark: (1) The reader can see the book, **Infinite Series by Chao Wen-Min, pp 355-366. (Chinese Version)**

(2) There are some special polynomials worth studying, such as **Legengre Polynomials.** The reader can see the book, **Essentials of Ordinary Differential Equations by Ravi P. Agarwal and Ramesh C. Gupta. pp 305-312.**

(3) The part (h) tells us one formula to calculte the value of the finite series$\sum_{k=1}^{m} k^n$. There is an interesting story from the mail that **Fermat, pierre de (1601-1665)** sent to **Blaise Pascal (1623-1662). Fermat** used the **Mathematical Induction** to show that

$$\sum_{k=1}^{n} k\,(k+1)\cdots(k+p) = \frac{n\,(n+1)\cdots(n+p+1)}{p+2}. \qquad (*)$$

In terms of (*), we can obtain another formula on $\sum_{k=1}^{m} k^n$.