The Riemann-Stieltjes Integral

Riemann-Stieltjes integrals

7.1 Prove that \( \int_a^b d\alpha = \alpha(b) - \alpha(a) \), directly from Definition 7.1.

**Proof:** Let \( f = 1 \) on \([a, b]\), then given any partition \( P = \{a = x_0, \ldots, x_n = b\} \), then we have

\[
S(P, 1, \alpha) = \sum_{k=1}^{n} f(t_k) \Delta \alpha_k, \text{ where } t_k \in [x_{k-1}, x_k]
\]

\[
= \sum_{k=1}^{n} \Delta \alpha_k
\]

\[
= \alpha(b) - \alpha(a).
\]

So, we know that \( \int_a^b d\alpha = \alpha(b) - \alpha(a) \).

7.2 If \( f \in R(\alpha) \) on \([a, b]\) and if \( \int_a^b f d\alpha = 0 \) for every \( f \) which is monotonic on \([a, b]\), prove that \( \alpha \) must be constant on \([a, b]\).

**Proof:** Use integration by parts, and thus we have

\[
\int_a^b f d\alpha = f(b) \alpha(b) - f(a) \alpha(a)
\]

Given any point \( c \in [a, b] \), we may choose a monotonic function \( f \) defined as follows.

\[
f = \begin{cases} 
0 & \text{if } x \leq c \\
1 & \text{if } x > c.
\end{cases}
\]

So, we have

\[
\int_a^b f d\alpha = \alpha(c) = \alpha(b).
\]

So, we know that \( \alpha \) is constant on \([a, b]\).

7.3 The following definition of a Riemann-Stieltjes integral is often used in the literature: We say that \( f \) is integrable with respect to \( \alpha \) if there exists a real number \( A \) having the property that for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for every partition \( P \) of \([a, b]\) with norm \( ||P|| < \delta \) and for every choice of \( t_k \) in \([x_{k-1}, x_k]\), we have

\[
|S(P, f, \alpha) - A| < \epsilon.
\]

(a) Show that if \( \int_a^b f d\alpha \) exists according to this definition, then it is also exists according to Definition 7.1 and the two integrals are equal.

**Proof:** Since refinement will decrease the norm, we know that if there exists a real number \( A \) having the property that for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for every partition \( P \) of \([a, b]\) with norm \( ||P|| < \delta \) and for every choice of \( t_k \) in \([x_{k-1}, x_k]\), we have

\[
|S(P, f, \alpha) - A| < \epsilon.
\]

Then choosing a \( P_\epsilon \) with \( ||P_\epsilon|| < \delta \), then for \( P \subseteq P_\epsilon \Rightarrow ||P|| < \delta \). So, we have

\[
|S(P, f, \alpha) - A| < \epsilon.
\]

That is, \( \int_a^b f d\alpha \) exists according to this definition, then it is also exists according to Definition 7.1 and the two integrals are equal.
(b) Let \( f(x) = a(x) = 0 \) for \( a \leq x < c \), \( f(x) = a(x) = 1 \) for \( c < x \leq b \), \( f(c) = 0, a(c) = 1 \). Show that \( \int_a^b f \, dx \) exists according to Definition 7.1 but does not exist by this second definition.

**Proof:** Note that \( \int_a^b f \, dx \) exists and equals 0 according to Definition 7.1. If \( \int_a^b f \, dx \) exists according to this definition, then given \( \varepsilon = 1 \), there exists a \( \delta > 0 \) such that for every partition \( P \) of \( [a, b] \) with norm \( ||P|| < \delta \) and for every choice of \( t_k \) in \( [x_{k-1}, x_k] \), we have \( |S(P, f, \alpha)| < 1 \). We may choose a partition \( P = \{a = x_0, \ldots, x_n = b\} \) with \( ||P|| < \delta \) and \( c \in (x_j, x_{j+1}) \), where \( j = 0, \ldots, n - 1 \). Then
\[
S(P, f, \alpha) = f(x)[a(x_{j+1}) - a(x_j)] = 1, \quad \text{where} \quad x \in (c, x_{j+1})
\]
which contradicts to \( |S(P, f, \alpha)| < 1 \).

**7.4** If \( f \in \mathbb{R} \) according to Definition 7.1, prove that \( \int_a^b f(x) \, dx \) also exists according to definition of Exercise 7.3. [Contrast with Exercise 7.3 (b).]

**Hint:** Let \( I = \int_a^b f(x) \, dx \), \( M = \sup \{|f(x)| : x \in [a, b]\} \). Given \( \varepsilon > 0 \), choose \( P_\varepsilon \) so that \( U(P_\varepsilon, f) < I + \varepsilon/2 \) (notation of section 7.11). Let \( N \) be the number of subdivision points in \( P_\varepsilon \) and let \( \delta = \varepsilon/(2MN) \). If \( ||P|| < \delta \), write
\[
U(P, f) = \sum M_k(f) \Delta x_k = S_1 + S_2,
\]
where \( S_1 \) is the sum of terms arising from those subintervals of \( P \) containing no points of \( P_\varepsilon \) and \( S_2 \) is the sum of remaining terms. Then
\[
S_1 \leq U(P_\varepsilon, f) < I + \varepsilon/2 \quad \text{and} \quad S_2 \leq NM||P|| < NM\delta = \varepsilon/2,
\]
and hence \( U(P_\varepsilon, f) < I + \varepsilon \). Similarly,
\[
L(P, f) > I - \varepsilon \quad \text{if} \quad ||P|| < \delta' \quad \text{for some} \quad \delta'.
\]
Hence \( |S(P, f) - I| < \varepsilon \) if \( ||P|| < \min(\delta, \delta') \).

**Proof:** The hint has proved it.

**Remark:** There are some exercises related with Riemann integrals, we write thme as references.

(1) Suppose that \( f \geq 0 \) and \( f \) is continuous on \( [a, b] \), and \( \int_a^b f(x) \, dx = 0 \). Prove that \( f(x) = 0 \) on \( [a, b] \).

**Proof:** Assume that there is a point \( c \in [a, b] \) such that \( f(c) > 0 \). Then by continuity of \( f \), we know that given \( \varepsilon = \frac{f(c)}{2} > 0 \), there is a \( \delta > 0 \) such that as \( |x - c| < \delta, \ x \in [a, b] \), we have
\[
|f(x) - f(c)| < \frac{f(c)}{2}
\]
which implies that
\[
\frac{f(c)}{2} < f(x) \quad \text{if} \quad x \in (c - \delta, c + \delta) \cap [a, b] := I
\]
So, we have
\[
0 < \frac{f(c)}{2} |I| \leq \int_I f(x) \, dx \leq \int_a^b f(x) \, dx = 0, \quad \text{where} \quad 0 < |I|, \quad \text{the length of} \ I
\]
which is absurd. Hence, we obtain that \( f(x) = 0 \) on \( [a, b] \).

(2) Let \( f \) be a continuous function defined on \( [a, b] \). Suppose that for every continuous function \( g \) defined on \( [a, b] \) which satisfies that
\[ \int_a^b g(x)dx = 0, \]
we always have
\[ \int_a^b f(x)g(x)dx = 0. \]
Show that \( f \) is a constant function on \([a, b]\).

**Proof:** Let \( \int_a^b f(x)dx = I \), and define \( g(x) = f(x) - \frac{I}{b-a} \), then we have
\[ \int_a^b g(x)dx = 0, \]
which implies that, by hypothesis,
\[ \int_a^b f(x)g(x)dx = 0 \]
which implies that
\[ \int_a^b (f(x) - c)g(x)dx = 0 \]
for any real \( c \).
So, we have
\[ \int_a^b (g(x))^2dx = 0 \]
if letting \( c = \frac{I}{b-a} \) on \([a, b]\).

(3) Define
\[ h(x) = \begin{cases} 
0 & \text{if } x \in [0, 1] - Q \\
\frac{1}{n} & \text{if } x \text{ is the rational number } m/n \text{ (in lowest terms)} \\
1 & \text{if } x = 0.
\end{cases} \]
Then \( h \in R([0, 1]) \).

**Proof:** Note that we have shown that \( h \) is continuous only at irrational numbers on \([0, 1] - Q\). We use it to show that \( h \) is Riemann integrable, i.e., \( h \in R([0, 1]) \). Consider the upper sum \( U(P, f) \) as follows.

Given \( \varepsilon > 0 \), there exists finitely many points \( x \) such that \( f(x) \geq \varepsilon/2 \). Consider a partition \( P_\varepsilon = \{x_0 = a, \ldots, x_n = b\} \) so that its subintervals \( I_j = [x_{j-1}, x_j] \) for some \( j \) containing those points and \( \sum |I_j| < \varepsilon/2 \). So, we have
\[ U(P, f) = \sum_{k=1}^n M_k \Delta x_k \]
\[ = \sum_1^1 + \sum_2^2 \]
\[ < \varepsilon/2 + \varepsilon/2 \]
\[ = \varepsilon \]
where \( \sum_1 = \sum_1 M_j I_j \), and \( \sum_2 \), is the sum of others.
So, we have shown that \( f \) satisfies the Riemann condition with respect to \( \alpha(x) = x \).

**Note:** (1) The reader can show this by **Theorem 7.48 (Lebesgue’s Criterion for Riemann Integrability).** Also, compare Exercise 7.32 and Exercise 4.16 with this.

(2) In **Theorem 7.19**, if we can make sure that there is a partition \( P_\varepsilon \) such that
\[ U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon, \]
then we automatically have, for any finer \( P \subset P \varepsilon \),
\[ U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \]
since the refinement makes \( U \) increase and \( L \) decrease.

(4) Assume that the function \( f(x) \) is differentiable on \([a, b]\), but not a constant and that \( f(a) = f(b) = 0 \). Then there exists at least one point \( \xi \) on \((a, b)\) for which
\[ |f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x) dx. \]

**Proof:** Consider \( \sup_{x \in [a, b]} |f'(x)| = M \) as follows.
(i) If \( M = +\infty \), then it is clear.
(ii) We may assume that \( M < +\infty \).

Let \( x \in [a, \frac{a+b}{2}] \), then
\[ f(x) = f(x) - f(a) = f'(y)(x-a) \leq M(x-a), \text{ where } y \in (a, x). \]
and let \( x \in [\frac{a+b}{2}, b] \), then
\[ f(x) = f(x) - f(b) = f'(z)(x-b) \leq M(b-x), \text{ where } z \in (x, b). \]

So, by (*) and (**), we know that
\[
\int_a^b f(x) dx = \int_a^{\frac{a+b}{2}} f(x) dx + \int_{\frac{a+b}{2}}^b f(x) dx
\leq M \int_a^{\frac{a+b}{2}} (x-a) dx + M \int_{\frac{a+b}{2}}^b (b-x) dx
= M \left( \frac{a-b}{2} \right)^2
\]
which implies that
\[ M \geq \frac{4}{(b-a)^2} \int_a^b f(x) dx. \]

Note that by (*) and (**), the equality does NOT hold since if it was, then we had \( f'(x) = M \) on \([a, b]\) which implies that \( f \) is a constant function. So, we have
\[ M > \frac{4}{(b-a)^2} \int_a^b f(x) dx. \]

By definition of supremum, we know that there exists at least one point \( \xi \) on \((a, b)\) for which
\[ |f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x) dx. \]

(5) **Gronwall Lemma:** Let \( f \) and \( g \) be continuous non-negative function defined on \([a, b]\), and \( c \geq 0 \). If
\[ f(x) \leq c + \int_a^x g(t) f(t) dt \text{ for all } x \in [a, b], \]
then
\[ f(x) \leq ce^{\int_a^x g(t) dt}. \]

In particular, as \( c = 0 \), we have \( f = 0 \) on \([a, b]\).

**Proof:** Let \( c > 0 \) and define
\[ F(x) = c + \int_a^x g(t)f(t)dt, \]

then we have

(i). \( F(a) = c > 0. \)

(ii). \( F'(x) = g(x)f(x) \geq 0 \Rightarrow F \) is increasing on \([a, b]\) by Mean Value Theorem

(iii). \( F(x) \geq f(x) \text{ on } [a, b] \Rightarrow F'(x) \leq g(x)F(x) \) by (ii).

So, from (iii), we know that

\[ F(x) \leq F(a)e^{\int_a^x g(t)dt} = ce^{\int_a^x g(t)dt} \text{ by (i)}. \]

For \( c = 0 \), we choose \( c_n = 1/n \rightarrow 0 \), then by preceding result,

\[ f(x) \leq \frac{1}{n}e^{\int_a^x g(t)dt} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

So, we have proved all.

(6) Define

\[ f(x) = \int_x^{x+1} \sin(t^2)dt. \]

(a) Prove that \(|f(x)| < 1/x \) if \( x > 0 \).

\textbf{Proof:} Let \( x > 0 \), then we have, by change of variable \((u = t^2)\), and integration by parts,

\[
\begin{align*}
    f(x) &= \frac{1}{2} \int_{x^2}^{(x+1)^2} \frac{\sin u}{\sqrt{u}} du \\
         &= -\frac{1}{2} \int_{x^2}^{(x+1)^2} \frac{d(\cos u)}{\sqrt{u}} \\
         &= -\frac{1}{2} \left[ \frac{\cos u}{\sqrt{u}} \right]_{x^2}^{(x+1)^2} + \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du \\
         &= \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du
\end{align*}
\]

which implies that,

\[
|f(x)| \leq \left| \frac{\cos(x^2)}{2x} \right| + \left| \frac{\cos[(x+1)^2]}{2(x+1)} \right| + \left| \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du \right|
\]

\[
< \frac{1}{2x} + \frac{1}{2(x+1)} + \frac{1}{4} \int_{x^2}^{(x+1)^2} \frac{du}{u^{3/2}}
\]

\[
= \frac{1}{2x} + \frac{1}{2(x+1)} - \frac{1}{2(x+1)} + \frac{1}{2x} = 1/x.
\]

\textbf{Note:} There is another proof by Second Mean Value Theorem to show above as follows. Since

\[ f(x) = \frac{1}{2} \int_{x^2}^{(x+1)^2} \frac{\sin u}{\sqrt{u}} du \text{ by (a)}, \]

we know that, by Second Mean Value Theorem,
\[ f(x) = \frac{1}{2} \left[ \frac{1}{x} \int_{x^2}^y \sin u du + \frac{1}{x+1} \int_y^{(x+1)^2} \sin u du \right] \]
\[ = \frac{1}{2} \left\{ \frac{1}{x} \left[ \cos(x^2) - \cos y \right] + \frac{1}{x+1} \left[ \cos y - \cos [(x+1)^2] \right] \right\} \]
\[ = \frac{1}{2} \left\{ \left( -\frac{1}{x} + \frac{1}{x+1} \right) \cos y + \frac{1}{x} \cos(x^2) - \frac{1}{x+1} \cos [(x+1)^2] \right\} \]
which implies that
\[ |f(x)| \leq \frac{1}{2} \left\{ \left| -\frac{1}{x} + \frac{1}{x+1} \right| |\cos y| + \left| \frac{\cos(x^2)}{x} \right| + \left| \frac{\cos [(x+1)^2]}{x+1} \right| \right\} \]
\[ \leq \frac{1}{2} \left\{ \left( \frac{1}{x} - \frac{1}{x+1} \right) + \left| \frac{\cos(x^2)}{x} \right| + \left| \frac{\cos [(x+1)^2]}{x+1} \right| \right\} \]
\[ < \frac{1}{2} \left\{ \left( \frac{1}{x} - \frac{1}{x+1} \right) + \frac{1}{x} + \frac{1}{x+1} \right\} \]
since no \( x \) makes \( |\cos(x^2)| = |\cos [(x+1)^2]| = 1 \)
\[ = \frac{1}{x}. \]

(b) Prove that \( 2xf(x) = \cos(x^2) - \cos \left[ (x+1)^2 \right] + r(x) \), where \( |r(x)| < c/x \) and \( c \) is a constant.

**Proof:** By (a), we have
\[ f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos \left[ (x+1)^2 \right]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du \]
which implies that
\[ 2xf(x) = \cos(x^2) - \frac{x}{x+1} \cos \left[ (x+1)^2 \right] + x \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du \]
\[ = \cos(x^2) - \cos \left[ (x+1)^2 \right] + \frac{1}{x+1} \cos \left[ (x+1)^2 \right] + x \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}} du \]
where
\[ r(x) = \frac{1}{x+1} \cos \left[ (x+1)^2 \right] + \frac{x}{2} \int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}} du \]
which implies that
\[ |r(x)| \leq \frac{1}{x+1} + \frac{x}{2} \int_{x^2}^{(x+1)^2} \frac{|\cos u|}{u^{3/2}} du \]
\[ < \frac{1}{x+1} + \frac{x}{2} \int_{x^2}^{(x+1)^2} u^{-3/2} du \]
\[ = \frac{1}{x+1} + \frac{1}{x+1} \]
\[ < \frac{2}{x}. \]

**Note:** Of course, we can use the note in (a) to show it. We write it as follows.

**Proof:** Since
\[ f(x) = \frac{1}{2} \left\{ \left( -\frac{1}{x} + \frac{1}{x+1} \right) \cos y + \frac{1}{x} \cos(x^2) - \frac{1}{x+1} \cos [(x+1)^2] \right\} \]
which implies that
\[2xf(x) = \left(\frac{x}{x+1} - 1\right)\cos(y) + \cos(x^2) - \frac{x}{x+1} \cos((x+1)^2)\]
\[= \cos(x^2) - \cos((x+1)^2) + \frac{1}{x+1} \cos((x+1)^2) + \left(\frac{x}{x+1} - 1\right) \cos(y)\]

where
\[r(x) = \frac{1}{x+1} \cos((x+1)^2) + \left(\frac{x}{x+1} - 1\right) \cos(y)\]

which implies that
\[|r(x)| \leq \frac{1}{x+1} + 1 - \frac{x}{x+1} = \frac{2}{x+1} < 2/x.\]

(c) Find the upper and lower limits of \(xf(x)\), as \(x \to \infty\).

**Proof:** Claim that \(\lim \sup_{x \to \infty} \cos(x^2) - \cos((x+1)^2) = 2\) as follows. Taking \(x = n\sqrt{2\pi}\), where \(n \in \mathbb{Z}\), then
\[\cos(x^2) - \cos((x+1)^2) = -\cos\left(n\sqrt{2\pi} + 1\right).\]

If we can show that \(\left\{n\sqrt{2\pi}\right\}\) is dense in \([0, 2\pi]\) modulus \(2\pi\). It is equivalent to show that \(\left\{n\sqrt{\frac{2}{\pi}}\right\}\) is dense in \([0, 1]\) modulus \(1\). So, by lemma \(\{ar : a \in \mathbb{Z}\}\), where \(r \in \mathbb{Q}\) is dense in \([0, 1]\) modulus \(1\), we have proved the claim. In other words, we have proved the claim.

**Note:** We use the lemma as follows. \(\{ar+b : a \in \mathbb{Z}, b \in \mathbb{Z}\}\), where \(r \in \mathbb{Q}\) is dense in \([0, 1]\) modulus \(1\). It is equivalent to \(\{ar : a \in \mathbb{Z}\}\), where \(r \in \mathbb{Q}\) is dense in \([0, 1]\) modulus \(1\).

**Proof:** Say \(\{ar+b : a \in \mathbb{Z}, b \in \mathbb{Z}\} = S\), and since \(r \in \mathbb{Q}\), then by Exercise 1.16, there are infinitely many rational numbers \(h/k\) with \(k > 0\) such that \(|kr - h| < \frac{1}{k}\). Consider \((x - \delta, x + \delta) := I\), where \(\delta > 0\), and thus choosing \(k_0\) large enough so that \(1/k_0 < \delta\). Define \(L = |k_0r - h_0|\), then we have \(sL \in I\) for some \(s \in \mathbb{Z}\). So, \(sL = (\pm)[(sk_0r - (sh_0)] \in S\). That is, we have proved that \(S\) is dense in \(R\).

(d) Does \(\int_{0}^{\infty} \sin(t^2)dt\) converge?

**Proof:** Yes,
\[\left|\int_{x}^{x'} \sin^2tdt\right| = \left|\frac{1}{2} \int_{x}^{x'} \frac{\sin u}{\sqrt{u}} du\right| \text{ by the process of (a)}\]
\[= \left|\frac{1}{2} \left[\frac{2}{x} \int_{x}^{x'} \sin u du + \frac{2}{x'} \int_{x'}^{x} \sin u du\right]\right| \text{ by Second Mean Value Theorem}\]
\[= \frac{1}{2} \left[\frac{2}{x} + \frac{2}{x'}\right]\]
\[< \frac{2}{x}\]
which implies that the integral exists.

**Note:** (i) We can show it without Second Mean Value Theorem by the method of (a). However Second Mean Value Theorem is more powerful for this exercise.

(ii) Here is the famous Integral named Dirichlet Integral used widely in the STUDY of Fourier Series. We write it as follows. Show that the Dirichlet Integral
\[\int_{0}^{\infty} \frac{\sin x}{x} dx\]
converges but not absolutely converges. In other words, the Dirichlet Integral converges conditionally.

**Proof:** Consider
\[
\int_{x}^{x'} \frac{\sin x}{x} \, dx = \frac{1}{x} \int_{x}^{y} \sin x \, dx + \frac{1}{x'} \int_{y}^{x'} \sin x \, dx
\]
by Second Mean Value Theorem; we have
\[
\left| \int_{x}^{x'} \frac{\sin x}{x} \, dx \right| \leq \frac{2}{x} + \frac{2}{x'} < \frac{4}{x}.
\]
So, we know that Dirichlet Integral converges.

Define \( I_n = [\frac{\pi}{4} + 2n\pi, \frac{\pi}{2} + 2n\pi] \), then
\[
\int_{0}^{\infty} \left| \frac{\sin x}{x} \right| \, dx \geq \int_{I_n} \left| \frac{\sin x}{x} \right| \, dx
\]
\[
\geq \int_{I_n} \left( \frac{\sqrt{2}}{\frac{\pi}{2} + 2n\pi} \right) \, dx
\]
\[
\geq \sum_{n=0}^{\infty} \left( \frac{\sqrt{2}}{\frac{\pi}{4} + 2n\pi} \right) \to \infty.
\]
So, we know that Dirichlet Integral does NOT converge absolutely.

(7) Deal similarity with
\[
f(x) = \int_{x}^{x+1} \sin(e^t) \, dt.
\]
Show that
\[
e^{x} |f(x)| < 2
\]
and that
\[
e^{x} f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) + r(x),
\]
where \(|r(x)| < \min(1, Ce^{-x})\), for all \(x\) and

**Proof:** Since
\[
f(x) = \int_{x}^{x+1} \sin(e^t) \, dt
\]
\[
= \int_{e^x}^{e^{x+1}} \frac{\sin u}{u} \, du \text{ by Change of Variable (let } u = e^t)\]
\[
= \frac{\cos e^x}{e^x} - \frac{\cos e^{x+1}}{e^{x+1}} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} \, du \text{ by Integration by parts}
\]
which implies that
\[
|f(x)| < \left| \frac{\cos e^x}{e^x} \right| + \left| \frac{\cos e^{x+1}}{e^{x+1}} \right| + \int_{e^x}^{e^{x+1}} \frac{du}{u^2} \text{ since } \cos u \text{ is not constant}
\]
\[
\leq \frac{1}{e^x} + \frac{1}{e^{x+1}} + \frac{1}{e^x} \left( 1 - \frac{1}{e} \right)
\]
which implies that
\[
e^{x} |f(x)| < 2.
\]
In addition, by (*), we have
\[
e^{x} f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) + r(x),
\]
where
\[ r(x) = -e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du \]

which implies that
\[ |r(x)| = 1 - e^{-1} < 1 \text{ for all } x \]

or which implies that, by Integration by parts,

\[ |r(x)| = e^x \left| \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du \right| \]
\[ = e^x \left( \frac{\sin e^{x+1}}{e^{2(x+1)}} - \frac{\sin e^x}{e^x} + 2 \int_{e^x}^{e^{x+1}} \frac{\sin u}{u^3} du \right) \]
\[ < e^x \left( \frac{1}{e^{2(x+1)}} + \frac{1}{e^{2x}} + 2 \int_{e^x}^{e^{x+1}} \frac{du}{u^3} \right) \text{ since } \sin u \text{ is not constant } 1 \]
\[ = 2e^{-x} \text{ for all } x. \]

By (**) and (***) we have proved that \(|r(x)| < \min(1, Ce^{-x})\) for all \(x\), where \(C = 2\).

**Note:** We give another proof on (7) by Second Mean Value Theorem as follows.

**Proof:** Since
\[ f(x) = \int_{e^x}^{e^{x+1}} \frac{\sin u}{u} du \]
\[ = \frac{1}{e^x} \int_{e^x}^{y} \sin u du + \frac{1}{e^{x+1}} \int_{y}^{e^{x+1}} \sin u du \text{ by Second Mean Value Theorem} \]
\[ = \frac{1}{e^x} \left( \cos e^x - \cos y \right) + \frac{1}{e^{x+1}} \left( \cos y - \cos e^{x+1} \right) \]

which implies that
\[ e^x |f(x)| = |\cos e^x - \cos y + e^{-1}(\cos y - \cos e^{x+1})| \]
\[ = |(\cos e^x - e^{-1} \cos e^{x+1}) + \cos y(1 - e^{-1})| \]
\[ \leq |\cos e^x - e^{-1} \cos e^{x+1}| + (1 - e^{-1}) \]
\[ < (1 + e^{-1}) + (1 - e^{-1}) \text{ since no } x \text{ makes } |\cos e^x| = |\cos e^{x+1}| = 1. \]
\[ = 2. \]

In addition, by (*), we know that
\[ e^x f(x) = \cos e^x - e^{-1} \cos e^{x+1} + r(x) \]

where
\[ r(x) = (e^{-1} \cos y - \cos y) \]

which implies that
\[ |r(x)| \leq 1 - e^{-1} < 1 \text{ for all } x. \]

In addition, from the proof of the process in (7), we know that
\[ |r(x)| = e^x \left| \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du \right| \]
\[ = e^x \left[ \frac{1}{e^{2x}} \int_{e^x}^{y} \cos u du + \frac{1}{e^{2(x+1)}} \int_{y}^{e^{x+1}} \cos u du \right] \]
\[ = e^{-x} |\sin y - \sin e^x + e^{-2}(\sin e^{x+1} - \sin y)| \]
\[ = e^{-x}|\sin y(1 - e^{-2}) + (e^{-2} \sin e^{x+1} - \sin e^x)| \]
\[ \leq e^{-x}(1 - e^{-2}) + e^{-x}|e^{-2} \sin e^{x+1}| + e^{-x}|\sin e^x| \]
\[ < e^{-x}(1 - e^{-2}) + e^{-x}(1 + e^{-2}) \text{ since no } x \text{ makes } |\sin e^{x+1}| = |\sin e^x| = 1 \]
\[ = 2e^{-x} \text{ for all } x. \quad *** \]

So, by (**) and (***) we have proved that \(|r(x)| < \min(1, Ce^{-x})\), where \(C = 2\).

(8) Suppose that \(f\) is real, continuously differentiable function on \([a, b]\),
\[ f(a) = f(b) = 0, \]
and
\[ \int_{a}^{b} f^2(x) dx = 1. \]

Prove that
\[ \int_{a}^{b} xf(x)f'(x) dx = -\frac{1}{2} \]
and that
\[ \int_{a}^{b} [f'(x)]^2 dx \cdot \int_{a}^{b} x^2 f^2(x) dx > \frac{1}{4}. \]

**Proof:** Consider
\[ \int_{a}^{b} xf(x)f'(x) dx = \int_{a}^{b} xf(x)df(x) \]
\[ = xf^2(x)\bigg|_{a}^{b} - \int_{a}^{b} f(x)df(x) \]
\[ = -\int_{a}^{b} f^2(x) dx + \int_{a}^{b} xf(x)f'(x) dx \text{ since } f(a) = f(b) = 0, \]
so we have
\[ \int_{a}^{b} xf(x)f'(x) dx = -\frac{1}{2}. \]

In addition, by **Cauchy-Schwarz Inequality**, we know that
\[ \int_{a}^{b} [f'(x)]^2 dx \cdot \int_{a}^{b} x^2 f^2(x) dx \geq \left( \int_{a}^{b} xf(x)f'(x) dx \right)^2 = \frac{1}{4}. \]

Note that the equality does NOT hold since if it was, then we have \(f'(x) = kxf(x)\). It implies that
\[ [f(x) - kxf(x)]e^{-\frac{4x^2}{y}} = 0 \]
which implies that
\[ (fe^{-\frac{4x^2}{y}})' = 0 \]
which implies that
\[ f(x) = Ce^{-\frac{4x^2}{y}}, \text{ a constant} \]
which implies that
\[ C = 0 \text{ since } f(a) = 0. \]

That is, \( f(x) = 0 \) on \([a, b] \) which is absurd.

7.5 Let \( \{a_n\} \) be a sequence of real numbers. For \( x \geq 0 \), define

\[ A(x) = \sum_{n \leq x} a_n = \sum_{n=1}^{[x]} a_n, \]

where \([x]\) is the largest integer in \(x\) and empty sums are interpreted as zero. Let \( f \) have a continuous derivative in the interval \( 1 \leq x \leq a \). Use Stieltjes integrals to derive the following formula:

\[ \sum_{n \leq a} a_n f(n) = -\int_1^a A(x)f'(x)dx + A(a)f(a). \]

**Proof:** Since

\[ \int_1^a A(x)f'(x)dx = \int_1^a A(x)df(x) \text{ since } f \text{ has a continuous derivative on } [1,a] \]

\[ = -\int_1^a f(x)dA(x) + A(a)f(a) - A(1)f(1) \text{ by integration by parts} \]

\[ = -\sum_{n \leq a} a_n f(n) + A(a)f(a) \text{ by } \int_1^a f(x)dA(x) = \sum_{n=2}^{[a]} a_n f(n) \text{ and } A(1) = a_1, \]

we know that

\[ \sum_{n \leq a} a_n f(n) = -\int_1^a A(x)f'(x)dx + A(a)f(a). \]

7.6 Use **Euler's summation formula, integration by parts** in a Stieltjes integral, to derive the following identities:

(a) \( \sum_{k=1}^n \frac{1}{k^s} = \frac{1}{n^{s+1}} + s \int_1^n \frac{[x]}{x^{s+1}}dx \) if \( s \neq 1 \).

**Proof:**

\[ \sum_{k=1}^n \frac{1}{k^s} = \int_1^n x^{-s}d[x] + 1 \]

\[ = -\int_1^n [x]x^{-s} + n^{-s}[n] - 1^{-s}[1] + 1 \]

\[ = s \int_1^n \frac{[x]}{x^{s+1}}dx + n^{-s} \]

\[ = \frac{1}{n^{s-1}} + s \int_1^n \frac{[x]}{x^{s+1}}dx \text{ if } s \neq 1. \]

(b) \( \sum_{k=1}^n \frac{1}{k} = \log n - \int_1^n \frac{x-[x]}{x^2} + 1. \)

**Proof:**
\[
\sum_{k=1}^{n} \frac{1}{k} = \int_{1}^{n} \frac{1}{x} \, dx + 1 \\
= -\int_{1}^{n} [x] \, dx^{-1} + n^{-1}[n] - 1^{-1}[1] + 1 \\
= \int_{1}^{n} x^{-1} \, dx - \int_{1}^{n} x^{-1} \, dx + \int_{1}^{n} \frac{[x]}{x^2} \, dx + 1 \\
= \log n - \int_{1}^{n} \frac{x - [x]}{x^2} + 1.
\]

7.7 Assume that \( f' \) is continuous on \([1, 2n]\) and use Euler’s summation formula or integration by parts to prove that

\[
\sum_{k=1}^{2n} (-1)^k f(k) = \int_{1}^{2n} f'(x) ([x] - 2[x/2]) \, dx.
\]

**Proof:**

\[
\sum_{k=1}^{2n} (-1)^k f(k) = -\sum_{k=1}^{2n} f(k) + 2\sum_{k=1}^{n} f(2k) \\
= -\left( \int_{1}^{2n} f(x) \, dx + f(1) \right) + 2\left( \int_{1}^{2n} f(x) \, dx/2 \right) \\
= -\left( -\int_{1}^{2n} [x] \, df(x) + [2n] f(2n) \right) + 2\left( -\int_{1}^{2n} [x/2] \, df(x) + f(2n) [2n/2] - f(1)[1/2] \right) \\
= \int_{1}^{2n} f'(x) [x] \, dx - 2nf(2n) - \int_{1}^{2n} f'(x) [x/2] \, dx + 2nf(2n) \\
= \int_{1}^{2n} f'(x) ([x] - 2[x/2]) \, dx.
\]

7.8 Let \( \phi_1 = x - [x] - \frac{1}{2} \) if \( x \neq \text{integer} \), and let \( \phi_1 = 0 \) if \( x = \text{integer} \). Also, let \( \phi_2 = \int_{0}^{x} \phi_1(t) \, dt \). If \( f'' \) is continuous on \([1,n]\) prove that Euler’s summation formula implies that

\[
\sum_{k=1}^{n} f(k) = \int_{1}^{n} f(x) \, dx - \int_{1}^{n} \phi_2(x) f''(x) \, dx + \frac{f(1) + f(n)}{2}.
\]

**Proof:** Using Theorem 7.13, then we have

\[
\sum_{k=1}^{n} f(k) = \int_{1}^{n} f(x) \, dx + \int_{1}^{n} f'(x) \phi_1(x) \, dx + \frac{f(1) + f(n)}{2} \\
= \int_{1}^{n} f(x) \, dx + \int_{1}^{n} f'(x) \phi_2(x) + \frac{f(1) + f(n)}{2} \\
= \int_{1}^{n} f(x) \, dx + \left( -\int_{1}^{n} \phi_2(x) f'(x) + f'(n) \phi_2(n) - f'(1) \phi_2(1) \right) + \frac{f(1) + f(n)}{2} \\
= \int_{1}^{n} f(x) \, dx - \int_{1}^{n} \phi_2(x) f'(x) + \frac{f(1) + f(n)}{2} \\
= \int_{1}^{n} f(x) \, dx - \int_{1}^{n} \phi_2(x) f''(x) + \frac{f(1) + f(n)}{2} \text{ since } f'' \text{ is continuous on } [1,n].
\]
7.9 Take \( f(x) = \log x \) in Exercise 7.8 and prove that
\[
\log n! = \left( n + \frac{1}{2} \right) \log n - n + 1 + \int_1^n \frac{\phi(t)}{t^2} \, dt.
\]

**Proof:** Let \( f(x) = \log x \), then by Exercise 7.8, it is clear. So, we omit the proof.

**Remark:** By Euler’s summation formula, we can show that
\[
\sum_{1 \leq k \leq n} \log k = \int_1^n \log x \, dx + \int_1^n \left( x - [x] - \frac{1}{2} \right) \, dx + \frac{\log n}{2}.
\]

Since
\[
\left| \left( x - [x] - \frac{1}{2} \right) \right| \leq 1/2
\]
and
\[
\int_a^{a+1} \left( x - [x] - \frac{1}{2} \right) \, dx = 0 \text{ for all real } a,
\]
we thus have the convergence of the improper integral
\[
\int_1^n \left( x - [x] - \frac{1}{2} \right) \, dx
\]
by Second Mean Value Theorem.

So, by (*), we have
\[
\log n! = \left( n + \frac{1}{2} \right) \log n - n + C + \gamma_n
\]
where
\[
C = 1 + \int_1^\infty \left( x - [x] - \frac{1}{2} \right) \, dx,
\]
and
\[
\gamma_n = - \int_n^{\infty} \left( x - [x] - \frac{1}{2} \right) \, dx.
\]

So,
\[
\lim_{n \to \infty} \frac{n!}{e^n n^{n+1/2}} = e^C := C_1.
\]

Now, using Wallis formula, we have
\[
\lim_{n \to \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (2n)(2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} = \pi/2
\]
which implies that
\[
\frac{(2n)!^4}{[(2n)!]^2(2n + 1)} (1 + o(1)) = \pi/2
\]
which implies that, by (**),
\[
\frac{C_1^4(2n^{n+1/2}e^{-n})^4}{C_1^4[(2n)^{2n+1/2}e^{-2n}](2n + 1)} (1 + o(1)) = \pi/2
\]
which implies that
\[
\frac{C_1^4 n}{2(2n + 1)} (1 + o(1)) = \pi/2.
\]
Let \( n \to \infty \), we have \( C_1 = \sqrt{2\pi} \), and \( \int_1^{\infty} \left( x - [x] - \frac{1}{2} \right) \, dx = \frac{1}{2} \log 2\pi - 1 \).

**Note:** In (**), the formula is called Stirling formula. The reader should be noted that Wallis formula is equivalent to Stirling formula.
7.10 If \( x \geq 1 \), let \( \pi(x) \) denote the number of primes \( p \leq x \), that is,

\[
\pi(x) = \sum_{p \leq x} 1,
\]

where the sum is extended over all primes \( p \leq x \). The prime number theorem states that

\[
\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.
\]

This is usually proved by studying a related function \( \vartheta \) given by

\[
\vartheta(x) = \sum_{p \leq x} \log p,
\]

where again the sum is extended over all primes \( p \leq x \). Both function \( \pi \) and \( \vartheta \) are step functions with jumps at the primes. This exercise shows how the Riemann-Stieltjes integral can be used to relate these two functions.

(a) If \( x \geq 2 \), prove that \( \pi(x) \) and \( \vartheta(x) \) can be expressed as the following Riemann-Stieltjes integrals:

\[
\vartheta(x) = \int_{\frac{3}{2}}^{x} \log t \, d\pi(t), \quad \pi(x) = \int_{\frac{3}{2}}^{x} \frac{1}{\log t} \, d\vartheta(t).
\]

Note. The lower limit can be replaced by any number in the open interval \((1, 2)\).

Proof: Since \( \vartheta(x) = \sum_{p \leq x} \log p \), we know that by Theorem 7.9,

\[
\vartheta(x) = \int_{\frac{3}{2}}^{x} \log t \, d\pi(t),
\]

and \( \pi(x) = \sum_{p \leq x} 1 \), we know that by Theorem 7.9,

\[
\pi(x) = \int_{\frac{3}{2}}^{x} \frac{1}{\log t} \, d\vartheta(t).
\]

(b) If \( x \geq 2 \), use integration by parts to show that

\[
\vartheta(x) = \pi(x) \log x - \int_{2}^{x} \frac{\pi(t)}{t} \, dt,
\]

\[
\pi(x) = \frac{\vartheta(x)}{\log x} + \int_{2}^{x} \frac{\vartheta(t)}{t \log^2 t} \, dt.
\]

These equations can be used to prove that the prime number theorem is equivalent to the relation \( \lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1 \).

Proof: Use integration by parts, we know that

\[
\vartheta(x) = \int_{\frac{3}{2}}^{x} \log t \, d\pi(t) = -\int_{\frac{3}{2}}^{x} \frac{\pi(t)}{t} \, dt + \log x \pi(x) - \log(3/2) \pi(3/2)
\]

\[
= -\int_{\frac{3}{2}}^{x} \frac{\pi(t)}{t} \, dt + \log x \pi(x) \text{ since } \pi(3/2) = 0
\]

\[
= \pi(x) \log x - \int_{2}^{x} \frac{\pi(t)}{t} \, dt \text{ since } \int_{\frac{3}{2}}^{2} \frac{\pi(t)}{t} \, dt = 0 \text{ by } \pi(x) = 0 \text{ on } [0, 2),
\]

and

\[
\int_{\frac{3}{2}}^{x} \frac{\pi(t)}{t} \, dt = \int_{2}^{x} \frac{\pi(t)}{t} \, dt.
\]
\[ \pi(x) = \int_{3/2}^{\infty} \frac{1}{\log t} d\vartheta(t) = \int_{3/2}^{\infty} \frac{\vartheta(t)}{t\log^2 t} dt + \frac{\vartheta(x)}{\log x} - \frac{\vartheta(3/2)}{\log(3/2)} \]

\[ = \int_{3/2}^{\infty} \frac{\vartheta(t)}{t\log^2 t} dt + \frac{\vartheta(x)}{\log x} \quad \text{since } \vartheta(3/2) = 0 \]

\[ = \frac{\vartheta(x)}{\log x} + \int_{2}^{\infty} \frac{\vartheta(t)}{t\log^2 t} dt \quad \text{since } \int_{3/2}^{2} \frac{\vartheta(t)}{t\log^2 t} dt = 0 \text{ by } \vartheta(x) = 0 \text{ on } [0, 2). \]

7.11 If \( a \neq b \) on \([a, b]\), prove that

(a) \( \int_{a}^{b} fda = \int_{a}^{c} fda + \int_{c}^{b} fda \), \((a < c < b)\)

**Proof:** Given \( \varepsilon > 0 \), there is a partition \( P \) such that

\[ U(P, f, a) < \bar{I}(a, b) + \varepsilon. \]  

Let \( P' = \{c\} \cup P = P_1 \cup P_2 \), where \( P_1 = \{a = x_0, \ldots, x_n = c\} \) and

\[ P_2 = \{x_n = c, \ldots, x_n \in b\} \] then we have

\[ \bar{I}(a, c) + \bar{I}(c, b) \leq U(P_1, f, a) + U(P_2, f, a) = U(P', f, a) \leq U(P, f, a). \]

So, by (1) and (2), we have

\[ \bar{I}(a, c) + \bar{I}(c, b) \leq \bar{I}(a, b) \]

since \( \varepsilon \) is arbitrary.

On the other hand, given \( \varepsilon > 0 \), there is a partition \( P_1 \) and \( P_2 \) such that

\[ U(P_1, f, a) + U(P_2, f, a) \leq \bar{I}(a, c) + \bar{I}(c, b) + \varepsilon \]

which implies that, let \( P = P_1 \cup P_2 \)

\[ U(P, f, a) = U(P_1, f, a) + U(P_2, f, a) \leq \bar{I}(a, c) + \bar{I}(c, b) + \varepsilon. \]

Also,

\[ \bar{I}(a, b) \leq U(P, f, a). \]

By (3) and (4), we have

\[ \bar{I}(a, b) \leq \bar{I}(a, c) + \bar{I}(c, b) \]

since \( \varepsilon \) is arbitrary.

So, by (*) and (**), we have proved it.

(b) \( \int_{a}^{b} (f + g)da \leq \int_{a}^{c} fda + \int_{c}^{b} gda \).

**Proof:** In any compact interval \( J \), we have

\[ \sup_{x \in J} (f + g) \leq \sup_{x \in J} f + \sup_{x \in J} g. \]

So, given \( \varepsilon > 0 \), there is a partition \( P_f \) and \( P_g \) such that

\[ \sum_{k=1}^{n_1} M_k(f) \Delta a_k \leq \int_{a}^{b} fda + \varepsilon/2 \]

and

\[ \sum_{k=1}^{n_2} M_k(g) \Delta a_k \leq \int_{a}^{b} gda + \varepsilon/2. \]

So, consider \( P = P_f \cup P_g \), then we have, by (1),

\[ U(P, f + g, \alpha) \leq U(P, f, \alpha) + U(P, g, \alpha) \]

along with
If \( U(P,f,\alpha) \leq \sum_{k=1}^{n_1} M_k(f)\Delta a_k \) and \( U(P,g,\alpha) \leq \sum_{k=1}^{n_2} M_k(g)\Delta a_k \)

which implies that, by (2) and (3),

\[
U(P,f+g,\alpha) \leq \int_a^b f\,da + \int_a^b g\,da + \varepsilon
\]

which implies that

\[
\int_a^b (f+g)\,da \leq \int_a^b f\,da + \int_a^b g\,da
\]

since \( \varepsilon \) is arbitrary.

(c) \( \int_a^b (f+g)\,da \geq \int_a^b f\,da + \int_a^b g\,da \)

Proof: Similarly by (b), so we omit the proof.

7.12 Give an example of bounded function \( f \) and an increasing function \( \alpha \) defined on \([a, b] \) such that \( |f| \in R(\alpha) \) but for which \( \int_a^b f\,da \) does not exist.

Solution: Let

\[
f(x) = \begin{cases} 
1 & \text{if } x \in [0, 1] \cap Q \\
-1 & \text{if } x \in [0, 1] \cap Q^c
\end{cases}
\]

and \( \alpha(x) = x \) on \([0, 1] \). Then it is clear that \( f \not\in R(\alpha) \) on \([a, b] \) and \( |f| \in R(\alpha) \) on \([a, b] \).

7.13 Let \( \alpha \) be a continuous function of bounded variation on \([a, b] \). Assume that \( g \in R(\alpha) \) on \([a, b] \) and define \( \beta(x) = \int_a^x g(t)\,da(t) \) if \( x \in [a, b] \). Show that:

(a) If \( f \not\in [a, b] \), there exists a point \( x_0 \) in \([a, b] \) such that

\[ \int_a^{x_0} f\,d\beta = f(a) \int_a^{x_0} g\,da + f(b) \int_{x_0}^b g\,da. \]

Proof: Since \( \alpha \) is a continuous function of bounded variation on \([a, b] \), and \( g \in R(\alpha) \) on \([a, b] \), we know that \( \beta(x) \) is a continuous function of bounded variation on \([a, b] \), by Theorem 7.32. Hence, by Second Mean-Value Theorem for Riemann-Stieltjes integrals, we know that

\[ \int_a^b f\,d\beta = f(a) \int_a^{x_0} g\,da + f(b) \int_{x_0}^b g\,da. \]

which implies that, by Theorem 7.26,

\[ \int_a^b f\,d\beta = f(a) \int_a^{x_0} g\,da + f(b) \int_{x_0}^b g\,da. \]

(b) If, in addition, \( f \) is continuous on \([a, b] \), we also have

\[ \int_a^b f(x)g(x)\,dx = f(a) \int_a^{x_0} g\,da + f(b) \int_{x_0}^b g\,da. \]

Proof: Since

\[ \int_a^b f\,d\beta = \int_a^b f(x)g(x)\,dx \]

we know that, by (a),
\[ \int_a^b f(x)g(x)dx = f(a) \int_a^{x_0} gda + f(b) \int_{x_0}^b gda. \]

**Remark:** We do NOT need the hypothesis that \( f \) is continuous on \([a, b]\).

7.14 Assume that \( f \in R(\alpha) \) on \([a, b]\), where \( \alpha \) is of bounded variation on \([a, b]\). Let \( V(x) \) denote the total variation of \( \alpha \) on \([a, b] \) for each \( x \) in \((a, b]\), and let \( V(a) = 0 \). Show that

\[ \left| \int_a^b f\alpha \right| \leq \int_a^b |f|dV \leq MV(b), \]

where \( M \) is an upper bound for \( |f| \) on \([a, b]\). In particular, when \( \alpha(x) = x \), the inequality becomes

\[ \left| \int_a^b f\alpha \right| \leq M(b - a). \]

**Proof:** Given \( \varepsilon > 0 \), there is a partition \( P = \{a = x_0, \ldots, x_n = b\} \) such that

\[ \int_a^b f\alpha - \varepsilon < S(P, f, \alpha) \]

\[ = \sum_{k=1}^n f(t_k)\Delta \alpha_k, \text{ where } t_k \in [x_{k-1}, x_k] \]

\[ \leq \sum_{k=1}^n |f(t_k)||\alpha(x_k) - \alpha(x_{k-1})| \]

\[ \leq \sum_{k=1}^n |f(t_k)|(V(x_k) - V(x_{k-1})) \]

\[ = S(P, |f|, V) \]

\[ \leq U(P, |f|, V) \text{ since } V \text{ is increasing on } [a, b] \]

which implies that, taking infimum,

\[ \int_a^b f\alpha - \varepsilon \leq \int_a^b |f|dV \]

since \( |f| \in R(V) \) on \([a, b]\).

So, we have

\[ \left| \int_a^b f\alpha \right| \leq \int_a^b |f|dV \]

since \( \int_a^b |f|dV \) is clear non-negative. If \( M \) is an upper bound for \( |f| \) on \([a, b]\), then (*) implies that

\[ \left| \int_a^b f\alpha \right| \leq \int_a^b |f|dV \leq MV(b) \]

which implies that

\[ \left| \int_a^b f\alpha \right| \leq M(b - a) \]

if \( \alpha(x) = x \).

7.15 Let \( \{\alpha_n\} \) be a sequence of functions of bounded variation on \([a, b]\). Suppose there exists a function \( \alpha \) defined on \([a, b]\) such that the total variation of \( \alpha - \alpha_n \) on \([a, b]\) tends to 0 as \( n \to \infty \). Assume also that \( \alpha(a) = \alpha_n(a) = 0 \) for each \( n = 1, 2, \ldots \). If \( f \) is
continuous on \([a,b]\), prove that
\[
\lim_{n\to\infty} \int_a^b f(x) \, d\alpha_n(x) = \int_a^b f(x) \, d\alpha(x).
\]

**Proof:** Use Exercise 7.14, we then have
\[
\left| \int_a^b f(x) d(\alpha - \alpha_n(x)) \right| \leq MV_n(b) \to 0 \text{ as } n \to \infty
\]
where \(V_n\) is the total variation of \(\alpha - \alpha_n\), and \(M = \sup_{x \in [a,b]} |f(x)|\).

So, we have
\[
\lim_{n\to\infty} \int_a^b f(x) \, d\alpha_n(x) = \int_a^b f(x) \, d\alpha(x).
\]

**Remark:** We do NOT need the hypothesis \(\alpha(a) = \alpha_n(a) = 0\) for each \(n = 1, 2, \ldots\).

**7.16** If \(f \in R(\alpha), f^2 \in R(\alpha), g \in R(\alpha),\) and \(g^2 \in R(\alpha)\) on \([a,b]\), prove that
\[
\frac{1}{2} \int_a^b \left[ \int_a^b f(x) \ g(x) \ v \ g(y) \ \, d\alpha(y) \right] \, d\alpha(x)
\]
\[
= \left( \int_a^b f^2(x) \, d\alpha(x) \right) \left( \int_a^b g^2(x) \, d\alpha(x) \right) - \left( \int_a^b f(x) g(x) \, d\alpha(x) \right)^2.
\]
When \(\alpha \neq \) on \([a,b]\), deduce the Cauchy-Schwarz inequality
\[
\left( \int_a^b f(x) g(x) \, d\alpha(x) \right)^2 \leq \left( \int_a^b f^2(x) \, d\alpha(x) \right) \left( \int_a^b g^2(x) \, d\alpha(x) \right).
\]
(Compare with Exercise 1.23.)

**Proof:** Consider
\[
\frac{1}{2} \int_a^b \left[ \int_a^b f(x) \ g(x) \ v \ g(y) \ \, d\alpha(y) \right] \, d\alpha(x)
\]
\[
= \left( \int_a^b f^2(x) \, d\alpha(x) \right) \left( \int_a^b g^2(y) \, d\alpha(y) \right) - \left( \int_a^b f(x) g(x) \, d\alpha(x) \right) \left( \int_a^b g(x) \, d\alpha(x) \right)
\]
if \(\alpha \neq \) on \([a,b]\), then we have
It implies that \[ R \text{ Riemann Integral} \]

Write \[ P \]

that implies that \[ \text{Conjugate} \]

In addition, in this sense, we have the \[ f \]

where \[ f \]

Note \[ \text{Remark} \]

Remark: (1) Here is another proof: Let \[ A = \int_a^b f(x) dx \], \[ B = \int_a^b f(x)g(x) dx \], and \[ C = \int_a^b g^2(x) dx \]. From the fact,

\[ 0 \leq \int_a^b [f(x)z + g(x)]^2 dx \] for any real \[ z \]

\[ = Az^2 + 2Bz + C. \]

It implies that \[ B^2 \leq AC. \]

That is,

\[ \left( \int_a^b f(x)g(x) dx \right)^2 \leq \left( \int_a^b f^2(x) dx \right) \left( \int_a^b g^2(x) dx \right). \]

Note: (1) The reader may recall the inner product in Linear Algebra. We often consider Riemann Integral by defining

\[ \langle f, g \rangle := \int_a^b f(x)g(x) dx \]

where \( f \) and \( g \) are real continuous functions defined on \( [a, b] \). This definition is a real case. For complex case, we need to preserve its positive definite. So, we define

\[ \langle f, g \rangle := \int_a^b f(x)\bar{g}(x) dx \]

where \( f \) and \( g \) are complex continuous functions defined on \( [a, b] \), and \( \bar{g} \) means its conjugate. In addition, in this sense, we have the triangular inequality:

\[ \|f - g\| \leq \|f - h\| + \|f - h\|, \] where \( \|f\| = \sqrt{\langle f, f \rangle} \).

(2) Suppose that \( f \in R(a) \) on \( [a, b] \) where \( \alpha \) on \( [a, b] \) and given \( \varepsilon > 0 \), then there exists a continuous function \( g \) on \( [a, b] \) such that

\[ \|f - g\| < \varepsilon. \]

Proof: Let \( K = \sup_{x \in [a,b]} |f(x)| \), and given \( \varepsilon > 0 \), we want to show that

\[ \|f - g\| < \varepsilon. \]

Since \( f \in R(a) \) on \( [a, b] \) where \( \alpha \) on \( [a, b] \), given \( (1 >)\varepsilon' > 0 \), there is a partition \( P = \{x_0 = a, \ldots, x_n = b\} \) such that

\[ U(P, f, a) - L(P, f, a) = \sum_{j=1}^n [M_j(f) - m_j(f)] \Delta a_j < (\varepsilon')^2. \]

Write \( P = A \cup B \), where \( A = \{x_j : M_j(f) - m_j(f) < \varepsilon'\} \) and \( B = \{x_j : M_j(f) - m_j(f) \geq \varepsilon'\} \), then
\[ \varepsilon \sum_{B} \Delta \alpha_j \leq \sum_{B} [M_j(f) - m_j(f)] \Delta \alpha_j < (\varepsilon')^2 \] by (1)

which implies that

\[ \sum_{B} \Delta \alpha_j < \varepsilon'. \]

For this partition \( P \), we define the function \( g \) as follows.

\[ g(t) = \frac{x_j - t}{x_j - x_{j-1}} f(x_{j-1}) + \frac{t - x_{j-1}}{x_j - x_{j-1}} f(x_j), \text{ where } x_{j-1} \leq t \leq x_j. \]

So, it is clear that \( g \) is continuous on \([a, b] \). In every subinterval \([x_{j-1}, x_j]\)

\[ |f(t) - g(t)| = \left| \frac{x_j - t}{x_j - x_{j-1}} [f(t) - f(x_{j-1})] + \frac{t - x_{j-1}}{x_j - x_{j-1}} [f(t) - f(x_j)] \right| \]

\[ \leq |f(t) - f(x_{j-1})| + |f(t) - f(x_j)| \]

\[ \leq 2[M_j(f) - m_j(f)] \]

Consider

\[ \sum_{A} \int_{[x_{j-1}, x_j]} |f(t) - g(t)|^2 \, da \leq \sum_{A} 4[M_j(f) - m_j(f)]^2 \Delta \alpha_j \] by (3)

\[ \leq 4 \sum_{A} \varepsilon'[M_j(f) - m_j(f)] \Delta \alpha_j \text{ by definition of } A \]

\[ \leq 4\varepsilon' \sum_{A} \Delta \alpha_j \text{ by } \varepsilon' < 1 \]

\[ = 4\varepsilon'[a(b) - a(a)] \]

and

\[ \sum_{B} \int_{[x_{j-1}, x_j]} |f(t) - g(t)|^2 \, da \leq \sum_{B} 4K^2 \Delta \alpha_j \]

\[ \leq 4K^2 \varepsilon' \text{ by (2).} \]

Hence,

\[ \int_{a}^{b} |f(t) - g(t)|^2 \, da \leq 4\varepsilon'[a(b) - a(a)] + 4K^2 \varepsilon' \]

\[ < \varepsilon^2 \]

if we choose \( \varepsilon' \) is small enough so that \( 4\varepsilon'[a(b) - a(a)] + 4K^2 \varepsilon' < \varepsilon^2 \). That is, we have proved that

\[ \|f - g\| < \varepsilon. \]

**P.S.:** The exercise tells us a Riemann-Stieltjes integrable function can be approximated (approached) by continuous functions.

(3)There is another important result called **Holder’s inequality.** It is useful in Analysis and more general than **Cauchy-Schwarz inequality.** In fact, it is the case \( p = q = 2 \) in **Holder’s inequality.** We consider the following results.

Let \( p \) and \( q \) be positive real numbers such that

\[ \frac{1}{p} + \frac{1}{q} = 1. \]

Prove that the following statements.

(a) If \( u \geq 0 \) and \( v \geq 0 \), then
Equality holds if and only if \( u^p = v^q \).

**Proof:** Let \( f(u) = \frac{u^p}{p} + \frac{v^q}{q} - uv \) be a function defined on \([0, +\infty)\), where \( \frac{1}{p} + \frac{1}{q} = 1 \), \( p > 0 \), \( q > 0 \) and \( v \geq 0 \), then \( f'(u) = u^{p-1} - v \). So, we know that

\[
f'(u) < 0 \text{ if } u \in \left(0, \sqrt[p-1]{v}\right) \text{ and } f'(u) > 0 \text{ if } u \in \left(\sqrt[p-1]{v}, +\infty\right)
\]

which implies that, by \( f\left(\sqrt[p-1]{v}\right) = 0 \), \( f(u) \geq 0 \). Hence, we know that \( f(u) \geq 0 \) for all \( u \geq 0 \). That is, \( uv \leq \frac{u^p}{p} + \frac{v^q}{q} \). In addition, \( f(u) = 0 \) if and only if \( u = \sqrt[p-1]{v} \) if and only if \( u^p = v^q \).

So, Equality holds if and only if \( u^p = v^q \).

**Note:** (1) Here is another good proof by using **Young’s Inequality**, let \( f(x) \) be an strictly increasing and continuous function defined on \( [x : x \geq 0] \), with \( f(0) = 0 \). Then we have, let \( a > 0 \) and \( b > 0 \),

\[
ab \leq \int_{0}^{a} f(x)dx + \int_{0}^{b} f^{-1}(x)dx, \text{ where } f^{-1} \text{ is the inverse function of } f.
\]

And the equality holds if and only if \( f(a) = b \).

**Proof:** The proof is easy by drawing the function \( f \) on \( x - y \) plane. So, we omit it.

So, by **Young’s Inequality**, let \( f(x) = x^a \), where \( a > 0 \), we have the **Holder’s inequality**.

(2) The reader should be noted that there are many proofs of (a), for example, using the concept of convex function, or using **A. P. ≥ G. P.** along with continuity.

(b) If \( f, g \in R(a) \) on \( [a, b] \) where \( a \not{\in} [a, b] \), \( f, g \geq 0 \) on \( [a, b] \), and

\[
\int_{a}^{b} f^p \alpha = 1 = \int_{a}^{b} g^q \alpha,
\]

then

\[
\int_{a}^{b} fg \alpha \leq 1.
\]

**Proof:** By **Holder’s inequality**, we have

\[
fg \leq \frac{f^p}{p} + \frac{g^q}{q}
\]

which implies that, by \( a \not{\in} [a, b] \), and

\[
\int_{a}^{b} f^p \alpha = 1 = \int_{a}^{b} g^q \alpha,
\]

\[
\int_{a}^{b} fg \alpha \leq \int_{a}^{b} \frac{f^p}{p} \alpha + \int_{a}^{b} \frac{g^q}{q} \alpha = \frac{1}{p} + \frac{1}{q} = 1.
\]

(c) If \( f \) and \( g \) are complex functions in \( R(a) \), where \( a \not{\in} [a, b] \), then

\[
\left| \int_{a}^{b} fg \alpha \right| \leq \left\{ \int_{a}^{b} |f|^p \alpha \right\}^{1/p} \left\{ \int_{a}^{b} |g|^q \alpha \right\}^{1/q}.
\]

**Proof:** First, we note that

\[
\left| \int_{a}^{b} fg \alpha \right| \leq \int_{a}^{b} |fg| \alpha.
\]

Also,

\[
\int_{a}^{b} |f|^p \alpha = M^p \Rightarrow \int_{a}^{b} \left( \frac{|f|}{M} \right)^p \alpha = 1
\]
and
\[ \int_a^b |g|^q da = N^q \Rightarrow \int_a^b \left( \frac{|g|}{N} \right)^q da = 1. \]

Then we have by (b),
\[ \int_a^b \frac{|f|}{M} \frac{|g|}{N} da \leq 1 \]
which implies that, by (*),
\[ \left| \int_a^b fg da \right| \leq MN = \left\{ \int_a^b |f|^p da \right\}^{1/p} \left\{ \int_a^b |g|^q da \right\}^{1/q}. \]

(d) Show that Holder’s inequality is also true for the "improper" integrals.

**Proof:** It is clear by (c), so we omit the proof.

7.17 Assume that \( f \in R(\alpha), g \in R(\alpha), \) and \( f \cdot g \in R(\alpha) \) on \([a, b]\). Show that
\[ \frac{1}{2} \int_a^b \left[ \int_a^b (f(y) - f(x))(g(y) - g(x))da(y) \right] da(x) \]
\[ = (a(b) - a(a)) \int_a^b f(x)g(x)da(x) - \left( \int_a^b f(x)da(x) \right) \left( \int_a^b g(x)da(x) \right). \]
If \( a \nearrow \) on \([a, b]\), deduce the inequality
\[ \left( \int_a^b f(x)da(x) \right) \left( \int_a^b g(x)da(x) \right) \leq (a(b) - a(a)) \int_a^b f(x)g(x)da(x) \]
when both \( f \) and \( g \) are increasing (or both are decreasing) on \([a, b]\). Show that the reverse inequality holds if \( f \) increases and \( g \) decreases on \([a, b]\).

**Proof:** Since
\[ \frac{1}{2} \int_a^b \left[ \int_a^b (f(y) - f(x))(g(y) - g(x))da(y) \right] da(x) \]
\[ = \frac{1}{2} \int_a^b \left[ \int_a^b f(y)g(y) - f(y)g(x) - f(x)g(y) + f(x)g(x)da(y) \right] da(x) \]
\[ = (a(b) - a(a)) \int_a^b f(x)g(y)da(y) - \left( \int_a^b f(x)da(x) \right) \left( \int_a^b g(x)da(x) \right) \]
which implies that, (let \( a, f, \) and \( g \nearrow \) on \([a, b]\)),
\[ 0 \leq \frac{1}{2} \int_a^b \left[ \int_a^b (f(y) - f(x))(g(y) - g(x))da(y) \right] da(x) \]
and (let \( a, \) and \( f \nearrow \) on \([a, b]\), \( g \searrow \) on \([a, b]\)),
\[ 0 \geq \frac{1}{2} \int_a^b \left[ \int_a^b (f(y) - f(x))(g(y) - g(x))da(y) \right] da(x), \]
we know that, (let \( a, f, \) and \( g \nearrow \) on \([a, b]\))
\[ \left( \int_a^b f(x)da(x) \right) \left( \int_a^b g(x)da(x) \right) \leq (a(b) - a(a)) \int_a^b f(x)g(x)da(x) \]
and (let \( a, \) and \( f \nearrow \) on \([a, b]\), \( g \searrow \) on \([a, b]\))
\[ \left( \int_a^b f(x)da(x) \right) \left( \int_a^b g(x)da(x) \right) \geq (a(b) - a(a)) \int_a^b f(x)g(x)da(x). \]

Riemann integrals
7.18 Assume \( f \in R(\alpha) \) on \([a, b]\). Use Exercise 7.4 to prove that the limit

\[
\lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left( a + k \frac{b-a}{n} \right)
\]

exists and has the value \( \int_{a}^{b} f(x)\,dx \). Deduce that

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2} = \frac{\pi^2}{8}, \quad \lim_{n \to \infty} \sum_{k=1}^{n} (n^2 + k^2)^{-1/2} = \log(1 + \sqrt{2}).
\]

**Proof:** Since \( f \in R(\alpha) \) on \([a, b]\), given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that as \( \|P\| < \delta \), we have

\[
\left| S(P, f) - \int_{a}^{b} f(x)\,dx \right| < \varepsilon.
\]

For this \( \delta \), we choose \( n \) large enough so that \( \frac{b-a}{n} < \delta \), that is, as \( n \geq N \), we have \( \frac{b-a}{n} < \delta \). So,

\[
\left| S(P, f) - \int_{a}^{b} f(x)\,dx \right| < \varepsilon
\]

which implies that

\[
\left| \frac{b-a}{n} \sum_{k=1}^{n} f\left( a + k \frac{b-a}{n} \right) - \int_{a}^{b} f(x)\,dx \right| < \varepsilon.
\]

That is,

\[
\lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left( a + k \frac{b-a}{n} \right)
\]

exists and has the value \( \int_{a}^{b} f(x)\,dx \).

Since \( \sum_{k=1}^{n} \frac{n}{k^2 + n^2} = \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{\left( \frac{k}{n} \right)^2 + 1} \), we know that by above result,

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2} = \lim_{n \to \infty} \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{\left( \frac{k}{n} \right)^2 + 1} = \int_{0}^{1} \frac{dx}{1 + x^2} = \arctan 1 - \arctan 0 = \pi/4.
\]

Since \( \sum_{k=1}^{n} (n^2 + k^2)^{-1/2} = \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{\left[ 1 + \left( \frac{k}{n} \right)^2 \right]^{1/2}} \), we know that by above result,
\[
\lim_{n \to \infty} \sum_{k=1}^{n} (n^2 + k^2)^{-1/2} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\left[1 + (\frac{k}{n})^2\right]^{1/2}} \\
= \int_{0}^{1} \frac{dx}{\left(1 + x^2\right)^{1/2}} \\
= \int_{0}^{\pi/4} \sec \theta d\theta, \ \text{let} \ x = \tan \theta \\
= \int_{0}^{\pi/4} \sec \theta \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta \\
= \int_{1}^{1 + \frac{\pi}{2}} \frac{du}{u}, \ \text{let} \ \sec \theta + \tan \theta = u \\
= \log \left(1 + \sqrt{2}\right).
\]

7.19 Define

\[
f(x) = \left(\int_{0}^{x} e^{-t^2} dt\right)^2, \ g(x) = \int_{0}^{1} \frac{e^{-x^2(\tau^2 + 1)}}{\tau^2 + 1} dt.
\]

(a) Show that \(g'(x) + f'(x) = 0\) for all \(x\) and deduce that \(f(x) + g(x) = \pi/4\).

**Proof:** Since

\[
f'(x) = 2 \left(\int_{0}^{x} e^{-t^2} dt\right) e^{-x^2}
\]

and note that if \(h(x, \tau) = \frac{e^{-x^2(\tau^2 + 1)}}{\tau^2 + 1}\), we know that \(h\) is continuous on \([0, a] \times [0, 1]\) for any real \(a > 0\), and \(h_x = -2xe^{-x^2(\tau^2 + 1)}\) is continuous on \([0, a] \times [0, 1]\) for any real \(a > 0\),

\[
g'(x) = \int_{0}^{1} h_x dt \\
= \int_{0}^{1} -2xe^{-x^2(\tau^2 + 1)} dt \\
= -2e^{-x^2} \int_{0}^{1} xe^{-(\tau^2)} dt \\
= -2e^{-x^2} \int_{0}^{1} e^{-(\tau^2)} du,
\]

we know that \(g'(x) + f'(x) = 0\) for all \(x\). Hence, we have \(f(x) + g(x) = C\) for all \(x\), constant. Since \(C = f(0) + g(0) = \int_{0}^{1} \frac{dt}{1+t^2} = \pi/4\), \(f(x) + g(x) = \pi/4\).

**Remark:** The reader should think it twice on how to find the auxiliary function \(g\).

(b) Use (a) to prove that

\[
\lim_{x \to \infty} \int_{0}^{x} e^{-t^2} dt = \frac{1}{2} \sqrt{\pi}.
\]

**Proof:** Note that

\[
|h(x, \tau)| = \frac{|e^{-x^2(\tau^2 + 1)}|}{\tau^2 + 1} \leq |e^{-x^2(\tau^2 + 1)}| \leq \frac{1}{x^2(\tau^2 + 1)} \text{ for all } x > 0;
\]

we know that

\[
\int_{0}^{1} \frac{e^{-x^2(\tau^2 + 1)}}{\tau^2 + 1} dt \leq \frac{1}{x^2} \int_{0}^{1} \frac{dt}{1 + t^2} \rightarrow 0 \text{ as } x \rightarrow \infty.
\]

So, by (a), we get
\[ \lim_{x \to \infty} f(x) = \pi/4 \]

which implies that
\[ \lim_{x \to \infty} \int_0^x e^{-t^2} \, dt = \frac{1}{2} \sqrt{\pi} \]
since \( \lim_{x \to \infty} \int_0^x e^{-t^2} \, dt \) exists by \( \int_0^x e^{-t^2} \, dt \leq \int_0^x \frac{dt}{1+t^2} = \arctan x \to \pi/2 \) as \( x \to \infty \).

**Remark:** (1) There are many methods to show this. But here is an elementary proof with help of Taylor series and Wallis formula. We prove it as follows. In addition, the reader will learn some beautiful and useful methods in the future. For example, use the application of Gamma function, and so on.

**Proof:** Note that two inequalities,
\[ 1 + x^2 \leq e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \text{ for all } x \]
and
\[ \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \leq \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2} \text{ if } |x| < 1 \]
which implies that
\[ 1 - x^2 \leq e^{-x^2} \text{ if } 0 \leq x \leq 1 \Rightarrow (1-x^2)^n \leq e^{-nx^2} \quad 1 \]
and
\[ e^{-x^2} \leq \frac{1}{1+x^2} \text{ if } x \leq 0 \Rightarrow e^{-nx^2} \leq \left(\frac{1}{1+x^2}\right)^n \quad 2 \]

So, we have, by (1) and (2),
\[ \int_0^1 (1-x^2)^n \, dx \leq \int_0^1 e^{-nx^2} \, dx \leq \int_0^\infty e^{-nx^2} \, dx \leq \int_0^\infty \left(\frac{1}{1+x^2}\right)^n \, dx. \quad 3 \]

Note that
\[ \int_0^\infty e^{-nx^2} \, dx = \frac{1}{\sqrt{n}} \int_0^\infty e^{-x^2} \, dx := K \quad \sqrt{n}. \]

Also,
\[ \int_0^1 (1-x^2)^n \, dx = \int_0^{\pi/2} \sin^{2n+1} t \, dt = \frac{2 \cdot 4 \cdot 6 \cdots (2n-2)(2n)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \]
and
\[ \int_0^\infty \left(\frac{1}{1+x^2}\right)^n \, dx = \int_0^{\pi/2} \sin^{2n-2} t \, dt = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} \frac{\pi}{2}, \]
so
\[ \sqrt{n} \frac{2 \cdot 4 \cdot 6 \cdots (2n-2)(2n)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \leq K \leq \sqrt{n} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} \frac{\pi}{2} \]
which implies that
\[ \frac{\left[2 \cdot 4 \cdot 6 \cdots (2n-2)(2n)\right]^2}{2n+1 \left[1 \cdot 3 \cdot 5 \cdots (2n-1)\right]^2} \leq K^2 \leq \frac{n}{2n-1} \frac{\left[1 \cdot 3 \cdot 5 \cdots (2n-3)\right]^2(2n-1)}{\left[2 \cdot 4 \cdot 6 \cdots (2n-2)\right]^2} \left(\frac{\pi}{2}\right)^2 \quad 4 \]

By Wallis formula, we know that, by (4)
\[ K = \frac{\sqrt{\pi}}{2}. \]

That is, we have proved that **Euler-Possion Integral**
\[ \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}. \]

**Note:** (Wallis formula)

\[
\lim_{n \to \infty} \frac{[2 \cdot 4 \cdot 6 \cdots (2n - 2)(2n)]^2}{[1 \cdot 3 \cdot 5 \cdots (2n - 1)]^2 (2n + 1)} = \frac{\pi}{2}.
\]

**Proof:** As \(0 \leq x \leq \pi/2\), we have

\[ \sin^{2n+1}t \leq \sin^{2n}t \leq \sin^{2n-1}t, \text{ where } n \in \mathbb{N}. \]

So, we know that

\[ \int_0^{\pi/2} \sin^{2n+1}tdt \leq \int_0^{\pi/2} \sin^{2n}tdt \leq \int_0^{\pi/2} \sin^{2n-1}tdt \]

which implies that

\[ \frac{(2n)(2n-2) \cdots 4 \cdot 2}{(2n+1)(2n-1) \cdots 3 \cdot 1} \leq \frac{(2n-1)(2n-3) \cdots 3 \cdot 1 \cdot \frac{\pi}{2}}{(2n-2)(2n-4) \cdots 4 \cdot 2} \leq \frac{(2n-2)(2n-4) \cdots 4 \cdot 2}{(2n-1)(2n-3) \cdots 3 \cdot 1}. \]

So,

\[
\left[ \frac{(2n)(2n-2) \cdots 4 \cdot 2}{(2n-1)(2n-3) \cdots 3 \cdot 1} \right]^2 \frac{1}{2n+1} \leq \frac{\pi}{2} \leq \left[ \frac{(2n-1)(2n-3) \cdots 3 \cdot 1}{(2n-2)(2n-4) \cdots 4 \cdot 2} \right]^2 \frac{1}{2n}.
\]

Hence, from

\[
\left[ \frac{(2n)(2n-2) \cdots 4 \cdot 2}{(2n-1)(2n-3) \cdots 3 \cdot 1} \right]^2 \left( \frac{1}{2n+1} - \frac{1}{2n+1} \right) \leq \frac{1}{2n} \frac{\pi}{2} \to 0,
\]

we know that

\[
\lim_{n \to \infty} \frac{[2 \cdot 4 \cdot 6 \cdots (2n - 2)(2n)]^2}{[1 \cdot 3 \cdot 5 \cdots (2n - 1)]^2 (2n + 1)} = \frac{\pi}{2}.
\]

(2) Here is another exercise from Hadamard’s result. We Write it as follows. Let \(f \in C^k(R)\) with \(f(0) = 0\). Prove that there exists an unique function \(g \in C^{k-1}(R)\) such that \(f = xg(x)\) on \(R\).

**Proof:** Consider

\[
f(x) = f(x) - f(0) = \int_0^1 df(xt) = \int_0^1 xf'(xt)dt = x \int_0^1 f'(xt)dt;
\]

we know that if \(g(x) := \int_0^1 f'(xt)dt\), then we have prove it.

**Note:** In fact, we can do this job by routine work. Define

\[
g(x) = \begin{cases} \frac{f(x)}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}
\]

However, it is too long to write. The trouble is to make sure that \(g \in C^{k-1}(R)\).

7.20 Assume \(g \in R\) on \([a, b]\) and define \(f(x) = \int_a^x g(t)dt\) if \(t \in [a, b]\). Prove that the
integral \( \int_a^x |g(t)| \, dt \) gives the total variation of \( f \) on \([a,x]\).

**Proof:** Since \( \int_a^x |g(t)| \, dt \) exists, given \( \varepsilon > 0 \), there exists a partition 
\[ P_1 = \{x_0 = a, \ldots, x_n = x\} \] such that 
\[ L(P_1, |g|) > \int_a^x |g(t)| \, dt - \varepsilon. \]  

So, for this \( P_1 \), we have 
\[ \sum_{k=1}^n |f(x_k) - f(x_{k-1})| = \sum_{k=1}^n \left| \int_{x_{k-1}}^{x_k} g(t) \, dt \right| = \sum_{k=1}^n |c_k(x_k - x_{k-1})| \quad \text{by Mean Value Theorem} \]

where \( \inf_{x \in [x_{k-1}, x_k]} |g(x)| \leq c_k \leq \sup_{x \in [x_{k-1}, x_k]} |g(x)| \).

Hence, we know that, by (1) and (2), 
\[ \sum_{k=1}^n |f(x_k) - f(x_{k-1})| > \int_a^x |g(t)| \, dt - \varepsilon \]

which implies that 
\[ V_f(a,b) \geq \int_a^x |g(t)| \, dt \]
since \( \varepsilon \) is arbitrary.

Conversely, since \( \int_a^x |g(t)| \, dt \) exists, given \( \varepsilon > 0 \), there exists a partition \( P_2 \) such that 
\[ U(P_2, |g|) < \int_a^x |g(t)| \, dt + \varepsilon /2. \]

Also, for the same \( \varepsilon \), there exists a partition \( P_3 = \{t_0 = a, \ldots, t_m = x\} \) such that 
\[ V_f(a,b) - \varepsilon /2 < \sum_{k=1}^m |f(t_k) - f(t_{k-1})|. \]

Let \( P = P_2 \cup P_3 = \{s_0 = a, \ldots, s_p = x\} \), then by (3) and (4), we have 
\[ U(P, |g|) < \int_a^x |g(t)| \, dt + \varepsilon /2 \]

and 
\[ V_f(a,b) - \varepsilon /2 < \sum_{k=1}^p |f(s_k) - f(s_{k-1})| \]
\[ = \sum_{k=1}^p \left| \int_{s_{k-1}}^{s_k} g(t) \, dt \right| \]
\[ = \sum_{k=1}^p |c_k(x_k - x_{k-1})| \]
\[ \leq U(P, |g|) \]

which imply that 
\[ V_f(a,b) \leq \int_a^x |g(t)| \, dt \]
since \( \varepsilon \) is arbitrary.

Therefore, from above discussion, we have proved that 
\[ V_f(a,b) = \int_a^x |g(t)| \, dt. \]

7.21 If \( f = (f_1, \ldots, f_n) \) be a vector-valued function with a continuous derivative \( f' \) on
[a, b]. Prove that the curve described by \( f \) has length
\[
\Lambda_f(a, b) = \int_a^b \|f'(t)\| \, dt.
\]

**Proof:** Since \( f' = (f_1, \ldots, f_n) \) is continuous on \([a, b]\), we know that
\[
\left[ \sum_{j=1}^n (f_j')^2(t) \right]^{1/2} = \|f'(t)\| \text{ is uniformly continuous on } [a, b].
\]
So, given \( \varepsilon > 0 \), there exists a \( \delta_1 > 0 \) such that for \( |x - y| < \delta_1 \), where \( x, y \in [a, b] \), we have
\[
\|f'(x)\| - \|f'(y)\| < \frac{\varepsilon}{3(b - a)}.
\]
Since \( \|f'(t)\| \in R \) on \([a, b]\), for the same \( \varepsilon \), there exists \( \delta_2 > 0 \) such that for \( \|P_1\| < \delta_2 \), where \( P_1 = \{x_0 = a, \ldots, x_n = b\} \) we have
\[
\left| S(P_1, \|f'\|) - \int_a^b \|f'(t)\| \, dt \right| < \varepsilon/3, \text{ where } S(P_1, \|f'\|) = \sum_{j=1}^n \|f'(t_j)\| \Delta x_j
\]
and \( \Lambda_f(a, b) \) exists by **Theorem 6.17**, for the same \( \varepsilon \), there exists a partition \( P_2 = \{s_0 = a, \ldots, s_m = b\} \) such that
\[
\Lambda_f(a, b) - \varepsilon/3 < \sum_{k=1}^m \|f(s_k) - f(s_{k-1})\|.
\]
Let \( \delta = \min(\delta_1, \delta_2) \) and \( P \subseteq P_2 \) so that \( \|P\| < \delta \), where \( P = \{y_0 = a, \ldots, y_q = b\} \) then by (1)-(3), we have
(i) As \( |x - y| < \delta \), where \( x, y \in [a, b] \), we have
\[
\|f'(x)\| - \|f'(y)\| < \frac{\varepsilon}{3(b - a)}.
\]
(ii) As \( \|P\| < \delta \), we have
\[
\left| S(P, \|f'\|) - \int_a^b \|f'(t)\| \, dt \right| < \varepsilon/3, \text{ where } S(P, \|f'\|) = \sum_{j=1}^q \|f'(\tilde{t}_j)\| \Delta y_j
\]
(iii) As \( \|P\| < \delta \), we have
\[
\Lambda_f(a, b) - \varepsilon/3 \leq \sum_{k=1}^m \left\{ \sum_{j=1}^n [(f_j(s_k) - (f_j)(s_{k-1})]^2 \right\}^{1/2}
\]
\[
\leq \sum_{k=1}^q \left\{ \sum_{j=1}^n [(f_j(y_k) - (f_j)(y_{k-1})]^2 \right\}^{1/2}
\]
\[
= \sum_{k=1}^q \left\{ \sum_{j=1}^n [f_j(z_k)]^2 \Delta y_j, \text{ by Mean Value Theorem} \right\}
\]
\[
= \sum_{k=1}^q \|f'(z_k)\| \Delta y_j
\]
\[
\leq A_f(a, b)
\]
By (ii) and (iii), we have
\[
\left| \sum_{k=1}^{q} g(z_k) \Delta y_j - S(P,g) \right| = \left| \sum_{k=1}^{q} g(z_k) \Delta y_j - \sum_{j=1}^{q} g(\tilde{y}_j) \Delta y_j \right|
\leq \sum_{k=1}^{q} |g(z_k) - g(\tilde{y}_j)| \Delta y_j
< \sum_{k=1}^{q} \frac{\varepsilon}{3(b-a)} \Delta y_j
= \varepsilon/3.
\]

Hence, (5)-(7) implies that
\[
\left| \int_{a}^{b} \|f'(t)\| dt - \Lambda_f(a,b) \right| < \varepsilon.
\]
Since \(\varepsilon\) is arbitrary, we have proved that
\[
\Lambda_f(a,b) = \int_{a}^{b} \|f'(t)\| dt.
\]

7.22 If \(f^{(n+1)}\) is continuous on \([a,x]\), define
\[
I_n(x) = \frac{1}{n!} \int_{a}^{x} (x-t)^n f^{(n+1)}(t) dt.
\]
(a) Show that
\[
I_{k-1}(x) - I_k(x) = \frac{f^{(k)}(a)(x-a)^k}{k!}, \quad k = 1, 2, \ldots, n.
\]

**Proof:** Since, for \(k = 1, 2, \ldots, n,\)
\[
I_k(x) = \frac{1}{k!} \int_{a}^{x} (x-t)^k f^{(k+1)}(t) dt
= \frac{1}{k!} \int_{a}^{x} (x-t)^k df^{(k)}(t)
= \frac{1}{k!} \left[ (x-t)^k f^{(k)}(t) \right]_{a}^{x} + k \int_{a}^{x} (x-t)^{k-1} f^{(k)}(t) dt
= \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{(k-1)!} \int_{a}^{x} (x-t)^{k-1} f^{(k)}(t) dt
= \frac{f^{(k)}(a)(x-a)^k}{k!} + I_{k-1}(x),
\]
we know that
\[
I_{k-1}(x) - I_k(x) = \frac{f^{(k)}(a)(x-a)^k}{k!}, \quad \text{for } k = 1, 2, \ldots, n.
\]
(b) Use (a) to express the remainder in Taylor’s formula (Theorem 5.19) as an integral.

**Proof:** Since \(f(x) - f(a) = I_0(x)\), we know that
\[ f(x) = f(a) + I_0(x) \]
\[ = f(a) + \sum_{k=1}^{n} [I_{k-1}(x) - I_k(x)] + I_n(x) \]
\[ = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{n!} \int_{a}^{x} (x-t)^n f^{(n+1)}(t) dt \] by (a).

So, by Taylor’s formula, we know that
\[ R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} = \frac{1}{n!} \int_{a}^{x} (x-t)^n f^{(n+1)}(t) dt, \]
for some \( c \in (a,x) \).

where \( R_n(x) \) is the remainder term.

**Remark:** 1. The reader should be noted that with help of **Mean Value Theorem**, we have
\[ \frac{1}{n!} \int_{a}^{x} (x-t)^n f^{(n+1)}(t) dt = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}. \]

2. Use **Integration by parts** repeatedly; we can show (*). Of course, there is other proofs such as **Mathematical Induction**.

**Proof:** Since
\[ \int uv^{(n+1)} dt = uv^{(n)} - u'v^{(n-1)} + u''v^{(n-2)} - \ldots + (-1)^n u^{(n)} v + (-1)^{n+1} \int u^{(n+1)} v dt, \]
letting \( v(t) = (x-t)^n \) and \( u(t) = f(t) \), then
\[ f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{1}{n!} \int_{a}^{x} (x-t)^n f^{(n+1)}(t) dt. \]

**Note:** The reader should give it a try to show it. Since it is not hard, we omit the detail.

3. The remainder term as an integral is useful; the reader should see the textbook in Ch9, pp242-244.

4. There is a good exercise related with an application of Taylor’s Remainder. We write it as a reference.

Let \( u''(t) + f(t)u(t) = 0 \), where \( f(t) \) is continuous and non-negative on \([0,c]\) If \( u \) is defined and not a zero function on \([0,c]\) and
\[ \int_{a}^{b} (b-t)(a-t)f(t) < b-a \] for all \( a, b \in [0,c] \), where \( a < b \).

Then \( u \) at most has one zero on \([0,c]\).

**Proof:** First, we note that \( u \) has at most finitely many zeros in the interval \([0,c]\) by **uniqueness theorem on O.D.E.** So, let \( u(a) = u(b) = 0 \), where \( a, b \in [0,c] \) with \( a < b \), and no point \( y \in (a,b) \) such that \( u(y) = 0 \). Consider \([a,b]\) and by Taylor’s Theorem with Remainder Term as an integral, we have
\[ u(x) = u(a) + u'(a)(x-a) + \int_{a}^{x} (x-t)u''(t) dt \]
\[ = u'(a)(x-a) + \int_{a}^{x} (x-t)u''(t) dt \]
\[ = u'(a)(x-a) - \int_{a}^{x} (x-t)u(t)f(t) dt. \]

Note that \( u(x) \) is positive on \((a,b)\) ( Or, \( u(x) \) is negative on \((a,b)\) ) So, we have
\[ |u(x)| \leq |u'(a)|(x-a). \]

By (\ref{eq:inequality}),
\[
0 = u(b) = u'(a)(b-a) - \int_a^b (b-t)u(t)f(t)dt
\]
which implies that
\[
u'(a)(b-a) = \int_a^b (b-t)u(t)f(t)\]
which implies that by (**), and note that \(u'(a) \neq 0\),
\[
b - a \leq \int_a^b (b-t)(t-a)f(t)dt
\]
which contradicts to (*). So, \(u\) at most has one zero on \([0,c]\).

Note: (i) In particular, let \(f(t) = e^{-t}\), we have (*) holds.

Proof: Since
\[
\int_a^b (b-t)(t-a)e^{-t}dt = e^{-a}(-2 + b - a) + e^{-b}(2 + b - a)
\]
by integration by parts twice, we have, (let \(b - a = x\)),
\[
e^{-a}(-2 + b - a) + e^{-b}(2 + b - a) - (b - a)
\]
\[
= e^{-a}(-2 + x) + e^{-x-a}(2 + x) - x
\]
\[
= x(e^{-a} - 1) + e^{-x-a}(-2e^x + x + 2)
\]
\[
< 0 \text{ since } a < b \text{ and } e^x > 1 + x.
\]

(ii) In the proof of exercise, we use the uniqueness theorem: If \(p(x)\) and \(q(x)\) are continuous on \([0,a]\), then
\[
y'' + p(x)y' + q(x)y = 0, \text{ where } y(0) = y_0 \text{ and } y'(0) = y'_0
\]
has one and only one solution. In particular, if \(y(0) = y'(0) = 0\), then \(y = 0 \text{ on } [0,a]\) is the only solution. We do NOT give a proof; the reader can see the book, Theory of Ordinary Differential Equation by Ince, section 3.32, or Theory of Ordinary Differential Equation by Coddington and Levison, Chapter 6.

However, we need use the uniqueness theorem to show that \(u\) (in the exercise) has at most finitely many zeros in \([0,c]\).

Proof: Let \(S = \{x : u(x) = 0, x \in [0,c]\}\). If \(#(S) = \infty\), then by Bolzano-Weierstrass Theorem, \(S\) has an accumulation point \(p\) in \([0,c]\). Then \(u(p) = 0\) by continuity of \(u\). In addition, let \(r_n \to p\), and \(u(r_n) = 0\), then
\[
\lim_{x\to p} u(x) - u(p) = \lim_{x\to r_n} u(r_n) - u(p) = 0.
\]
(Note that if \(p\) is the endpoint of \([0,c]\), we may consider \(x \to p^+\) or \(x \to p^-\)). So, by uniqueness theorem, we then have \(u = 0\) on \([0,c]\) which contradicts to the hypothesis, \(u\) is not a zero function on \([0,c]\). So, \(#(S) < \infty\).

7.23 Let \(f\) be continuous on \([0,a]\). If \(x \in [0,a]\), define \(f_0(x) = f(x)\) and let
\[
f_{n+1}(x) = \frac{1}{n!} \int_0^x (x-t)^nf(t)dt, \ n = 0, 1, 2, \ldots
\]

(a) Show that the \(n\)th derivative of \(f_n\) exists and equals \(f\).

Proof: Consider, by Chain Rule,
\[
\frac{f'_n}{(n-1)!} \int_0^x (x-t)^{n-1}f(t)dt = f_{n-1} \quad \text{for all } n \in \mathbb{N},
\]
we have
\[
f^{(n)}_n = f.
\]
That is, \(n\)th derivative of \(f_n\) exists and equals \(f\).

**Remark:** (1) There is another proof by **Mathematical Induction** and **Integration by parts**. It is not hard; we omit the proof.

(2) The reader should note that the exercise tells us that given any continuous function \(f\) on \([a,b]\), there exists a function \(g_n\) on \([a,b]\) such that \(g_n^{(n)} = f\), where \(n \in \mathbb{N}\). In fact, the function
\[
g_n = \frac{1}{n!} \int_a^x (x-t)^n f(t)dt, n = 0, 1, 2, \ldots
\]

(3) The reader should compare the exercise with 7.22. At the same time, look at two integrands in both exercises.

(b) Prove the following **theorem of M. Fekete**: The number of changes in sign of \(f\) in \([0,a]\) is not less than the number of changes in sign in the ordered set of numbers
\[
f(a), f_1(a), \ldots, f_n(a).
\]

**Proof**: Let \(T(f)\) denote the number of changes in sign of \(f\) on \([0,a]\) and \(S_n(f)\) the number of changes in sign in the ordered set of numbers
\[
f(a), f_1(a), \ldots, f_n(a).
\]
We prove \(T(f) \geq S_n(f)\) for each \(n\) by **Mathematical Induction** as follows. Note that \(S_n(f) \leq n\).

As \(n = 1\), if \(S_1(f) = 0\), then there is nothing to prove it. If \(S_1(f) = 1\), it means that \(f(a)f_1(a) < 0\). Without loss of generality, we may assume that \(f(a) > 0\), so \(f_1(a) < 0\) which implies that
\[
0 > f_1(a) = \frac{1}{0!} \int_0^a f(t)dt
\]
which implies that there exists a point \(y \in [0,a]\) such that \(f(y) < 0\). Hence, \(T(f) \geq S_1(f)\) holds for any continuous functions defined on \([0,a]\).

Assume that \(n = k\) holds for any continuous functions defined on \([0,a]\), As \(n = k + 1\), we consider the ordered set of numbers
\[
f(a), f_1(a), \ldots, f_k(a), f_{k+1}(a).
\]

Note that
\[
f_{n+1}(a) = (f_1)_n(a) \quad \text{for all } n \in \mathbb{N},
\]
so by induction hypothesis,
\[
T(f_i) \geq S_k(f_i)
\]
Suppose \(S_k(f_i) = p\), and \(f_1(0) = 0\), then \(f'_1 = f\) at least has \(p\) zeros by **Rolle’s Theorem**. Hence,
\[
T(f) \geq T(f_1) \geq S_k(f_1) = p
\]
We consider two cases as follows.

(i) \(f(a)f_1(a) \geq 0\) : With help of (*),
\[
T(f) \geq S_k(f_1) = S_{k+1}(f).
\]
(ii) $f(a)f_1(a) < 0$ : Claim that

$$T(f) > S_k(f_i) = p$$

as follows. Suppose **NOT**, it means that $T(f) = T(f_i) = p$ by (*). Say

$$f(a_1) = f(a_2) = \ldots = f(a_p) = 0,$$

where $0 < a_1 < a_2 < \ldots < a_p < 1.$

and

$$f_1(b_1) = f_1(b_2) = \ldots = f_1(b_p) = 0,$$

where $0 < b_1 < b_2 < \ldots < b_p < 1.$

By $f(a)f_1(a) < 0$, we know that

$$f(x)f_1(x) < 0 \text{ where } x \in (0, c), \ c = \min(a_1, b_1)$$

which is impossible since

$$f(x)f_1(x) = f(x)[f_1(x) - f_1(0)] \text{ by } f_1(0) = 0$$

$$= f(x)f_1(y), \text{ where } y \in (0, x) \subseteq (0, c)$$

$$= f(x)f(y)$$

$$> 0 \text{ since } f(x) \text{ and } f(y) \text{ both positive or negative.}$$

So, we obtain that $T(f) > S_k(f_i) = p$. That is, $T(f) \geq S_k(f_i) + 1 = S_{k+1}(f)$.

From above results, we have proved it by **Mathematical Induction**.

(c) Use (b) to prove the following theorem of **Feje’r**. The number of changes in sign of $f$ in $[0, a]$ is not less than the number of changes in sign in the ordered set

$$f(0), \int_0^a f(t)dt, \int_0^a tf(t)dt, \ldots, \int_0^a t^nf(t)dt.$$  

**Proof:** Let $g(x) = f(a - x)$, then, define $g_0(x) = g(x)$, and for $n = 0, 1, 2, \ldots,$

$$g_{n+1}(a) = \frac{1}{n!} \int_0^a (a - t)^n g(t)dt$$

$$= \frac{1}{n!} \int_0^a u^n f(u)du \text{ by change of variable } (u = a - t).$$

So, by (b), the number of changes in sign of $g$ in $[0, a]$ is not less than the number of changes in sign in the ordered set

$$g(a), g_1(a), \ldots, g_{n+1}(a).$$

That is, the number of changes in sign of $g$ in $[0, a]$ is not less than the number of changes in sign in the ordered set

$$f(0), \int_0^a f(t)dt, \int_0^a tf(t)dt, \ldots, \int_0^a t^nf(t)dt.$$  

Note that the number of changes in sign of $g$ in $[0, a]$ equals the number of changes in sign of $f$ in $[0, a]$, so we have proved the **Feje’r Theorem**.

### 7.24

Let $f$ be a positive continuous function in $[a, b]$. Let $M$ denote the maximum value of $f$ on $[a, b]$. Show that

$$\lim_{n \to \infty} \left( \int_a^b f(x)^n dx \right)^{1/n} = M.$$  

**Proof:** Since $f$ is a positive continuous function in $[a, b]$, there exists a point $c \in [a, b]$ such that $f(c) = M = \sup_{x \in [a, b]} f(x) > 0.$ Then given $(M > )\varepsilon > 0$, there is a $\delta > 0$ such that as $x \in B(c, \delta) \cap [a, b] := I$, we have

$$(0 < )M - \varepsilon < f(x) < M + \varepsilon.$$  

Hence, we have
\[|I|^{1/n}(M - \varepsilon) \leq \left( \int_a^b f^n(x) \, dx \right)^{1/n} \leq \left( \int_a^b f(x)^n \, dx \right)^{1/n} \leq (b - a)^{1/n}M\]

which implies that

\[M - \varepsilon \leq \lim_{n \to \infty} \inf \left( \int_a^b f(x)^n \, dx \right)^{1/n} \leq M.\]

So, \(\lim_{n \to \infty} \inf \left( \int_a^b f(x)^n \, dx \right)^{1/n} = M\) since \(\varepsilon\) is arbitrary. Similarly, we can show that \(\lim_{n \to \infty} \sup \left( \int_a^b f(x)^n \, dx \right)^{1/n} = M\). So, we have proved that \(\lim_{n \to \infty} \left( \int_a^b f(x)^n \, dx \right)^{1/n} = M\).

**Remark:** There is good exercise; we write it as a reference. Let \(f(x)\) and \(g(x)\) are continuous and non-negative function defined on \([a, b]\). Then

\[\lim_{n \to \infty} \left( \int_a^b f(x)^n g(x) \, dx \right)^{1/n} = \max_{x \in [a,b]} f(x).\]

Since the proof is similar, we omit it. (The reader may let \(a(x) = \int_a^x g(t) \, dt\).

7.25 A function \(f\) of two real variables is defined for each point \((x, y)\) in the unit square 0 \(\leq x \leq 1, 0 \leq y \leq 1\) as follows:

\[f(x,y) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 2y & \text{if } x \text{ is irrational.} \end{cases}\]

(a) Compute \(\int_{Q}^1 f(x,y) \, dx\) and \(\int_{Q}^1 f(x,y) \, dx\) in terms of \(y\).

**Proof:** Consider two cases for upper and lower Riemann-Stieltjes integrals as follows.

(i) As \(y \in [0, 1/2]\) : Given any partition \(P = \{x_0 = 0, \ldots, x_n = 1\}\), we have

\[\sup_{x \in [x_j, x_{j+1}]} f(x,y) = 1, \quad \inf_{x \in [x_j, x_{j+1}]} f(x,y) = 2y.\]

Hence, \(\int_{0}^1 f(x,y) \, dx = 1\), and \(\int_{0}^1 f(x,y) \, dx = 2y\).

(ii) As \(y \in (1/2, 1]\) : Given any partition \(P = \{x_0 = 0, \ldots, x_n = 1\}\), we have

\[\sup_{x \in [x_{j-1}, x_j]} f(x,y) = 2y, \quad \inf_{x \in [x_{j-1}, x_j]} f(x,y) = 1.\]

Hence, \(\int_{0}^1 f(x,y) \, dx = 2y\), and \(\int_{0}^1 f(x,y) \, dx = 1\).

(b) Show that \(\int_{0}^1 f(x,y) \, dy\) exists for each fixed \(x\) and compute \(\int_{Q}^x f(x,y) \, dy\) in terms of \(x\) and \(t\) for 0 \(\leq x \leq 1, 0 \leq t \leq 1\).

**Proof:** If \(x \in Q \cap [0, 1]\), then \(f(x,y) = 1\). And if \(x \in Q^c \cap [0, 1]\), then \(f(x,y) = 2y\). So, for each fixed \(x\), we have

\[\int_{0}^1 f(x,y) \, dy = \int_{0}^1 1 \, dy = 1 \text{ if } x \in Q \cap [0, 1]\]

and

\[\int_{0}^1 f(x,y) \, dy = \int_{0}^1 2y \, dy = 1 \text{ if } x \in Q^c \cap [0, 1].\]

In addition,

\[\int_{Q}^x f(x,y) \, dy = \int_{Q}^x 1 \, dy = t \text{ if } x \in Q \cap [0, 1].\]
and
\[ \int_0^t f(x,y) dy = \int_0^t 2y dy = t^2 \text{ if } x \in Q^c \cap [0,1]. \]

(c) Let \( F(x) = \int_0^1 f(x,y) dy \). Show that \( \int_0^1 F(x) dx \) exists and find its value.

**Proof:** By (b), we have
\[ F(x) = 1 \text{ on } [0,1]. \]
So, \( \int_0^1 F(x) dx \) exists and
\[ \int_0^1 F(x) dx = 1. \]

**7.26** Let \( f \) be defined on \([0,1]\) as follows: \( f(0) = 0 \); if \( 2^{-n-1} < x \leq 2^{-n} \), then \( f(x) = 2^{-n} \) for \( n = 0, 1, 2, \ldots \)

(a) Give two reasons why \( \int_0^1 f(x) dx \) exists.

**Proof:** (i) \( f(x) \) is monotonic decreasing on \([0,1]\). (ii) \( \{x : f \text{ is discontinuous at } x\} \) has measure zero.

**Remark:** We compute the value of the integral as follows.

**Solution:** Consider the interval \( I_n = [2^{-n}, 1] \) where \( n \in N \), then we have \( f \in R \) on \( I_n \) for each \( n \), and
\[ \int_{2^{-n}}^1 f(x) dx = \sum_{k=1}^{n} \int_{2^{-k+1}}^{2^{-k}} f(x) dx \]
\[ = \sum_{k=1}^{n} 2^{-k+1} \int_{2^{-k}}^{2^{-k+1}} dx \]
\[ = \sum_{k=1}^{n} (2^{-k+1})(2^{-k}) \]
\[ = 2^{-k} \left[ 1 - \left( \frac{1}{4} \right)^n \right] \frac{1 - \frac{1}{4}}{1 - \frac{1}{4}} \]
\[ = \frac{2}{3} \left[ 1 - \left( \frac{1}{4} \right)^n \right] \rightarrow \frac{2}{3} \text{ as } n \rightarrow \infty. \]
So, the integral \( \int_0^1 f(x) dx = \frac{2}{3} \).

**Note:** In the remark, we use the following fact. If \( f \in R \) on \([a,b]\), then
\[ \int_a^b f(x) dx = \lim_{n \to \infty} \int_{a_n}^{b_n} f(x) dx \]
where \( \{a_n\} \) is a sequence with \( a_n \to a \), and \( a_n \geq a \) for all \( n \).

**Proof:** Since \( a_n \to a \), given \( \varepsilon > 0 \), there is a positive integer \( N \) such that as \( n \geq N \), we have
\[ |a_n - a| < \varepsilon/M, \text{ where } M = \sup_{x \in [a,b]} |f(x)| \]
So,
\[
\left| \int_a^b f(x)dx - \int_{a_n}^{b} f(x)dx \right| = \left| \int_{a_n}^a f(x)dx \right| \\
\leq M|a_n - a| < \varepsilon.
\]

That is, \( \int_a^b f(x)dx = \lim_{n \to \infty} \int_{a_n}^{b} f(x)dx \).

(b) Let \( F(x) = \int_0^x f(t)dt \). Show that for \( 0 < x \leq 1 \) we have
\[
F(x) = xA(x) - \frac{1}{3}A(x)^2,
\]
where \( A(x) = 2^{-\lceil \log_{\frac{\log 2}{x}} \rceil} \) and where \( \lceil y \rceil \) is the greatest integer in \( y \).

**Proof:** First, we note that \( F(x) = \int_0^x f(t)dt - \int_x^1 f(t)dt = \frac{2}{3} - \int_x^1 f(t)dt \). So, it suffices to consider the value of the integral \( \int_x^1 f(t)dt \).

Given any \( x \in [0, 1] \), then there exists a positive \( N \) such that \( 2^{-N-1} < x \leq 2^{-N} \). So,
\[
\int_x^1 f(t)dt = \int_{2^{-N}} f(t)dt + \int_x^{2^{-N}} f(t)dt
\]
\[
= \frac{2}{3} \left[ 1 - \left( \frac{1}{4} \right)^N \right] + 2^{-N}(2^{-N} - x) \text{ by Remark in (a)}
\]
\[
= \frac{2}{3} + \frac{1}{3} \left( \frac{1}{4} \right)^N - \left( \frac{1}{2} \right)^N x.
\]

So,
\[
F(x) = \left( \frac{1}{2} \right)^N x - \frac{1}{3} \left( \frac{1}{4} \right)^N
\]
\[
= 2^{-N}x - \frac{1}{3}2^{-2N}
\]
\[
= xA(x) - \frac{1}{3}A(x)^2
\]
where \( A(x) = 2^{-N} \). Note that \( 2^{-N-1} < x \leq 2^{-N} \), we have
\[
N = \lceil \log_{1/2} x \rceil, \text{ where } \lceil y \rceil \text{ is the Gauss symbol.}
\]

Hence,
\[
A(x) = 2^{-\lceil \log_{\frac{\log 2}{x}} \rceil}.
\]

**Remark:** (1) The reader should give it a try to show it directly by considering \([0,x] \), where \( 0 \leq x \leq 1 \).

(2) Here is a good exercise. We write it as a reference. Suppose that \( f \) is defined on \([0, 1] \) by the following
\[
f(x) = \begin{cases} 
\frac{1}{2^n} & \text{if } x = \frac{j}{2^n} \text{ where } j \text{ is an odd integer and } 0 < j < 2^n, n = 1, 2, \ldots, \\\n0 & \text{otherwise.}
\end{cases}
\]
Show that \( f \in R [0, 1] \) and has the value of the integral 0.

**Proof:** In order to show this, we consider the Riemann’s condition with respect to \( a(x) = x \) as follows. Given a partition
\[
P = \left\{ x_0 = 0 = \frac{0}{2^n}, x_1 = \frac{1}{2^n}, x_2 = \frac{2}{2^n}, \ldots, x_j = \frac{j}{2^n}, \ldots, x_{2^n} = \frac{2^n}{2^n} = 1 \right\},
\]
then the upper sum
\[
U(P, f) = \sum_{k=1}^{2^n} M_k \Delta x_k \\
= \frac{1}{2^n} \sum_{k=1}^{2^n} M_k \\
= \frac{1}{2^n} \left( \frac{2}{2^n} + n - 1 \right) \to 0 \text{ as } n \to \infty.
\]

So, \( f \) satisfies the Riemann’s condition on \([0, 1]\).

**Note:** (1) The reader should give it a try to show that the set of discontinuities of \( f \) has measure zero. Thus by **Theorem 7.48 (Lebesgue’s Criterion for Riemann Integral)**, we know that \( f \in R \) on \([0, 1]\). In addition, by the fact, the lower Riemann integral equals the Riemann integral, we know that its integral is zero.

(2) For the existence of Riemann integral, we summarize to be the theorem: Let \( f \) be a bounded function on \([a, b]\). Then the following statements are equivalent:

(i) \( f \in R \) on \([a, b]\).

(ii) \( f \) satisfies Riemann’s condition on \([a, b]\).

(iii) \( \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx \)

(iv) the set of discontinuities of \( f \) on \( I \) has measure zero.

**P.S.:** The reader should see the textbook, pp 391; we have the general discussion.

### 7.27
Assume \( f \) has a derivative which is monotonic decreasing and satisfies \( f'(x) \geq m > 0 \) for all \( x \) in \([a, b]\). Prove that

\[
\left| \int_{a}^{b} \cos f(x) \, dx \right| \leq \frac{2}{m}.
\]

**Hint:** Multiply and divide the integrand by \( f'(x) \) and use **Theorem 7.37(ii)**.

**Proof:** Since \( f'(x) \geq m > 0 \), and \( \frac{1}{f'} \) is monotonic increasing on \([a, b]\), we consider

\[
\int_{a}^{b} \cos f(x) \, dx = \int_{a}^{b} \frac{\cos f(x)}{f'(x)} \, f'(x) \, dx \\
= \frac{1}{f(b)} \int_{c}^{f(b)} [\cos f(x)] f'(x) \, dx, \text{ by **Theorem 7.37(ii)**} \\
= \frac{1}{f(b)} \int_{f(c)}^{f(b)} \cos u \, du, \text{ by Change of Variable} \\
= \frac{\sin f(b) - \sin f(c)}{f'(b)}
\]

which implies that

\[
\left| \int_{a}^{b} \cos f(x) \, dx \right| \leq \frac{2}{m}.
\]

### 7.28
Given a decreasing sequence of real numbers \( \{G(n)\} \) such that \( G(n) \to 0 \) as \( n \to \infty \). Define a function \( f \) on \([0, 1]\) in terms of \( \{G(n)\} \) as follows: \( f(0) = 1 \); if \( x \) is irrational, then \( f(x) = 0 \); if \( x \) is rational \( m/n \) (in lowest terms), then \( f(m/n) = G(n) \).

Compute the oscillation \( \omega_f(x) \) at each \( x \) in \([0, 1]\) and show that \( f \in R \) on \([0, 1]\).

**Proof:** Let \( x_0 \in Q^c \cap [0, 1] \). Since \( \lim_{n \to \infty} G(n) = 0 \), given \( \varepsilon > 0 \), there exists a positive integer \( K \) such that as \( n \geq K \), we have \( |G(n)| < \varepsilon \). So, there exists a finite number
of positive integers \( n \) such that \( G(n) \geq \varepsilon \). Denote \( S = \{ x : |f(x)| \geq \varepsilon \} \), then \( \#(S) < \infty \).

Choose a \( \delta > 0 \) such that \((x_0 - \delta, x_0 + \delta) \subseteq [0, 1] \) does NOT contain all points of \( S \). Note that \( f(x_0) = 0 \). Hence, we know that \( f \) is continuous at \( x_0 \). That is, \( \omega_f(x) = 0 \) for all \( x \in Q^c \cap [0, 1] \).

Let \( x_0 = 0 \), then it is clear that \( \omega_f(0) = 1 = f(0) > 0 \). So, \( f \) is not continuous at \( 0 \in Q \cap [0, 1] \).

Let \( x_0 \in Q \cap (0, 1) \), say \( x_0 = \frac{n}{N} \) (in lowest terms). Since \( \{ G(n) \} \) is monotone decreasing, there exists a finite number of positive integers \( n \) such that \( G(n) \geq G(N) \).

Denote \( T = \{ x : |f(x)| \geq G(N) \} \), then \( \#(T) < \infty \). Choose a \( \delta > 0 \) such that \((x_0 - \delta, x_0 + \delta) \cap [0, 1] \) does NOT contain all points of \( T \). Let \( h \in (0, \delta) \), then

\[
\sup \{ |f(x) - f(y)| : x, y \in (x_0 - h, x_0 + h) \cap [0, 1] \} = f(x_0) = G(N).
\]

So, \( \omega_f(x_0) = G(N) \). That is, \( \omega_f(x) = f(x) \) for all \( x \in Q \cap (0, 1) \).

**Remark:** (1) If we have proved \( f \) is continuous on \( Q^c \cap [0, 1] \), then \( f \) is automatically Riemann integrable on \([0, 1]\) since \( D(\subseteq Q \cap [0, 1] \subseteq Q) \), the set of discontinuities of \( f \) has measure zero.

(2) Here is a good exercise. We write it as a reference. Given a function \( f \) defined on \((a, b)\), then the set of continuities of \( f \) on \((a, b)\) is \( G_\delta \) set.

**Proof:** Let \( C \) denote the set of continuities of \( f \) on \((a, b)\), then

\[
C = \{ x : \omega_f(x) = 0 \} = \bigcap_{k=1}^\infty \{ x : \omega_f(x) < 1/k \}
\]

and \( \{ x : \omega_f(x) < 1/k \} \) is open. We know that \( C \) is a \( G_\delta \) set.

**Note:** (i) We call \( S \) a \( G_\delta \) set if \( S = \bigcap_{n=1}^\infty O_n \), where \( O_n \) is open for each \( n \).

(ii) Given \( y \in \{ x \in (a, b) : \omega_f(x) < 1/k \} := I \), then \( \omega_f(y) < 1/k \). Hence, there exists a \( d > 0 \), such that

\[
\Omega_f(B(y, d)) < 1/k, \text{ where } B(y, d) \subseteq (a, b)
\]

For \( z \in B(y, d) \), consider a smaller \( \delta \) so that \( B(z, \delta) \subseteq B(y, d) \). Hence,

\[
\Omega_f(B(z, \delta)) < 1/k
\]

which implies that

\[
\omega_f(z) < 1/k.
\]

Hence, \( B(y, d) \subseteq I \). That is, \( y \) is an interior point of \( I \). That is, \( I \) is open since every point of \( I \) is interior.

**7.29** Let \( f \) be defined as in Exercise 7.28 with \( G(n) = 1/n \). Let \( g(x) = 1 \) if \( 0 < x \leq 1 \), \( g(0) = 0 \). Show that the composite function \( h \) defined by \( h(x) = g[f(x)] \) is not Riemann-integrable on \([0, 1]\), although both \( f \in R \) and \( g \in R \) on \([0, 1]\).

**Proof:** By Exercise 7.28, we know that

\[
h(x) = \begin{cases} 
0 & \text{if } x \in Q^c \cap [0, 1] \\
1 & \text{if } x \in Q \cap [0, 1] 
\end{cases}
\]

which is discontinuous everywhere on \([0, 1]\). Hence, the function \( h \) (Dirichlet Function) is not Riemann-integrable on \([0, 1]\).

**7.30** Use Lebesgue’s theorem to prove Theorem 7.49.

(a) If \( f \) is of bounded variation on \([a, b]\), then \( f \in R \) on \([a, b]\).
Proof: Since \( f \) is of bounded variation on \([a,b]\), by Theorem 6.13 (Jordan Theorem), \( f = f_1 - f_2 \), where \( f_1 \) and \( f_2 \) are increasing on \([a,b]\). Let \( D_i \) denote the set of discontinuities of \( f_i \) on \([a,b]\), \( i = 1, 2 \). Hence, \( D \), the set of discontinuities of \( f \) on \([a,b]\) is \( D \subseteq D_1 \cup D_2 \).

Since \( |D_1| = |D_2| = 0 \), we know that \( |D| = 0 \). In addition, \( f \) is of bounded on \([a,b]\) since \( f \) is bounded variation on \([a,b]\). So, by Theorem 7.48, \( f \in R \) on \([a,b]\).

(b) If \( f \in R \) on \([a,b]\), then \( f \in R \) on \([c,d]\) for every subinterval \([c,d] \subseteq [a,b]\), \(|f| \in R \) and \( f^2 \in R \) on \([a,b]\). Also, \( f \cdot g \in R \) on \([a,b]\) whenever \( g \in R \) on \([a,b]\).

Proof: (i) Let \( D_{[a,b]} \) and \( D_{[c,d]} \) denote the set of discontinuities of \( f \) on \([a,b]\) and \([c,d]\), respectively. Then \( D_{[c,d]} \subseteq D_{[a,b]} \).

Since \( f \in R \) on \([a,b]\), and use Theorem 7.48, \( |D_{[a,b]}| = 0 \) which implies that \( |D_{[c,d]}| = 0 \).

In addition, since \( f \) is bounded on \([a,b]\), \( f \) is automatically is bounded on \([c,d]\) for every compact subinterval \([c,d]\). So, by Theorem 7.48, \( f \in R \) on \([c,d]\).

(ii) Let \( D_f \) and \( D_{|f|} \) denote the set of discontinuities of \( f \) and \(|f| \) on \([a,b]\), respectively, then \( D_{|f|} \subseteq D_f \).

Since \( f \in R \) on \([a,b]\), and use Theorem 7.48, \( |D_f| = 0 \) which implies that \( |D_{|f|}| = 0 \). In addition, since \( f \) is bounded on \([a,b]\), it is clear that \(|f| \) is bounded on \([a,b]\). So, by Theorem 7.48, \(|f| \in R \) on \([a,b]\).

(iii) Let \( D_f \) and \( D_{f^2} \) denote the set of discontinuities of \( f \) and \( f^2 \) on \([a,b]\), respectively, then \( D_{f^2} \subseteq D_f \).

Since \( f \in R \) on \([a,b]\), and use Theorem 7.48, \( |D_f| = 0 \) which implies that \( |D_{f^2}| = 0 \). In addition, since \( f \) is bounded on \([a,b]\), it is clear that \( f^2 \) is bounded on \([a,b]\). So, by Theorem 7.48, \( f^2 \in R \) on \([a,b]\).

(iv) Let \( D_f \) and \( D_g \) denote the set of discontinuities of \( f \) and \( g \) on \([a,b]\), respectively. Let \( D_{fg} \) denote the set of discontinuities of \( fg \) on \([a,b]\), then \( D_{fg} \subseteq D_f \cup D_g \).

Since \( f,g \in R \) on \([a,b]\), and use Theorem 7.48, \( |D_f| = |D_g| = 0 \) which implies that \( |D_{fg}| = 0 \). In addition, since \( f \) and \( g \) are bounded on \([a,b]\), it is clear that \( fg \) is bounded on \([a,b]\). So, by Theorem 7.48, \( fg \in R \) on \([a,b]\).

(c) If \( f \in R \) and \( g \in R \) on \([a,b]\), then \( fg \in R \) on \([a,b]\) whenever \( g \) is bounded away from 0.

Proof: Let \( D_f \) and \( D_g \) denote the set of discontinuities of \( f \) and \( g \) on \([a,b]\), respectively. Since \( g \) is bounded away from 0, we know that \( fg \) is well-defined and \( fg \) is also bounded on \([a,b]\). Consider \( D_{fg} \), the set of discontinuities of \( fg \) on \([a,b]\) is \( D_{fg} \subseteq D_f \cup D_g \).

Since \( f \in R \) and \( g \in R \) on \([a,b]\), by Theorem 7.48, \( |D_f| = |D_g| = 0 \) which implies that \( |D_{fg}| = 0 \). Since \( fg \) is bounded on \([a,b]\) with \( |D_{fg}| = 0 \), \( fg \in R \) on \([a,b]\) by Theorem 7.48.

Remark: The condition that the function \( g \) is bounded away from 0 CANNOT omit. For example, say \( g(x) = x \) on \((0,1]\) and \( g(0) = 1 \). Then it is clear that \( g \in R \) on \([0,1]\), but
1/g \not\in R on [0,1]. In addition, the reader should note that when we ask whether a function is Riemann-integrable or not, we always assume that f is **BOUNDED** on a **COMPACT INTERVAL** [a,b].

(d) If f and g are bounded functions having the same discontinuities on [a,b], then f \in R on [a,b] if, and only if, g \in R on [a,b].

**Proof:** (⇒) Suppose that f \in R on [a,b], then D_f, the set of discontinuities of f on [a,b] has measure zero by **Theorem 7.48**. From hypothesis, D_f = D_g, the set of discontinuities of g on [a,b], we know that g \in R on [a,b] by **Theorem 7.48**.

(⇐) If we change the roles of f and g, we have proved it.

(e) Let g \in R on [a,b] and assume that m \leq g(x) \leq M for all x \in [a,b]. If f is continuous on [m,M], the composite function h defined by h(x) = f[g(x)] is Riemann-integrable on [a,b].

**Proof:** Note that h is bounded on [a,b]. Let D_h and D_g denote the set of discontinuities of h and g on [a,b], respectively. Then

D_h \subseteq D_g.

Since g \in R on [a,b], then |D_g| = 0 by **Theorem 7.48**. Hence, |D_h| = 0 which implies that h \in R on [a,b] by **Theorem 7.48**.

**Remark:** (1) There is a more general theorem related with **Riemann-Stieltjes Integral**. We write it as a reference.

(2) **Theorem** Suppose g \in R(\alpha) on [a,b], m \leq g(x) \leq M for all x \in [a,b]. If f is continuous on [m,M], the composite function h defined by h(x) = f[g(x)] \in R(\alpha) on [a,b].

**Proof:** It suffices to consider the case that \alpha is increasing on [a,b]. If \alpha(a) = \alpha(b), there is nothing to prove. So, we assume that \alpha(a) < \alpha(b). In addition, let K = \sup_{x \in [a,b]} |h(x)|. We claim that h satisfies Riemann condition with respect to \alpha on [a,b]. That is, given \varepsilon > 0, we want to find a partition P such that

\[ U(P, h, \alpha) - L(P, h, \alpha) = \sum_{k=1}^{n} [M_k(h) - m_k(h)] \Delta \alpha_k < \varepsilon. \]

Since f is uniformly continuous on [m,M], for this \varepsilon > 0, there is a \left( \frac{\varepsilon}{2^{n+1}} \right) \delta > 0 such that as |x - y| < \delta where x, y \in [m,M], we have

\[ |f(x) - f(y)| < \frac{\varepsilon}{2(\alpha(b) - \alpha(a))}. \] 1

Since g \in R(\alpha) on [a,b], for this \delta > 0, there exists a partition P such that

\[ U(P, g, \alpha) - L(P, g, \alpha) = \sum_{k=1}^{n} [M_k(g) - m_k(g)] \Delta \alpha_k < \delta^2. \] 2

Let \(P = A \cup B\), where \(A = \{x_j : M_k(g) - m_k(g) < \delta\}\) and \(B = \{x_j : M_k(g) - m_k(g) \geq \delta\}\), then

\[ \delta \sum_B \Delta \alpha_k \leq \sum_B [M_k(h) - m_k(h)] \Delta \alpha_k \leq \delta^2 \text{ by (2)} \]

which implies that

\[ \sum_B \Delta \alpha_k \leq \delta. \]
So, we have

\[ U(A, h, \alpha) - L(A, h, \alpha) \leq \sum_A [M_k(h) - m_k(h)] \Delta \alpha_k \]

\[ \leq \varepsilon/2 \text{ by (1)} \]

and

\[ U(B, h, \alpha) - L(B, h, \alpha) \leq \sum_B [M_k(h) - m_k(h)] \Delta \alpha_k \]

\[ \leq 2K \sum_B \Delta \alpha_k \]

\[ \leq 2K\delta \]

\[ < \varepsilon/2. \]

It implies that

\[ U(P, h, \alpha) - L(P, h, \alpha) = U(A, h, \alpha) - L(A, h, \alpha) + U(B, h, \alpha) - L(B, h, \alpha) \]

\[ < \varepsilon. \]

That is, we have proved that \( h \) satisfies the Riemann condition with respect to \( \alpha \) on \([a, b]\). So, \( h \in R(\alpha) \) on \([a, b]\).

**Note:** We mention that if we change the roles of \( f \) and \( g \), then the conclusion does **NOT** hold. Since the counterexample is constructed by some conclusions that we will learn in Real Analysis, we do **NOT** give it a proof. Let \( C \) be the standard Cantor set in \([0, 1]\) and \( C' \) the Cantor set with positive measure in \([0, 1]\). Use similar method on defining Cantor Lebesgue Function, then there is a continuous function \( f : [0, 1] \rightarrow [0, 1] \) such that \( f(C) = C' \). And Choose \( g = X_C \) on \([0, 1]\). Then

\[ h = g \circ f = X_{C'}, \]

which is **NOT** Riemann integrable on \([0, 1]\).

(2) The reader should note the followings. Since these proofs use the **exercise 7.30** and **Theorem 7.49**, we omit it.

(i) If \( f \in R \) on \([a, b]\), then \( |f| \in R \) on \([a, b]\), and \( f^r \in R \) on \([a, b]\), where \( r \in [0, \infty) \).

(ii) If \( |f| \in R \) on \([a, b]\), it does **NOT** implies \( f \in R \) on \([a, b]\). And if \( f^2 \in R \) on \([a, b]\), it does **NOT** implies \( f \in R \) on \([a, b]\). For example,

\[ f = \begin{cases} 
1 & \text{if } x \in Q \cap [a, b] \\
-1 & \text{if } x \in Q^c \cap [a, b]
\end{cases} \]

(iii) If \( f^3 \in R \) on \([a, b]\), then \( f \in R \) on \([a, b]\).

7.31 Use Lebesgue’s theorem to prove that if \( f \in R \) and \( g \in R \) on \([a, b]\) and if \( f(x) \geq m > 0 \) for all \( x \) in \([a, b]\), then the function \( h \) defined by

\[ h(x) = f(x)^{g(x)} \]

is Riemann-integrable on \([a, b]\).

**Proof:** Consider

\[ h(x) = \exp(h \log f), \]

then by **Theorem 7.49**, \( f \in R \Rightarrow \log f \in R \Rightarrow h \log f \in R \Rightarrow \exp(h \log f) = h \in R. \)

7.32 Let \( I = [0, 1] \) and let \( A_1 = I - (\frac{1}{4}, \frac{2}{3}) \) be the subset of \( I \) obtained by removing
those points which lie in the open middle third of $I$; that is, $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Let $A_2$ be the subset of $A_1$ obtained by removing the open middle third of $[0, \frac{1}{3}]$ and of $[\frac{2}{3}, 1]$. Continue this process and define $A_3, A_4, \ldots$ The set $C = \cap_{n=1}^{\infty} A_n$ is called the **Cantor set**. Prove that

(a) $C$ is compact set having measure zero.

**Proof:** Write $C = \cap_{n=1}^{\infty} A_n$. Note that every $A_n$ is closed, so $C$ is closed. Since $A_1(\neq \phi)$ is closed and bounded, $A_1$ is compact and $C \subseteq A_1$, we know that $C$ is compact by **Theorem 3.39**.

In addition, it is clear that $|A_n| = \left(\frac{2}{3}\right)^n$ for each $n$. Hence, $|C| \leq \lim_{n \to \infty} |A_n| = 0$, which implies that $C$ has a measure zero.

(b) $x \in C$ if, and only if, $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where each $a_n$ is either 0 or 2.

**Proof:** ($\Rightarrow$) Let $x \in C = \cap_{n=1}^{\infty} A_n$, then $x \in A_n$ for all $n$. Consider the followings.

(i) Since $x \in A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, then it implies that $a_1 = 0$ or 2.

(ii) Since $x \in A_2 = ([0, \frac{1}{3}] \cup [\frac{2}{3}, 1]) \cup ([\frac{4}{9}, \frac{2}{3}] \cup [\frac{2}{3}, \frac{8}{9}])$, then it implies that $a_2 = 0$ or 2.

Inductively, we have $a_n = 0$ or 2. So, $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where each $a_n$ is either 0 or 2.

($\Leftarrow$) If $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where each $a_n$ is either 0 or 2, then it is clear that $x \in A_n$ for each $n$. Hence, $x \in C$.

(c) $C$ is uncountable.

**Proof:** Suppose that $C$ is countable, write $C = \{x_1, x_2, \ldots\}$. We consider unique ternary expansion: if $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, then $x := (a_1, \ldots, a_n, \ldots)$. From this definition, by (b), we have

$$x_k = (x_{k1}, x_{k2}, \ldots, x_{kk}, \ldots)$$

where each component is 0 or 2. Choose $y = (y_1, y_2, \ldots)$ where

$$y_j = \begin{cases} 
2 & \text{if } x_{jj} = 0 \\
0 & \text{if } x_{jj} = 2. 
\end{cases}$$

By (b), $y \in C$. It implies that $y = x_k$ for some $k$ which contradicts to the choice of $y$. Hence, $C$ is uncountable.

**Remark:** (1) In fact, $C = C'$ means that $C$ is a perfect set. Hence, $C$ is uncountable. The reader can see the book, *Principles of Mathematical Analysis* by Walter Rudin, pp 41-42.

(2) Let $C = \{x : x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where each $a_n$ is either 0 or 2)$. Define a new function $\phi : C \to [0, 1]$ by

$$\phi(x) = \sum_{n=1}^{\infty} \frac{(a_n)/2}{2^n},$$

then it is clear that $\phi$ is 1-1 and onto. So, $C$ is equivalent to $[0, 1]$. That is, $C$ is uncountable.

(d) Let $f(x) = 1$ if $x \in C$, $f(x) = 0$ if $x \notin C$. Prove that $f \in R$ on $[0, 1]$.

**Proof:** In order to show that $f \in R$ on $[0, 1]$, it suffices to show that, by **Theorem 7.48**, $f$ is continuous on $[0, 1] - C$ since it implies that $D \subseteq C$, where $D$ is the set of discontinuities of $f$ on $[0, 1]$. Let $x_0 \in [0, 1] - C$, and note that $C = C'$, so there is a $\delta > 0$ such that
(x_0 - \delta, x_0 + \delta) \cap C = \phi$, where $(x_0 - \delta, x_0 + \delta) \subseteq [0, 1]$. Then given $\varepsilon > 0$, there is a 
$\delta > 0$ such that as $x \in (x_0 - \delta, x_0 + \delta)$, we have 
$$|f(x) - f(x_0)| = 0 < \varepsilon.$$ 

Remark: (1) $C = C'$ :Given $x \in C = \cap_{n=1}^{\infty} A_n$, and note that every endpoints of $A_n$ 
belong to $C$. So, $x$ is an accumulation point of the set $\{y : y$ is the endpoints of $A_n\}$. So, 
$C \subseteq C'$. In addition, $C' \subseteq C$ since $C$ is closed. Hence, $C = C'$. 

(2) In fact, we have 
$f$ is continuous on $[0, 1] - C$ and $f$ is not continuous on $C$. 

Proof: In (d), we have proved that $f$ is continuous on $[0, 1] - C$, so it remains to show that $f$ is not continuous on $C$. Let $x_0 \in C$, if $f$ is continuous at $x_0$, then given $\varepsilon = 1/2$, there is a 
$\delta > 0$ such that as $x \in (x_0 - \delta, x_0 + \delta) \cap [0, 1]$, we have 
$$|f(x) - f(x_0)| < 1/2$$ 
which is absurd since we can choose $y \in (x_0 - \delta, x_0 + \delta) \cap [0, 1]$ and $y \not\in C$ by the fact $C$ 
does NOT contain an open interval since $C$ has measure zero. So, we have proved that $f$ is 
not continuous on $C$.

Note: In a metric space $M$, a set $S(\subseteq M)$ is called nowhere dense if $int(cl(S)) = \phi$. 
Hence, we know that $C$ is a nowhere dense set. 

Supplement on Cantor set.

From the exercise 7.32, we have learned what the Cantor set is. We write some 
conclusions as a reference.

(1) The Cantor set $C$ is compact and perfect.
(2) The Cantor set $C$ is uncountable. In fact, $\#(C) = \#(R)$.
(3) The Cantor set $C$ has measure zero.
(4) The Cantor set $C$ is nowhere dense.
(5) Every point $x$ in $C$ can be expressed as $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where each $a_n$ is either 0 or 
2.
(6) $X_C : [0, 1] \to \{0, 1\}$ the characteristic function of $C$ on $[0, 1]$ is Riemann integrable.

The reader should be noted that Cantor set $C$ in the exercise is 1 –dimensional case. We 
can use the same method to construct a $n$ –dimensional Cantor set in the set 
$\{(x_1, \ldots, x_n) : 0 \leq x_j \leq 1, j = 1, 2, \ldots n\}$. In addition, there are many researches on Cantor 
set. For example, we will learn so called Space-Filling Curve on the textbook, Ch9, pp 
224-225.

In addition, there is an important function called Cantor-Lebesgue Function related 
with Cantor set. The reader can see the book, Measure and Integral (An Introduction to 
Real Analysis) written by Richard L. Wheeden and Antoni Zygmund, pp 35.

7.33 The exercise outlines a proof (due to Ivan Niven) that $\pi^2$ is irrational. Let 
$f(x) = x^n(1-x)^n/n!$. Prove that:
(a) $0 < f(x) < 1/n!$ if $0 < x < 1$.

Proof: It is clear.
(b) Each $4\text{th}$ derivative $f^{(4)}(0)$ and $f^{(4)}(1)$ is an integer.

Proof: By Leibnitz Rule,
\[ f^{(k)}(x) = \frac{1}{n!} \sum_{j=0}^{k} \binom{k}{j} \{n \cdots (n-j)(n-j+1)\} \{(-1)^{k-j} [n \cdots (n-k+j+1)(1-x)^{n-k+j}] \} \]

which implies that

\[ f^{(k)}(0) = \begin{cases} 
0 & \text{if } k < n \\
1 & \text{if } k = n \\
\left(\frac{1}{n!}\right) (-1)^{k-n} [n \cdots (2n-k+1)] & \text{if } k > n 
\end{cases}. \]

So, \( f^{(k)}(0) \in \mathbb{Z} \) for each \( k \in \mathbb{N} \). Similarly, \( f^{(k)}(1) \in \mathbb{Z} \) for each \( k \in \mathbb{Z} \).

Now assume that \( \pi^2 = a/b \), where \( a \) and \( b \) are positive integers, and let

\[ F(x) = b^n \sum_{k=0}^{n} (-1)^k f^{(2k)}(x) x^{2n-2k}. \]

\textbf{Proof:} By (b), it is clear.

(c) \( F(0) \) and \( F(1) \) are integers.

\textbf{Proof:} By (b), it is clear.

(d) \( \pi^2 a^n f(x) \sin \pi x = \frac{d}{dx} \{ F'(x) \sin \pi x - \pi F(x) \cos \pi x \} \)

\textbf{Proof:} Note that

\[ F''(x) + \pi^2 F(x) = b^n \sum_{k=0}^{n} (-1)^k f^{(2k+2)}(x) x^{2n-2k} + \pi^2 b^n \sum_{k=0}^{n} (-1)^k f^{(2k)}(x) x^{2n-2k} \]

\[ = b^n \sum_{k=0}^{n} (-1)^k f^{(2k+2)}(x) x^{2n-2k} + b^n (-1)^n f^{(2n+2)}(x) \]

\[ + \pi^2 b^n \sum_{k=1}^{n} (-1)^k f^{(2k)}(x) x^{2n-2k} + \pi^2 b^n f(x) x^{2n} \]

\[ = b^n \sum_{k=0}^{n} (-1)^k f^{(2k+2)}(x) x^{2n-2k} + b^n (-1)^n f^{(2n+2)}(x) + \pi^2 b^n f(x) x^{2n} \]

\[ = \pi^2 a^n f(x) \text{ since } f \text{ is a polynomial of degree } 2n. \]

So,

\[ \frac{d}{dx} \{ F'(x) \sin \pi x - \pi F(x) \cos \pi x \} = (\sin \pi x)[F''(x) + \pi^2 F(x)] \]

\[ = \pi^2 a^n f(x) \sin \pi x. \]

(e) \( F(1) + F(0) = \pi a^n \int_0^1 f(x) \sin \pi x dx \).

\textbf{Proof:} By (d), we have

\[ \pi^2 a^n \int_0^1 f(x) \sin \pi x dx = F'(x) \sin \pi x - \pi F(x) \cos \pi x \bigg|_0^1 \]

\[ = [F'(1) \sin \pi - \pi F(1) \cos \pi] - [F'(0) \sin 0 - \pi F(0) \cos 0] \]

\[ = \pi [F(1) + F(0)] \]

which implies that
\[ F(1) + F(0) = \pi a^n \int_0^1 f(x) \sin \pi x dx. \]

(f) Use (a) in (e) to deduce that \( 0 < F(1) + F(0) < 1 \) if \( n \) is sufficiently large. This contradicts (c) and show that \( \pi^2 \) (and hence \( \pi \)) is irrational.

**Proof:** By (a), and \( \sin x \in [0, 1] \) on \([0, \pi]\), we have

\[
0 < \pi a^n \int_0^1 f(x) \sin \pi x dx \leq \frac{\pi a^n}{n!} \int_0^1 \sin \pi x dx = \frac{2a^n}{n!} \to 0 \quad \text{as} \quad n \to \infty.
\]

So, as \( n \) is sufficiently large, we have, by (d),

\[
0 < F(1) + F(0) < 1
\]

which contradicts (c). So, we have proved that \( \pi^2 \) (and hence \( \pi \)) is irrational.

**Remark:** The reader should know that \( \pi \) is a transcendental number. (Also, so is \( e \).) It is well-known that a transcendental number must be an irrational number.

In 1900, David Hilbert asked 23 problems, the 7th problem is that, if \( a\to 0, 1 \) is an algebraic number and \( \beta \) is an algebraic number but not rational, then is it true that \( a^\beta \) is a transcendental number. The problem is completely solved by Israil Moiseevic Gelfand in 1934. There are many open problem now on algebraic and transcendental numbers. For example, It is an open problem: Is the Euler Constant

\[
\gamma = \lim \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \right)
\]

a transcendental number.

7.34 Given a real-valued function \( \alpha \), continuous on the interval \([a, b]\) and having a finite bounded derivative \( \alpha' \) on \((a, b)\). Let \( f \) be defined and bounded on \([a, b]\) and assume that both integrals

\[
\int_a^b f(x) \, d\alpha(x) \quad \text{and} \quad \int_a^b f(x) \alpha'(x) \, dx
\]

exists. Prove that these integrals are equal. (It is not assumed that \( \alpha' \) is continuous.)

**Proof:** Since both integrals exist, given \( \varepsilon > 0 \), there exists a partition

\( P = \{x_0 = a, \ldots, x_n = b\} \)

such that

\[
\left| S(P, f, \alpha) - \int_a^b f(x) \, d\alpha(x) \right| < \varepsilon / 2
\]

where

\[
S(P, f, \alpha) = \sum_{j=1}^{n} f(t_j) \Delta \alpha_j \quad \text{for} \quad t_j \in [x_{j-1}, x_j]
\]

\[
= \sum_{j=1}^{n} f(t_j) \alpha'(s_j) \Delta x_j \quad \text{by Mean Value Theorem, where} \quad s_j \in (x_{j-1}, x_j)
\]

and

\[
\left| S(P, f\alpha') - \int_a^b f(x) \alpha'(x) \, dx \right| < \varepsilon / 2
\]

where

\[
S(P, f\alpha') = \sum_{j=1}^{n} f(t_j) \alpha'(t_j) \Delta x_j \quad \text{for} \quad t_j \in [x_{j-1}, x_j]
\]

So, let \( t_j = s_j \), then we have
\[ S(P, f, a) = S(P, fa') \].

Hence,
\[
\left| \int_a^b f(x) \, dx - \int_a^b f(x) \, dx \right| 
\leq \left| S(P, f, a) - \int_a^b f(x) \, dx \right| + \left| S(P, fa') - \int_a^b f(x) \, dx \right| 
< \varepsilon.
\]

So, we have proved that both integrals are equal.

**7.35** Prove the following theorem, which implies that a function with a positive integral must itself be positive on some interval. Assume that \( f \in R \) on \([a, b]\) and that \( 0 \leq f(x) \leq M \) on \([a, b]\), where \( M > 0 \). Let \( I = \int_a^b f(x) \, dx \), let \( h = \frac{1}{2} I/(M + b - a) \), and assume that \( I > 0 \). Then the set \( T = \{ x : f(x) \geq h \} \) contains a finite number of intervals, the sum of whose lengths is at least \( h \).

Hint. Let \( P \) be a partition of \([a, b]\) such that every Riemann sum \( S(P, f) = \sum_{k=1}^{n} f(t_k) \Delta x_k \) satisfies \( S(P, f) > I/2 \). Split \( S(P, f) \) into two parts, \( S(P, f) = \sum_{k \in A} + \sum_{k \in B} \), where
\[ A = \{ k : [x_{k-1}, x_k] \subseteq T \}, \quad B = \{ k : k \notin A \}. \]
If \( k \in A \), use the inequality \( f(t_k) \leq M \); if \( k \in B \), choose \( t_k \) so that \( f(t_k) < h \). Deduce that \( \sum_{k \in A} \Delta x_k > h \).

**Proof:** It is clear by Hint, so we omit the proof.

**Remark:** There is another proof about that a function with a positive integral must itself be positive on some interval.

**Proof:** Suppose NOT, it means that in every subinterval, there is a point \( p \) such that \( f(p) \leq 0 \). So,
\[ L(P, f) = \sum_{j=1}^{n} m_j \Delta x_j \leq 0 \text{ since } m_j \leq 0 \]
for any partition \( P \). Then it implies that
\[ \sup_P L(P, f) = \int_a^b f(x) \, dx \leq 0 \]
which contradicts to a function with a positive integral. Hence, we have proved that a function with a positive integral must itself be positive on some interval.

Supplement on integration of vector-valued functions.

**(Definition)** Given \( f_1, \ldots, f_n \) real valued functions defined on \([a, b]\), and let \( f = (f_1, \ldots, f_n) : [a, b] \to R^n \). If \( \alpha \not\parallel \) on \([a, b]\). We say that \( f \in R(\alpha) \) on \([a, b]\) means that \( f_j \in R(\alpha) \) on \([a, b]\) for \( j = 1, 2, \ldots, n \). If this is the case, we define
\[ \int_a^b f \, da = \left( \int_a^b f_1 \, da, \ldots, \int_a^b f_n \, da \right). \]

From the definition, the reader should find that the definition is NOT stranger for us. When we talk \( f = (f_1, \ldots, f_n) \in R(\alpha) \) on \([a, b]\), it suffices to consider each \( f_j \in R(\alpha) \) on \([a, b]\) for \( j = 1, 2, \ldots, n \).

For example, if \( f \in R(\alpha) \) on \([a, b]\) where \( \alpha \not\parallel \) on \([a, b]\), then \( \|f\| \in R(\alpha) \) on \([a, b]\).

**Proof:** Since \( f \in R(\alpha) \) on \([a, b]\), we know that \( f_j \in R(\alpha) \) on \([a, b]\) for \( j = 1, 2, \ldots, n \). Hence,
\[
\sum_{k=1}^{n} f_j^2 \in R(a) \text{ on } [a, b]
\]
which implies that, by Remark (1) in Exercise 7.30,
\[
\|f\| = \left(\sum_{k=1}^{n} f_j^2 \in R(a) \text{ on } [a, b]\right)^{1/2} \in R(a) \text{ on } [a, b].
\]

**Remark:** In the case above, we have
\[
f_j \leq \frac{1}{2}\|f\| \text{ on } [a, b].
\]

**Proof:** Consider
\[
\|y\|^2 = \left< \int_a^b f_j da, \ldots, \int_a^b f_n da, \int_a^b f_1 da, \ldots, \int_a^b f_n da \right>
= \sum_{j=1}^{n} \left(\int_a^b f_j da\right)\left(\int_a^b f_j da\right),
\]
which implies that, (let \(y_j = \int_a^b f_j da, \ y = (y_1, \ldots, y_n)\)),
\[
\|y\|^2 = \sum_{j=1}^{n} y_j \left(\int_a^b f_j da\right)
= \sum_{j=1}^{n} \int_a^b f_j y_j da
= \int_a^b \left(\sum_{j=1}^{n} f_j y_j\right) da
\leq \int_a^b \|f\| \|y\| da
= \|y\| \int_a^b \|f\| da
\]
which implies that
\[
\|y\| \leq \int_a^b \|f\| da.
\]

**Note:** The equality holds if, and only if, \(f(t) = k(t)y\).

Existence theorems for integral and differential equations

The following exercises illustrate how the fixed-point theorem for contractions, (Theorem 4.48) is used to prove existence theorems for solutions of certain integral and differential equations. We denote by \(C[a, b]\) the metric space of all real continuous functions on \([a, b]\) with the metric
\[
d(f, g) = \|f - g\| = \max_{x \in [a, b]} |f(x) - g(x)|,
\]
and recall the \(C[a, b]\) is a complete metric space.

7.36 Given a function \(g\) in \(C[a, b]\), and a function \(K\) is continuous on the rectangle \(Q = [a, b] \times [a, b]\), consider the function \(T\) defined on \(C[a, b]\) by the equation
\[
T(\phi)(x) = g(x) + \lambda \int_a^b K(x, t) \phi(t) dt,
\]
where $\lambda$ is a given constant.

(a) Prove that $T$ maps $C[a,b]$ into itself.

**Proof:** Since $K$ is continuous on the rectangle $Q = [a, b] \times [a, b]$, and $\varphi(x) \in C[a,b]$, we know that

$$\int_a^b K(x, t)\varphi(t)dt \in C[a,b].$$

Hence, we prove that $T(\varphi)(x) \in C[a,b]$. That is, $T$ maps $C[a,b]$ into itself.

(b) If $|K(x, y)| \leq M$ on $Q$, where $M > 0$, and if $|\lambda| < M^{-1}(b-a)^{-1}$, prove that $T$ is a contraction of $C[a,b]$ and hence has a fixed point $\varphi$ which is a solution of the integral equation

$$\varphi(x) = g(x) + \lambda \int_a^b K(x, t)\varphi(t)dt.$$

**Proof:** Consider

$$\|T(\varphi_1)(x) - T(\varphi_2)(x)\| = \left\| \lambda \int_a^b K(x, t)[\varphi_1(t) - \varphi_2(t)]dt \right\| \leq |\lambda| \int_a^b |K(x, t)| |\varphi_1(t) - \varphi_2(t)|dt \leq |\lambda|M \int_a^b |\varphi_1(t) - \varphi_2(t)|dt \leq |\lambda|M(b-a)\|\varphi_1(t) - \varphi_2(t)\|.$$

Since $|\lambda| < M^{-1}(b-a)^{-1}$, then there exists $c$ such that $|\lambda| < c < M^{-1}(b-a)^{-1}$. Hence, by (*), we know that

$$\|T(\varphi_1)(x) - T(\varphi_2)(x)\| < c \|\varphi_1(t) - \varphi_2(t)\|$$

where $0 < cM(b-a) := \gamma < 1$. So, $T$ is a contraction of $C[a,b]$ and hence has a fixed point $\varphi$ which is a solution of the integral equation $\varphi(x) = g(x) + \lambda \int_a^b K(x, t)\varphi(t)dt$.

7.37 Assume $f$ is continuous on a rectangle $Q = [a-h, a+h] \times [b-k, b+k]$, where $h > 0, k > 0$.

(a) Let $\varphi$ be a function, continuous on $[a-h, a+h]$, such that $(x, \varphi(x)) \in Q$ for all $x$ in $[a-h, a+h]$. If $0 < c \leq h$, prove that $\varphi$ satisfies the differential equation $y' = f(x, y)$ on $(a-c, a+c)$ and the initial condition $\varphi(a) = b$ if, and only if, $\varphi$ satisfies the integral equation

$$\varphi(x) = b + \int_a^x f(t, \varphi(t))dt \text{ on } (a-c, a+c).$$

**Proof:** ($\Rightarrow$) Since $\varphi'(t) = f(t, \varphi(t))$ on $(a-c, a+c)$ and $\varphi(a) = b$, we have, $x \in (a-c, a+c)$

$$\varphi(x) = \varphi(a) + \int_a^x \varphi'(t)dt = \varphi(a) + \int_a^x f(t, \varphi(t))dt \text{ on } (a-c, a+c).$$

($\Leftarrow$) Assume

$$\varphi(x) = b + \int_a^x f(t, \varphi(t))dt \text{ on } (a-c, a+c),$$

then

$$\varphi'(x) = f(t, \varphi(x)) \text{ on } (a-c, a+c).$$

(b) Assume that $|f(x, y)| \leq M$ on $Q$, where $M > 0$, and let $c = \min \{h, k/M\}$. Let $S$
denote the metric subspace of $C[a - c, a + c]$ consisting of all $\varphi$ such that $|\varphi(x) - b| \leq Mc$ on $[a - c, a + c]$. Prove that $S$ is closed subspace of $C[a - c, a + c]$ and hence that $S$ is itself a complete metric space.

**Proof:** Since $C[a - c, a + c]$ is complete, if we can show that $S$ is closed, then $S$ is complete. Hence, it remains to show that $S$ is closed.

Given $f \in S'$, then there exists a sequence of functions $\{f_n\}$ such that $f_n \to f$ under the sup norm $\|\cdot\|$. So, given $\varepsilon > 0$, there exists a positive integer $N$ such that as $n \geq N$, we have

$$\max_{x \in [a - c, a + c]} |f_n(x) - f(x)| < \varepsilon.$$ 

Consider

$$|f(x) - b| \leq |f(x) - f_N(x)| + |f_N(x) - b|$$

$$\leq \|f(x) - f_N(x)\| + \|f_N(x) - b\|$$

$$< \varepsilon + Mc$$

which implies that

$$|f(x) - b| \leq Mc$$

for all $x$ since $\varepsilon$ is arbitrary. So, $f \in S$. It means that $S$ is closed.

(c) Prove that the function $T$ defined on $S$ by the equation

$$T(\varphi)(x) = b + \int_a^x f(t, \varphi(t))dt$$

maps $S$ into itself.

**Proof:** Since

$$|T(\varphi)(x) - b| = \left|\int_a^x f(t, \varphi(t))dt\right|$$

$$\leq \int_a^x |f(t, \varphi(t))|dt$$

$$\leq (x - a)M$$

$$\leq Mc$$

we know that $T(\varphi)(x) \in S$. That is, $T$ maps $S$ into itself.

(d) Now assume that $f$ satisfies a Lipschitz condition of the form

$$|f(x, y) - f(x, z)| \leq A|y - z|$$

for every pair of points $(x, y)$ and $(x, z)$ in $Q$, where $A > 0$. Prove that $T$ is a contraction of $S$ if $h < 1/A$. Deduce that for $h < 1/A$ the differential equation $y' = f(x, y)$ has exactly one solution $y = \varphi(x)$ on $(a - c, a + c)$ such that $\varphi(a) = b$.

**Proof:** Note that $h < 1/A$, there exists $\lambda$ such that $h < \lambda < 1/A$. Since

$$\|T(\varphi_1)(x) - T(\varphi_2)(x)\|$$

$$\leq \int_a^x |f(t, \varphi_1(t)) - f(t, \varphi_2(t))|dt$$

$$\leq A \int_a^x |\varphi_1(t) - \varphi_2(t)|dt$$

by $|f(x, y) - f(x, z)| \leq A|y - z|$,

$$\leq A(x - a)\|\varphi_1(t) - \varphi_2(t)\|$$

$$\leq Ah\|\varphi_1(t) - \varphi_2(t)\|$$

$$< \gamma\|\varphi_1(t) - \varphi_2(t)\|$$

where $0 < \lambda A := \gamma < 1$. Hence, $T$ is a contraction of $S$. It implies that there exists one and
only one \( \varphi \in S \) such that
\[
\varphi(x) = b + \int_a^x f(t, \varphi(t))dt
\]
which implies that
\[
\varphi'(x) = f(x, \varphi(x)).
\]
That is, the differential equation \( y' = f(x, y) \) has exactly one solution \( y = \varphi(x) \) on \((a - c, a + c)\) such that \( \varphi(a) = b \).

**Supplement on Riemann Integrals**

1. The reader should be noted that the metric space \((R([a, b]), d)\) is **NOT** complete, where
\[
d(f, g) = \int_a^b |f(x) - g(x)|dx.
\]
We do **NOT** give it a proof. The reader can see the book, *Measure and Integral (An Introduction to Real Analysis)* by Richard L. Wheeden and Antoni Zygmund, Ch5.

2. The reader may recall the **Mean Value Theorem**: Let \( f \) be a continuous function on \([a, b]\). Then
\[
\int_a^b f(x)dx = f(x_0)(b - a)
\]
where \( x_0 \in [a, b] \). In fact, the point \( x_0 \) can be chosen to be interior of \([a, b]\). That is, \( x_0 \in (a, b) \).

**Proof**: Let \( M = \sup_{x \in [a, b]} f(x) \), and \( m = \inf_{x \in [a, b]} f(x) \). If \( M = m \), then it is clear. So, we may assume that \( M \neq m \) as follows. Suppose **NOT**, it means that \( x_0 = a \) or \( b \). Note that,
\[
f(x_1) = m \leq f(x_0) := r \leq M = f(x_2)
\]
by continuity of \( f \) on \([a, b]\). Then we claim that
\[
f(x_0) = m \text{ or } M.
\]
If **NOT**, i.e.,
\[
f(x_1) < r < f(x_2)
\]
it means that there exists a point \( p \in (x_1, x_2) \) such that \( f(p) = r \) by **Intermediate Value Theorem**. It contradicts to \( p = a \) or \( b \). So, we have proved the claim. If \( f(a) = m \), then
\[
\int_a^b f(x)dx = m(b - a) \Rightarrow 0 = \int_a^b (f(x) - m)dx
\]
which implies that, by \( f(x) - m \geq 0 \) on \([a, b]\],
\[
f(x) = m \text{ for all } x \in [a, b].
\]
So, it is impossible. Similarly for other cases.

**Remark**: (1) The reader can give it a try to consider the Riemann-Stieltjes Integral as follows. Let \( \alpha \) be a continuous and increasing function on \([a, b]\). If \( f \) is continuous on \([a, b]\), then
\[
\int_a^b f(x)d\alpha(x) = f(c)[\alpha(b) - \alpha(a)]
\]
where \( c \in (a, b) \).

**Note**: We do **NOT** omit the continuity of \( \alpha \) on \([a, b]\) since
\( f(x) = x \) on \([0, 1]\); \( \alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in (0, 1) \end{cases} \).

(2) The reader can see the textbook, exercise 14.13 pp 404.

**Exercise:** Show that
\[
\frac{\pi}{2} < \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \frac{1}{2}\sin^2 x}} < \frac{\pi}{\sqrt{2}}.
\]

**Proof:** It is clear by the choice of \( x_0 \in (0, \pi/2) \).

3. Application on Integration by parts for Riemann-integrable function. It is well-known that
\[
\int f(x)dx = xf(x) - \int xdf(x).
\]
If \( f(x) \) has the inverse function \( g(y) = x \), then (*) implies that
\[
\int f(x)dx = xf(x) - \int g(y)dy.
\]
For example,
\[
\int \arcsin xdx = x\arcsin x - \int \sin ydy.
\]

4. Here is an observation on Series, Differentiation and Integration. We write it as a table to make the reader think it twice.

| Series | Summation by parts | Cesaro Sum | ? |
|--------|--------------------|------------|
| Differentiation | \((fg)' = f'g + fg'\) | Mean Value Theorem | Chain Rule |
| Integration | Integration by parts | Mean Value Theorem | Change of Variable |