The Riemann-Stieltjes Integral

Riemann-Stieltjes integrals

7.1 Prove that $\int_a^b d\alpha = \alpha(b) - \alpha(a)$, directly from Definition 7.1.

Proof: Let $f = 1$ on $[a,b]$, then given any partition $P = \{a = x_0, \ldots, x_n = b\}$, then we have

$$\begin{aligned} S(P,1,\alpha) &= \sum_{k=1}^{n} f(t_k)\Delta\alpha_k, \text{ where } t_k \in [x_{k-1}, x_k] \\ &= \sum_{k=1}^{n} \Delta\alpha_k \\ &= \alpha(b) - \alpha(a). \end{aligned}$$

So, we know that $\int_a^b d\alpha = \alpha(b) - \alpha(a)$.

7.2 If $f \in R(\alpha)$ on $[a,b]$ and if $\int_a^b f d\alpha = 0$ for every f which is monotonic on $[a,b]$, prove that α must be constant on $[a,b]$.

Proof: Use integration by parts, and thus we have

$$\int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a)$$

Given any point $c \in [a,b)$, we may choose a monotonic function f defined as follows.

$$f = \begin{cases} 0 \text{ if } x \leq c \\ 1 \text{ if } x > c. \end{cases}$$

So, we have

$$\int_a^b \alpha df = \alpha(c) = \alpha(b).$$

So, we know that α is constant on $[a,b]$.

7.3 The following definition of a Riemann-Stieltjes integral is often used in the literature: We say that f is integrable with respect to α if there exists a real number A having the property that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every partition P of $[a,b]$ with norm $\|P\| < \delta$ and for every choice of t_k in $[x_{k-1}, x_k]$, we have $|S(P,f,\alpha) - A| < \varepsilon$.

(a) Show that if $\int_a^b f d\alpha$ exists according to this definition, then it is also exists according to Definition 7.1 and the two integrals are equal.

Proof: Since refinement will decrease the norm, we know that if there exists a real number A having the property that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every partition P of $[a,b]$ with norm $\|P\| < \delta$ and for every choice of t_k in $[x_{k-1}, x_k]$, we have $|S(P,f,\alpha) - A| < \varepsilon$. Then choosing a P_ε with $\|P_\varepsilon\| < \delta$, then for $P \subseteq P_\varepsilon \Rightarrow \|P\| < \delta$. So, we have

$$|S(P,f,\alpha) - A| < \varepsilon.$$

That is, $\int_a^b f d\alpha$ exists according to this definition, then it is also exists according to Definition 7.1 and the two integrals are equal.

(b) Let $f(x) = \alpha(x) = 0$ for $a \le x < c$, $f(x) = \alpha(x) = 1$ for $c < x \le b$, $f(c) = 0, \alpha(c) = 1$. Show that $\int_a^b f d\alpha$ exists according to Definition 7.1 but does not exist by this second definition.

Proof: Note that $\int_a^b f d\alpha$ exists and equals 0 according to Definition 7.1If $\int_a^b f d\alpha$ exists according to this definition, then given $\varepsilon = 1$, there exists a $\delta > 0$ such that for every partition P of $[a,b]$ with norm $\|P\| < \delta$ and for every choice of t_k in $[x_{k-1}, x_k]$, we have $|S(P,f,\alpha)| < 1$. We may choose a partition $P = \{a = x_0, \ldots, x_n = b\}$ with $\|P\| < \delta$ and $c \in (x_j, x_{j+1})$, where $j = 0, \ldots, n-1$. Then

$$S(P,f,\alpha) = f(x)[\alpha(x_{j+1}) - \alpha(x_j)] = 1, \text{ where } x \in (c, x_{j+1})$$

which contradicts to $|S(P,f,\alpha)| < 1$.

7.4 If $f \in R$ according to Definition 7.1, prove that $\int_a^b f(x)dx$ also exists according to definition of Exercise 7.3. [Contrast with Exercise 7.3 (b).]

Hint: Let $I = \int_a^b f(x)dx$, $M = \sup\{|f(x)| : x \in [a,b].\}$ Given $\varepsilon > 0$, choose P_ε so that $U(P_\varepsilon, f) < I + \varepsilon/2$ (notation of section 7.11). Let N be the number of subdivision points in P_ε and let $\delta = \varepsilon/(2MN)$. If $\|P\| < \delta$, write

$$U(P,f) = \sum M_k(f)\Delta x_k = S_1 + S_2,$$

where S_1 is the sum of terms arising from those subintervals of P containing no points of P_ε and S_2 is the sum of remaining terms. Then

$$S_1 \le U(P_\varepsilon, f) < I + \varepsilon/2 \text{ and } S_2 \le NM\|P\| < NM\delta = \varepsilon/2,$$

and hence $U(P_\varepsilon, f) < I + \varepsilon$. Similarly,

$$L(P,f) > I - \varepsilon \text{ if } \|P\| < \delta' \text{ for some } \delta'.$$

Hence $|S(P,f) - I| < \varepsilon$ if $\|P\| < \min(\delta, \delta')$.

Proof: The hint has proved it.

Remark: There are some exercises related with Riemann integrals, we write thme as references.

(1) Suppose that $f \ge 0$ and f is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ on $[a,b]$.

Proof: Assume that there is a point $c \in [a,b]$ such that $f(c) > 0$. Then by continuity of f, we know that given $\varepsilon = \frac{f(c)}{2} > 0$, there is a $\delta > 0$ such that as $|x - c| < \delta$, $x \in [a,b]$, we have

$$|f(x) - f(c)| < \frac{f(c)}{2}$$

which implies that

$$\frac{f(c)}{2} < f(x) \text{ if } x \in (c - \delta, c + \delta) \cap [a,b] := I$$

So, we have

$$0 < \frac{f(c)}{2}|I| \le \int_I f(x)dx \le \int_a^b f(x)dx = 0, \text{ where } 0 < |I|, \text{ the length of } I$$

which is absurb. Hence, we obtain that $f(x) = 0$ on $[a,b]$.

(2) Let f be a continuous function defined on $[a,b]$. Suppose that for every continuous function g defined on $[a,b]$ which satisfies that

$$\int_a^b g(x)dx = 0,$$

we always have

$$\int_a^b f(x)g(x)dx = 0.$$

Show that f is a constant function on $[a,b]$.

Proof: Let $\int_a^b f(x)dx = I$, and define $g(x) = f(x) - \frac{I}{b-a}$, then we have

$$\int_a^b g(x)dx = 0,$$

which implies that, by hypothesis,

$$\int_a^b f(x)g(x)dx = 0$$

which implies that

$$\int_a^b (f(x) - c)g(x)dx = 0 \text{ for any real } c.$$

So, we have

$$\int_a^b (g(x))^2 dx = 0 \text{ if letting } c = \frac{I}{b-a}$$

which implies that $g(x) = 0$ forall $x \in [a,b]$ by (1). That is, $f(x) = \frac{I}{b-a}$ on $[a,b]$.

(3) Define

$$h(x) = \begin{cases} 0 \text{ if } x \in [0,1] - Q \\ \frac{1}{n} \text{ if } x \text{ is the rational number } m/n \text{ (in lowest terms)} \\ 1 \text{ if } x = 0. \end{cases}$$

Then $h \in R([0,1])$.

Proof: Note that we have shown that h is continuous only at irrational numbers on $[0,1] - Q$. We use it to show that h is Riemann integrable, i.e., $h \in R([0,1])$. Consider the upper sum $U(P,f)$ as follows.

Given $\varepsilon > 0$, there exists finitely many points x such that $f(x) \geq \varepsilon/2$. Consider a partition $P_\varepsilon = \{x_0 = a, \ldots, x_n = b\}$ so that its subintervals $I_j = [x_{j-1}, x_j]$ for some j containing those points and $\sum |I_j| < \varepsilon/2$. So, we have

$$\begin{aligned} U(P,f) &= \sum_{k=1}^{n} M_k \Delta x_k \\ &= \sum_1 + \sum_2 \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

where $\sum_1 = \sum_1 M_j I_j$, and $\sum_2$, is the sum of others.

So, we have shown that f satisfies the Riemann condition with respect to $\alpha(x) = x$.

Note: (1) The reader can show this by **Theorem 7.48 (Lebesgue's Criterion for Riemann Integrability).** Also, compare **Exercise 7.32** and **Exercise 4.16** with this.

(2) In **Theorem 7.19**, if we can make sure that there is a partition P_ε such that

$$U(P_\varepsilon,f,\alpha) - L(P_\varepsilon,f,\alpha) < \varepsilon,$$

then we automatically have, for any finer $P(\subseteq P_\varepsilon)$,

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

since the refinement makes U increase and L decrease.

(4) Assume that the function $f(x)$ is differentiable on $[a,b]$, but not a constant and that $f(a) = f(b) = 0$. Then there exists at least one point ξ on (a,b) for which

$$|f'(\xi)| > \frac{4}{(b-a)^2}\int_a^b f(x)dx.$$

Proof: Consider $\sup_{x\in[a,b]}|f'(x)| := M$ as follows.

(i) If $M = +\infty$, then it is clear.

(ii) We may assume that $M < +\infty$.

Let $x \in [a, \frac{a+b}{2}]$, then

$$f(x) = f(x) - f(a) = f'(y)(x-a) \le M(x-a), \text{ where } y \in (a,x). \qquad *$$

and let $x \in [\frac{a+b}{2}, b]$, then

$$f(x) = f(x) - f(b) = f'(z)(x-b) \le M(b-x), \text{ where } z \in (x,b). \qquad **$$

So, by (*) and (**), we know that

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx \\ &\le M\int_a^{\frac{a+b}{2}} (x-a)dx + M\int_{\frac{a+b}{2}}^b (b-x)dx \\ &= M\left(\frac{a-b}{2}\right)^2\end{aligned}$$

which implies that

$$M \ge \frac{4}{(b-a)^2}\int_a^b f(x)dx.$$

Note that by (*) and (**), the equality does **NOT** hold since if it was, then we had $f'(x) = M$ on $[a,b]$ which implies that f is a constant function. So, we have

$$M > \frac{4}{(b-a)^2}\int_a^b f(x)dx.$$

By definition of supremum, we know that there exists at least one point ξ on (a,b) for which

$$|f'(\xi)| > \frac{4}{(b-a)^2}\int_a^b f(x)dx.$$

(5) **Gronwall Lemma**: Let f and g be continuous non-negative function defined on $[a,b]$, and $c \ge 0$. If

$$f(x) \le c + \int_a^x g(t)f(t)dt \text{ for all } x \in [a,b],$$

then

$$f(x) \le ce^{\int_a^x g(t)dt}.$$

In particular, as $c = 0$, we have $f = 0$ on $[a,b]$.

Proof: Let $c > 0$ and define

$$F(x) = c + \int_a^x g(t)f(t)dt,$$

then we have

(i). $F(a) = c > 0.$

(ii). $F'(x) = g(x)f(x) \geq 0 \Rightarrow F$ is increasing on $[a,b]$ by **Mean Value Theorem**

(iii). $F(x) \geq f(x)$ on $[a,b] \Rightarrow F'(x) \leq g(x)F(x)$ by (ii).

So, from (iii), we know that

$$F(x) \leq F(a)e^{\int_a^x g(t)dt} = ce^{\int_a^x g(t)dt} \text{ by (i).}$$

For $c = 0,$ we choose $c_n = 1/n \to 0,$ then by preceding result,

$$f(x) \leq \frac{1}{n}e^{\int_a^x g(t)dt} \to 0 \text{ as } n \to \infty.$$

So, we have proved all.

(6) Define

$$f(x) = \int_x^{x+1} \sin(t^2)dt.$$

(a) Prove that $|f(x)| < 1/x$ if $x > 0.$

Proof: Let $x > 0,$ then we have, by **change of variable(**$u = t^2$**)**, and **integration by parts**,

$$\begin{aligned} f(x) &= \frac{1}{2}\int_{x^2}^{(x+1)^2} \frac{\sin u}{\sqrt{u}}du \\ &= \frac{-1}{2}\int_{x^2}^{(x+1)^2} \frac{d(\cos u)}{\sqrt{u}} \\ &= \frac{-1}{2}\left[\left.\frac{\cos u}{\sqrt{u}}\right|_{x^2}^{(x+1)^2} + \int_{x^2}^{(x+1)^2} \frac{\cos u}{2u^{3/2}}du \right] \\ &= \frac{\cos(x^2)}{2x} - \frac{\cos\left[(x+1)^2\right]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du \end{aligned}$$

which implies that,

$$\begin{aligned} |f(x)| &\leq \left|\frac{\cos(x^2)}{2x}\right| + \left|\frac{\cos\left[(x+1)^2\right]}{2(x+1)}\right| + \left|\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du\right| \\ &< \frac{1}{2x} + \frac{1}{2(x+1)} + \frac{1}{4}\int_{x^2}^{(x+1)^2} \frac{du}{u^{3/2}} \\ &= \frac{1}{2x} + \frac{1}{2(x+1)} - \frac{1}{2(x+1)} + \frac{1}{2x} \\ &= 1/x. \end{aligned}$$

Note: There is another proof by **Second Mean Value Theorem** to show above as follows. Since

$$f(x) = \frac{1}{2}\int_{x^2}^{(x+1)^2} \frac{\sin u}{\sqrt{u}}du \text{ by (a),}$$

we know that, by **Second Mean Value Theorem,**

$$f(x) = \frac{1}{2}\left[\frac{1}{x}\int_{x^2}^{y}\sin u du + \frac{1}{x+1}\int_{y}^{(x+1)^2}\sin u du\right]$$
$$= \frac{1}{2}\left\{\frac{1}{x}[\cos(x^2) - \cos y] + \frac{1}{x+1}\left[\cos y - \cos\left((x+1)^2\right)\right]\right\}$$
$$= \frac{1}{2}\left\{\left(-\frac{1}{x} + \frac{1}{x+1}\right)\cos(y) + \frac{1}{x}\cos(x^2) - \frac{1}{x+1}\cos\left((x+1)^2\right)\right\}$$

which implies that

$$|f(x)| \le \frac{1}{2}\left\{\left|-\frac{1}{x} + \frac{1}{x+1}\right||\cos y| + \left|\frac{\cos(x^2)}{x}\right| + \left|\frac{\cos\left((x+1)^2\right)}{x+1}\right|\right\}$$
$$\le \frac{1}{2}\left\{\left(\frac{1}{x} - \frac{1}{x+1}\right) + \left|\frac{\cos(x^2)}{x}\right| + \left|\frac{\cos\left((x+1)^2\right)}{x+1}\right|\right\}$$
$$< \frac{1}{2}\left\{\left(\frac{1}{x} - \frac{1}{x+1}\right) + \frac{1}{x} + \frac{1}{x+1}\right\}$$

since no x makes $|\cos(x^2)| = \left|\cos\left((x+1)^2\right)\right| = 1$

$$= 1/x.$$

(b) Prove that $2xf(x) = \cos(x^2) - \cos\left[(x+1)^2\right] + r(x)$, where $|r(x)| < c/x$ and c is a constant.

Proof: By (a), we have

$$f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos\left[(x+1)^2\right]}{2(x+1)} - \int_{x^2}^{(x+1)^2}\frac{\cos u}{4u^{3/2}}du$$

which implies that

$$2xf(x) = \cos(x^2) - \frac{x}{x+1}\cos\left[(x+1)^2\right] + x\int_{x^2}^{(x+1)^2}\frac{\cos u}{2u^{3/2}}du$$
$$= \cos(x^2) - \cos\left[(x+1)^2\right] + \frac{1}{x+1}\cos\left[(x+1)^2\right] + x\int_{x^2}^{(x+1)^2}\frac{\cos u}{2u^{3/2}}du$$

where

$$r(x) = \frac{1}{x+1}\cos\left[(x+1)^2\right] + \frac{x}{2}\int_{x^2}^{(x+1)^2}\frac{\cos u}{u^{3/2}}du$$

which implies that

$$|r(x)| \le \frac{1}{x+1} + \frac{x}{2}\int_{x^2}^{(x+1)^2}\frac{|\cos u|}{u^{3/2}}du$$
$$< \frac{1}{x+1} + \frac{x}{2}\int_{x^2}^{(x+1)^2}u^{-3/2}du$$
$$= \frac{1}{x+1} + \frac{1}{x+1}$$
$$< \frac{2}{x}.$$

Note: Of course, we can use the note in (a) to show it. We write it as follows.

Proof: Since

$$f(x) = \frac{1}{2}\left\{\left(-\frac{1}{x} + \frac{1}{x+1}\right)\cos(y) + \frac{1}{x}\cos(x^2) - \frac{1}{x+1}\cos\left((x+1)^2\right)\right\}$$

which implies that

$$
\begin{aligned}
2xf(x) &= \left(\frac{x}{x+1} - 1\right)\cos(y) + \cos(x^2) - \frac{x}{x+1}\cos\left((x+1)^2\right) \\
&= \cos(x^2) - \cos\left((x+1)^2\right) + \frac{1}{x+1}\cos\left((x+1)^2\right) + \left(\frac{x}{x+1} - 1\right)\cos(y)
\end{aligned}
$$

where

$$r(x) = \frac{1}{x+1}\cos\left((x+1)^2\right) + \left(\frac{x}{x+1} - 1\right)\cos(y)$$

which implies that

$$
\begin{aligned}
|r(x)| &\le \frac{1}{x+1} + 1 - \frac{x}{x+1} \\
&= \frac{2}{x+1} \\
&< 2/x.
\end{aligned}
$$

(c) Find the upper and lower limits of $xf(x)$, as $x \to \infty$.

Proof: Claim that $\limsup_{x\to\infty} \cos(x^2) - \cos\left((x+1)^2\right) = 2$ as follows. Taking $x = n\sqrt{2\pi}$, where $n \in Z$, then

$$\cos(x^2) - \cos\left((x+1)^2\right) = -\cos\left(n\sqrt{8\pi} + 1\right).$$ 1

If we can show that $\left\{n\sqrt{8\pi}\right\}$ is dense in $[0, 2\pi]$ modulus 2π. It is equivalent to show that $\left\{n\sqrt{\frac{2}{\pi}}\right\}$ is dense in $[0, 1]$ modulus 1. So, by lemma $\{ar : a \in Z\}$, where $r \in Q^c$ is dense in $[0, 1]$ modulus 1, we have proved the claim. In other words, we have proved the claim.

Note: We use the lemma as follows. $\{ar + b : a \in Z, b \in Z\}$, where $r \in Q^c$ is dense in R. It is equivalent to $\{ar : a \in Z\}$, where $r \in Q^c$ is dense in $[0, 1]$ modulus 1.

Proof: Say $\{ar + b : a \in Z, b \in Z\} = S$, and since $r \in Q^c$, then by **Exercise 1.16**, there are infinitely many rational numbers h/k with $k > 0$ such that $|kr - h| < \frac{1}{k}$. Consider $(x - \delta, x + \delta) := I$, where $\delta > 0$, and thus choosing k_0 large enough so that $1/k_0 < \delta$. Define $L = |k_0 r - h_0|$, then we have $sL \in I$ for some $s \in Z$. So, $sL = (\pm)[(sk_0)r - (sh_0)] \in S$. That is, we have proved that S is dense in R.

(d) Does $\int_0^\infty \sin(t^2)dt$ converge?

Proof: Yes,

$$
\begin{aligned}
\left|\int_x^{x'} \sin^2 t\,dt\right| &= \left|\frac{1}{2}\int_x^{x'} \frac{\sin u}{\sqrt{u}}du\right| \text{ by the process of (a)} \\
&= \left|\frac{1}{2}\left[\frac{1}{x}\int_x^y \sin u\,du + \frac{1}{x'}\int_y^{x'} \sin u\,du\right]\right| \text{ by } \textbf{Second Mean Value Theorem} \\
&= \frac{1}{2}\left[\frac{2}{x} + \frac{2}{x'}\right] \\
&< \frac{2}{x}
\end{aligned}
$$

which implies that the integral exists.

Note: (i) We can show it without **Second Mean Value Theorem** by the method of (a). However **Second Mean Value Theorem** is more powerful for this exercise.

(ii) Here is the famous Integral named **Dirichlet Integral** used widely in the STUDY of **Fourier Series**. We write it as follows. Show that the **Dirichlet Integral**

$$\int_0^\infty \frac{\sin x}{x}dx$$

converges but not absolutely converges. In other words, the **Dirichlet Integral** converges conditionally.

Proof: Consider

$$\int_x^{x'} \frac{\sin x}{x} dx = \frac{1}{x}\int_x^{y} \sin x dx + \frac{1}{x'}\int_y^{x'} \sin x dx \text{ by } \textbf{Second Mean Value Theorem};$$

we have

$$\left|\int_x^{x'} \frac{\sin x}{x} dx\right| \leq \frac{2}{x} + \frac{2}{x'} < \frac{4}{x}.$$

So, we know that **Dirichlet Integral** converges**.**

Define $I_n = [\frac{\pi}{4} + 2n\pi, \frac{\pi}{2} + 2n\pi]$, then

$$\begin{aligned}\int_0^{\infty}\left|\frac{\sin x}{x}\right|dx &\geq \int_{I_n}\left|\frac{\sin x}{x}\right|dx \\ &\geq \int_{I_n}\frac{\left(\frac{\sqrt{2}}{2}\right)}{\frac{\pi}{4}+2n\pi}dx \\ &\geq \sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}+2n\pi} \to \infty.\end{aligned}$$

So, we know that **Dirichlet Integral** does **NOT** converges absolutely.

(7) Deal similarity with

$$f(x) = \int_x^{x+1} \sin(e^t)dt.$$

Show that

$$e^x|f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-1}\cos(e^{x+1}) + r(x),$$

where $|r(x)| < \min(1, Ce^{-x})$, for all x and

Proof: Since

$$\begin{aligned}f(x) &= \int_x^{x+1}\sin(e^t)dt \\ &= \int_{e^x}^{e^{x+1}}\frac{\sin u}{u}du \text{ by } \textbf{Change of Variable} \text{ (let } u = e^t) \\ &= \frac{\cos e^x}{e^x} - \frac{\cos e^{x+1}}{e^{x+1}} - \int_{e^x}^{e^{x+1}}\frac{\cos u}{u^2}du \text{ by } \textbf{Integration by parts} \qquad *\end{aligned}$$

which implies that

$$\begin{aligned}|f(x)| &< \left|\frac{\cos e^x}{e^x}\right| + \left|\frac{\cos e^{x+1}}{e^{x+1}}\right| + \int_{e^x}^{e^{x+1}}\frac{du}{u^2} \text{ since } \cos u \text{ is not constant } 1 \\ &\leq \frac{1}{e^x} + \frac{1}{e^{x+1}} + \frac{1}{e^x}\left(1 - \frac{1}{e}\right)\end{aligned}$$

which implies that

$$e^x|f(x)| < 2.$$

In addition, by (*), we have

$$e^x f(x) = \cos(e^x) - e^{-1}\cos(e^{x+1}) + r(x),$$

where

$$r(x) = -e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du$$

which implies that

$$|r(x)| = 1 - e^{-1} < 1 \text{ for all } x \qquad **$$

or which implies that, by **Integration by parts,**

$$\begin{aligned} |r(x)| &= e^x \left| \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du \right| \\ &= e^x \left| \frac{\sin e^{x+1}}{e^{2(x+1)}} - \frac{\sin e^x}{e^x} + 2\int_{e^x}^{e^{x+1}} \frac{\sin u}{u^3} du \right| \\ &< e^x \left(\frac{1}{e^{2(x+1)}} + \frac{1}{e^{2x}} + 2\int_{e^x}^{e^{x+1}} \frac{du}{u^3} \right) \text{ since } \sin u \text{ is not constant 1} \\ &= 2e^{-x} \text{ for all } x. \qquad *** \end{aligned}$$

By (**) and (***), we have proved that $|r(x)| < \min(1, Ce^{-x})$ for all x, where $C = 2$.

Note: We give another proof on (7) by **Second Mean Value Theorem** as follows.

Proof: Since

$$\begin{aligned} f(x) &= \int_{e^x}^{e^{x+1}} \frac{\sin u}{u} du \\ &= \frac{1}{e^x} \int_{e^x}^{y} \sin u du + \frac{1}{e^{x+1}} \int_{y}^{e^{x+1}} \sin u du \text{ by } \textbf{Second Mean Value Theorem} \\ &= \frac{1}{e^x}(\cos e^x - \cos y) + \frac{1}{e^{x+1}}(\cos y - \cos e^{x+1}) \qquad * \end{aligned}$$

which implies that

$$\begin{aligned} e^x|f(x)| &= |\cos e^x - \cos y + e^{-1}(\cos y - \cos e^{x+1})| \\ &= |(\cos e^x - e^{-1}\cos e^{x+1}) + \cos y(1 - e^{-1})| \\ &\leq |\cos e^x - e^{-1}\cos e^{x+1}| + (1 - e^{-1}) \\ &< (1 + e^{-1}) + (1 - e^{-1}) \text{ since no } x \text{ makes } |\cos e^x| = |\cos e^{x+1}| = 1. \\ &= 2. \end{aligned}$$

In addition, by (*), we know that

$$e^x f(x) = \cos e^x - e^{-1}\cos e^{x+1} + r(x)$$

where

$$r(x) = (e^{-1}\cos y - \cos y)$$

which implies that

$$|r(x)| \leq 1 - e^{-1} < 1 \text{ for all } x. \qquad **$$

In addition, from the proof of the process in (7), we know that

$$
\begin{aligned}
|r(x)| &= e^x \left| \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du \right| \\
&= e^x \left| \frac{1}{e^{2x}} \int_{e^x}^{y} \cos u du + \frac{1}{e^{2(x+1)}} \int_{y}^{e^{x+1}} \cos u du \right| \\
&= e^{-x} |\sin y - \sin e^x + e^{-2}(\sin e^{x+1} - \sin y)| \\
&= e^{-x} |\sin y(1 - e^{-2}) + (e^{-2} \sin e^{x+1} - \sin e^x)| \\
&\leq e^{-x}(1 - e^{-2}) + e^{-x} |e^{-2} \sin e^{x+1}| + e^{-x} |\sin e^x| \\
&< e^{-x}(1 - e^{-2}) + e^{-x}(1 + e^{-2}) \text{ since no } x \text{ makes } |\sin e^{x+1}| = |\sin e^x| = 1 \\
&= 2e^{-x} \text{ for all } x. \qquad ***
\end{aligned}
$$

So, by (**) and (***), we have proved that $|r(x)| < \min(1, Ce^{-x})$, where $C = 2$.

(8) Suppose that f is real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x) dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = \frac{-1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x) dx > \frac{1}{4}.$$

Proof: Consider

$$
\begin{aligned}
\int_a^b x f(x) f'(x) dx &= \int_a^b x f(x) df(x) \\
&= x f^2(x)|_a^b - \int_a^b f(x) d(x f(x)) \\
&= -\int_a^b f^2(x) dx + \int_a^b x f(x) f'(x) dx \text{ since } f(a) = f(b) = 0,
\end{aligned}
$$

so we have

$$\int_a^b x f(x) f'(x) dx = \frac{-1}{2}.$$

In addition, by **Cauchy-Schwarz Inequality**, we know that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x) dx \geq \left(\int_a^b x f(x) f'(x) dx \right)^2 = \frac{1}{4}.$$

Note that the equality does **NOT** hold since if it was, then we have $f'(x) = kxf(x)$. It implies that

$$[f'(x) - kxf(x)] e^{\frac{-kx^2}{2}} = 0$$

which implies that

$$\left(f e^{\frac{-kx^2}{2}} \right)' = 0$$

which implies that

$$f(x) = Ce^{\frac{-kx^2}{2}}, \text{ a constant}$$

which implies that

$$C = 0 \text{ since } f(a) = 0.$$

That is, $f(x) = 0$ on $[a,b]$ which is absurb.

7.5 Let $\{a_n\}$ be a sequence of real numbers. For $x \geq 0$, define

$$A(x) = \sum_{n \leq x} a_n = \sum_{n=1}^{[x]} a_n,$$

where $[x]$ is the largest integer in x and empty sums are interpreted as zero. Let f have a continuous derivative in the interval $1 \leq x \leq a$. Use Stieltjes integrals to derive the following formula:

$$\sum_{n \leq a} a_n f(n) = -\int_1^a A(x) f'(x) dx + A(a) f(a).$$

Proof: Since

$$\int_1^a A(x) f'(x) dx = \int_1^a A(x) df(x) \text{ since } f \text{ has a continous derivative on } [1,a]$$

$$= -\int_1^a f(x) dA(x) + A(a) f(a) - A(1) f(1) \text{ by } \textbf{integration by parts}$$

$$= -\sum_{n \leq a} a_n f(n) + A(a) f(a) \text{ by } \int_1^a f(x) dA(x) = \sum_{n=2}^{[a]} a_n f(n) \text{ and } A(1) = a_1,$$

we know that

$$\sum_{n \leq a} a_n f(n) = -\int_1^a A(x) f'(x) dx + A(a) f(a).$$

7.6 Use **Euler's summation formula**, **integration by parts** in a Stieltjes integral, to derive the following identities:

(a) $\sum_{k=1}^n \frac{1}{k^s} = \frac{1}{n^{s-1}} + s \int_1^n \frac{[x]}{x^{s+1}} dx$ if $s \neq 1$.

Proof:

$$\sum_{k=1}^n \frac{1}{k^s} = \int_1^n x^{-s} d[x] + 1$$

$$= -\int_1^n [x] dx^{-s} + n^{-s}[n] - 1^{-s}[1] + 1$$

$$= s \int_1^n \frac{[x]}{x^{s+1}} dx + n^{1-s}$$

$$= \frac{1}{n^{s-1}} + s \int_1^n \frac{[x]}{x^{s+1}} dx \text{ if } s \neq 1.$$

(b) $\sum_{k=1}^n \frac{1}{k} = \log n - \int_1^n \frac{x-[x]}{x^2} + 1$.

Proof:

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k} &= \int_1^n \frac{1}{x}d[x]+1\\ &= -\int_1^n [x]dx^{-1}+n^{-1}[n]-1^{-1}[1]+1\\ &= \int_1^n x^{-1}dx-\int_1^n x^{-1}dx+\int_1^n \frac{[x]}{x^2}dx+1\\ &= \log n-\int_1^n \frac{x-[x]}{x^2}+1.\end{aligned}$$

7.7 Assume that f' is continuous on $[1,2n]$ and use Euler's summation formula or integration by parts to prove that

$$\sum_{k=1}^{2n}(-1)^k f(k) = \int_1^{2n} f'(x)([x]-2[x/2])dx.$$

Proof:

$$\begin{aligned}\sum_{k=1}^{2n}(-1)^k f(k) &= -\sum_{k=1}^{2n} f(k)+2\sum_{k=1}^{n} f(2k)\\ &= -\left(\int_1^{2n} f(x)d[x]+f(1)\right)+2\left(\int_1^{2n} f(x)d[x/2]\right)\\ &= -\left(-\int_1^{2n}[x]df(x)+[2n]f(2n)\right)+2\left(-\int_1^{2n}[x/2]df(x)+f(2n)[2n/2]-f(1)[1/2]\right)\end{aligned}$$

since f' is continuous on $[1,2n]$

$$\begin{aligned}&= \int_1^{2n} f'(x)[x]dx-2nf(2n)-\int_1^{2n} f'(x)[x/2]dx+2nf(2n)\\ &= \int_1^{2n} f'(x)([x]-2[x/2])dx.\end{aligned}$$

7.8 Let $\phi_1 = x-[x]-\frac{1}{2}$ if $x \neq$ integer, and let $\phi_1 = 0$ if $x =$ integer. Also, let $\phi_2 = \int_0^x \phi_1(t)dt$. If f'' is continuous on $[1,n]$ prove that Euler's summation formula implies that

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)dx-\int_1^n \phi_2(x)f''(x)dx+\frac{f(1)+f(n)}{2}.$$

Proof: Using Theorem 7.13, then we have

$$\begin{aligned}\sum_{k=1}^{n} f(k) &= \int_1^n f(x)dx+\int_1^n f'(x)\phi_1(x)dx+\frac{f(1)+f(n)}{2}\\ &= \int_1^n f(x)dx+\int_1^n f'(x)d\phi_2(x)+\frac{f(1)+f(n)}{2}\\ &= \int_1^n f(x)dx+\left(-\int_1^n \phi_2(x)df'(x)+f'(n)\phi_2(n)-f'(1)\phi_2(1)\right)+\frac{f(1)+f(n)}{2}\\ &= \int_1^n f(x)dx-\int_1^n \phi_2(x)df'(x)+\frac{f(1)+f(n)}{2}\\ &= \int_1^n f(x)dx-\int_1^n \phi_2(x)f''(x)dx+\frac{f(1)+f(n)}{2}\end{aligned}$$

since f'' is continuous on $[1,n]$.

7.9 Take $f(x) = \log x$ in Exercise 7.8 and prove that

$$\log n! = \left(n + \frac{1}{2}\right)\log n - n + 1 + \int_1^n \frac{\phi_2(t)}{t^2}dt.$$

Proof: Let $f(x) = \log x$, then by Exercise 7.8, it is clear. So, we omit the proof.

Remark: By **Euler's summation formula**, we can show that

$$\sum_{1<k\leq n} \log k = \int_1^n \log x dx + \int_1^n \left(x - [x] - \frac{1}{2}\right)\frac{dx}{x} + \frac{\log n}{2}. \quad *$$

Since

$$\left|\left(x - [x] - \frac{1}{2}\right)\right| \leq 1/2$$

and

$$\int_a^{a+1} \left(x - [x] - \frac{1}{2}\right)dx = 0 \text{ for all real } a, \quad **$$

we thus have the convergence of the improper integral

$\int_1^\infty \left(x - [x] - \frac{1}{2}\right)dx$ by **Second Mean Value Theorem.**

So, by (*), we have

$$\log n! = \left(n + \frac{1}{2}\right)\log n - n + C + \gamma_n$$

where

$$C = 1 + \int_1^\infty \left(x - [x] - \frac{1}{2}\right)\frac{dx}{x},$$

and

$$\gamma_n = -\int_n^\infty \left(x - [x] - \frac{1}{2}\right)\frac{dx}{x}.$$

So,

$$\lim_{n\to\infty} \frac{n!}{e^{-n}n^{n+1/2}} = e^C := C_1. \quad ***$$

Now, using **Wallis formula**, we have

$$\lim_{n\to\infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (2n)(2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2n-1)(2n+1)} = \pi/2$$

which implies that

$$\frac{(2^n n!)^4}{[(2n)!]^2(2n+1)}(1 + o(1)) = \pi/2$$

which implies that, by (***),

$$\frac{C_1^4(2^n n^{n+1/2}e^{-n})^4}{C_1^2\left[(2n)^{2n+1/2}e^{-2n}\right](2n+1)}(1 + o(1)) = \pi/2$$

which implies that

$$\frac{C_1^2 n}{2(2n+1)}(1 + o(1)) = \pi/2.$$

Let $n \to \infty$, we have $C_1 = \sqrt{2\pi}$, and $\int_1^\infty (x - [x] - \frac{1}{2})dx = \frac{1}{2}\log 2\pi - 1$.

Note: In (***), the formula is called **Stirling formula**. The reader should be noted that **Wallis formula is equivalent to Stirling formula.**

7.10 If $x \geq 1$, let $\pi(x)$ denote the number of primes $p \leq x$, that is,

$$\pi(x) = \sum_{p \leq x} 1,$$

where the sum is extended over all primes $p \leq x$. The prime number theorem states that

$$\lim_{x \to \infty} \pi(x) \frac{\log x}{x} = 1.$$

This is usually proved by studying a related function ϑ given by

$$\vartheta(x) = \sum_{p \leq x} \log p,$$

where again the sum is extended over all primes $p \leq x$. Both function π and ϑ are step functions with jumps at the primes. This exercise shows how the Riemann-Stieltjes integral can be used to relate these two functions.

(a) If $x \geq 2$, prove that $\pi(x)$ and $\vartheta(x)$ can be expressed as the following Riemann-Stieltjes integrals:

$$\vartheta(x) = \int_{3/2}^{x} \log t d\pi(t), \ \pi(x) = \int_{3/2}^{x} \frac{1}{\log t} d\vartheta(t).$$

Note. The lower limit can be replaced by any number in the open interval $(1,2)$.

Proof: Since $\vartheta(x) = \sum_{p \leq x} \log p$, we know that by **Theorem 7.9**,

$$\vartheta(x) = \int_{3/2}^{x} \log t d\pi(t),$$

and $\pi(x) = \sum_{p \leq x} 1$, we know that by **Theorem 7.9**,

$$\pi(x) = \int_{3/2}^{x} \frac{1}{\log t} d\vartheta(t).$$

(b) If $x \geq 2$, use integration by parts to show that

$$\vartheta(x) = \pi(x) \log x - \int_{2}^{x} \frac{\pi(t)}{t} dt,$$

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_{2}^{x} \frac{\vartheta(t)}{t \log^2 t} dt.$$

These equations can be used to prove that the prime number theorem is equivalent to the relation $\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1$.

Proof: Use integration by parts, we know that

$$\begin{aligned}
\vartheta(x) &= \int_{3/2}^{x} \log t d\pi(t) = -\int_{3/2}^{x} \frac{\pi(t)}{t} dt + \log x \pi(x) - \log(3/2)\pi(3/2) \\
&= -\int_{3/2}^{x} \frac{\pi(t)}{t} dt + \log x \pi(x) \text{ since } \pi(3/2) = 0 \\
&= \pi(x) \log x - \int_{2}^{x} \frac{\pi(t)}{t} dt \text{ since } \int_{3/2}^{2} \frac{\pi(t)}{t} dt = 0 \text{ by } \pi(x) = 0 \text{ on } [0,2),
\end{aligned}$$

and

$$\pi(x) = \int_{3/2}^{x} \frac{1}{\log t} d\vartheta(t) = \int_{3/2}^{x} \frac{\vartheta(t)}{t\log^2 t} dt + \frac{\vartheta(x)}{\log x} - \frac{\vartheta(3/2)}{\log(3/2)}$$
$$= \int_{3/2}^{x} \frac{\vartheta(t)}{t\log^2 t} dt + \frac{\vartheta(x)}{\log x} \text{ since } \vartheta(3/2) = 0$$
$$= \frac{\vartheta(x)}{\log x} + \int_{2}^{x} \frac{\vartheta(t)}{t\log^2 t} dt \text{ since } \int_{3/2}^{2} \frac{\vartheta(t)}{t\log^2 t} dt = 0 \text{ by } \vartheta(x) = 0 \text{ on } [0,2).$$

7.11 If $\alpha \nearrow$ on $[a,b]$, prove that
(a) $\bar{\int_a^b} f d\alpha = \bar{\int_a^c} f d\alpha + \bar{\int_c^b} f d\alpha$, $(a < c < b)$

Proof: Given $\varepsilon > 0$, there is a partition P such that
$$U(P,f,\alpha) < \bar{I}(a,b) + \varepsilon. \quad 1$$
Let $P' = \{c\} \cup P = P_1 \cup P_2$, where $P_1 = \{a = x_0, \ldots, x_{n_1} = c\}$ and $P_2 = \{x_{n_1} = c, \ldots, x_{n_2} = b\}$then we have
$$\bar{I}(a,c) + \bar{I}(c,b) \leq U(P_1,f,\alpha) + U(P_2,f,\alpha) = U(P',f,\alpha) \leq U(P,f,\alpha). \quad 2$$
So, by (1) and (2), we have
$$\bar{I}(a,c) + \bar{I}(c,b) \leq \bar{I}(a,b) \quad *$$
since ε is arbitrary.

On the other hand, given $\varepsilon > 0$, there is a partition P_1 and P_2 such that
$$U(P_1,f,\alpha) + U(P_2,f,\alpha) \leq \bar{I}(a,c) + \bar{I}(c,b) + \varepsilon$$
which implies that, let $P = P_1 \cup P_2$
$$U(P,f,\alpha) = U(P_1,f,\alpha) + U(P_2,f,\alpha) \leq \bar{I}(a,c) + \bar{I}(c,b) + \varepsilon. \quad 3$$
Also,
$$\bar{I}(a,b) \leq U(P,f,\alpha). \quad 4$$
By (3) and (4), we have
$$\bar{I}(a,b) \leq \bar{I}(a,c) + \bar{I}(c,b) \quad **$$
since ε is arbitrary.

So, by (*) and (**), we have proved it.

(b) $\bar{\int_a^b}(f+g)d\alpha \leq \bar{\int_a^b} f d\alpha + \bar{\int_a^b} g d\alpha.$

Proof: In any compact interval J, we have
$$\sup_{x\in J}(f+g) \leq \sup_{x\in J} f + \sup_{x\in J} g. \quad 1$$
So, given $\varepsilon > 0$, there is a partition P_f and P_g such that
$$\sum_{k=1}^{n_1} M_k(f)\Delta\alpha_k \leq \bar{\int_a^b} f d\alpha + \varepsilon/2 \quad 2$$
and
$$\sum_{k=1}^{n_2} M_k(g)\Delta\alpha_k \leq \bar{\int_a^b} g d\alpha + \varepsilon/2. \quad 3$$
So, consider $P = P_f \cup P_g$, then we have, by (1),
$$U(P,f+g,\alpha) \leq U(P,f,\alpha) + U(P,g,\alpha)$$
along with

$$U(P,f,\alpha) \leq \sum_{k=1}^{n_1} M_k(f)\Delta\alpha_k \text{ and } U(P,g,\alpha) \leq \sum_{k=1}^{n_2} M_k(g)\Delta\alpha_k$$

which implies that, by (2) and (3),

$$U(P,f+g,\alpha) \leq \int_a^{\bar{b}} fd\alpha + \int_a^{\bar{b}} gd\alpha + \varepsilon$$

which implies that

$$\int_a^{\bar{b}} (f+g)d\alpha \leq \int_a^{\bar{b}} fd\alpha + \int_a^{\bar{b}} gd\alpha$$

since ε is arbitrary.

(c) $\int_{a_-}^{b} (f+g)d\alpha \geq \int_{a_-}^{b} fd\alpha + \int_{a_-}^{b} gd\alpha$

Proof: Similarly by (b), so we omit the proof.

7.12 Give an example of bounded function f and an increasing function α defined on $[a,b]$ such that $|f| \in R(\alpha)$ but for which $\int_a^b fd\alpha$ does not exist.

Solution: Let

$$f(x) = \begin{cases} 1 \text{ if } x \in [0,1] \cap Q \\ -1 \text{ if } x \in [0,1] \cap Q^c \end{cases}$$

and $\alpha(x) = x$ on $[0,1]$. Then it is clear that $f \notin R(\alpha)$ on $[a,b]$ and $|f| \in R(\alpha)$ on $[a,b]$.

7.13 Let α be a continuous function of bounded variation on $[a,b]$. Assume that $g \in R(\alpha)$ on $[a,b]$ and define $\beta(x) = \int_a^x g(t)d\alpha(t)$ if $x \in [a,b]$. Show that:

(a) If $f \nearrow$ on $[a,b]$, there exists a point x_0 in $[a,b]$ such that $\int_a^b fd\beta = f(a)\int_a^{x_0} gd\alpha + f(b)\int_{x_0}^b gd\alpha.$

Proof: Since α is a continuous function of bounded variation on $[a,b]$, and $g \in R(\alpha)$ on $[a,b]$, we know that $\beta(x)$ is a continuous function of bounded variation on $[a,b]$, by **Theorem 7.32**. Hence, by **Second Mean-Value Theorem for Riemann-Stieltjes integrals**, we know that

$$\int_a^b fd\beta = f(a)\int_a^{x_0} d\beta(x) + f(b)\int_{x_0}^b d\beta(x)$$

which implies that, by **Theorem 7.26,**

$$\int_a^b fd\beta = f(a)\int_a^{x_0} gd\alpha + f(b)\int_{x_0}^b gd\alpha.$$

(b) If, in addition, f is continuous on $[a,b]$, we also have

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{x_0} gd\alpha + f(b)\int_{x_0}^b gd\alpha.$$

Proof: Since

$$\int_a^b fd\beta = \int_a^b f(x)g(x)dx \text{ by Theorem 7.26,}$$

we know that, by (a),

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{x_0} g d\alpha + f(b)\int_{x_0}^b g d\alpha.$$

Remark: We do NOT need the hypothesis that f is continuous on $[a,b]$.

7.14 Assume that $f \in R(\alpha)$ on $[a,b]$, where α is of bounded variation on $[a,b]$. Let $V(x)$ denote the total variation of α on $[a,x]$ for each x in $(a,b]$, and let $V(a) = 0$. Show that

$$\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| dV \leq MV(b),$$

where M is an upper bound for $|f|$ on $[a,b]$. In particular, when $\alpha(x) = x$, the inequality becomes

$$\left|\int_a^b f d\alpha\right| \leq M(b-a).$$

Proof: Given $\varepsilon > 0$, there is a partition $P = \{a = x_0, \ldots, x_n = b\}$ such that

$$\begin{aligned}
\int_a^b f d\alpha - \varepsilon &< S(P, f, \alpha) \\
&= \sum_{k=1}^n f(t_k)\Delta\alpha_k, \text{ where } t_k \in [x_{k-1}, x_k] \\
&\leq \sum_{k=1}^n |f(t_k)||\alpha(x_k) - \alpha(x_{k-1})| \\
&\leq \sum_{k=1}^n |f(t_k)|(V(x_k) - V(x_{k-1})) \\
&= S(P, |f|, V) \\
&\leq U(P, |f|, V) \text{ since } V \text{ is increasing on } [a,b]
\end{aligned}$$

which implies that, taking infimum,

$$\int_a^b f d\alpha - \varepsilon \leq \int_a^b |f| dV$$

since $|f| \in R(V)$ on $[a,b]$.

So, we have

$$\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| dV \qquad *$$

since $\int_a^b |f| dV$ is clear non-negative. If M is an upper bound for $|f|$ on $[a,b]$, then (*) implies that

$$\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| dV \leq MV(b)$$

which implies that

$$\left|\int_a^b f d\alpha\right| \leq M(b-a)$$

if $\alpha(x) = x$.

7.15 Let $\{\alpha_n\}$ be a sequence of functions of bounded variation on $[a,b]$. Suppose there exists a function α defined on $[a,b]$ such that the total variation of $\alpha - \alpha_n$ on $[a,b]$ tends to 0 as $n \to \infty$. Assume also that $\alpha(a) = \alpha_n(a) = 0$ for each $n = 1, 2, \ldots$. If f is continuous on $[a,b]$, prove that

$$\lim_{n\to\infty}\int_a^b f(x)d\alpha_n(x) = \int_a^b f(x)d\alpha(x).$$

Proof: Use **Exercise 7.14**, we then have

$$\left|\int_a^b f(x)d(\alpha - \alpha_n(x))\right| \leq MV_n(b) \to 0 \text{ as } n \to \infty$$

where V_n is the total variation of $\alpha - \alpha_n$, and $M = \sup_{x\in[a,b]}|f(x)|$.

So, we have

$$\lim_{n\to\infty}\int_a^b f(x)d\alpha_n(x) = \int_a^b f(x)d\alpha(x).$$

Remark: We do **NOT** need the hypothesis $\alpha(a) = \alpha_n(a) = 0$ for each $n = 1,2,\ldots.$

7.16 If $f \in R(\alpha)$, $f^2 \in R(\alpha)$, $g \in R(\alpha)$, and $g^2 \in R(\alpha)$ on $[a,b]$, prove that

$$\frac{1}{2}\int_a^b\left[\int_a^b\left|\begin{matrix} f(x) & g(x) \\ f(y) & g(y)\end{matrix}\right|^2 d\alpha(y)\right]d\alpha(x)$$
$$= \left(\int_a^b f^2(x)d\alpha(x)\right)\left(\int_a^b g^2(x)d\alpha(x)\right) - \left(\int_a^b f(x)g(x)d\alpha(x)\right)^2.$$

When $\alpha \nearrow$ on $[a,b]$, deduce the Cauchy-Schwarz inequality

$$\left(\int_a^b f(x)g(x)d\alpha(x)\right)^2 \leq \left(\int_a^b f^2(x)d\alpha(x)\right)\left(\int_a^b g^2(x)d\alpha(x)\right).$$

(Compare with Exercise 1.23.)

Proof: Consider

$$\frac{1}{2}\int_a^b\left[\int_a^b\left|\begin{matrix} f(x) & g(x) \\ f(y) & g(y)\end{matrix}\right|^2 d\alpha(y)\right]d\alpha(x)$$
$$= \frac{1}{2}\int_a^b\left[\int_a^b (f(x)g(y) - f(y)g(x))^2 d\alpha(y)\right]d\alpha(x)$$
$$= \frac{1}{2}\int_a^b\left[\int_a^b (f^2(x)g^2(y) - 2f(x)g(y)f(y)g(x) + f^2(y)g^2(x))d\alpha(y)\right]d\alpha(x)$$
$$= \frac{1}{2}\int_a^b f^2(x)d\alpha(x)\int_a^b g^2(y)d\alpha(y)$$
$$-\int_a^b f(x)g(x)d\alpha(x)\int_a^b f(y)g(y)d\alpha(y)$$
$$+\int_a^b g^2(x)d\alpha(x)\int_a^b f^2(y)d\alpha(y)$$
$$= \int_a^b f^2(x)d\alpha(x)\int_a^b g^2(y)d\alpha(y) - \left[\int_a^b f(x)g(x)d\alpha(x)\right]^2,$$

if $\alpha \nearrow$ on $[a,b]$, then we have

$$0 \le \frac{1}{2}\int_a^b \left[\int_a^b \begin{vmatrix} f(x) & g(x) \\ f(y) & g(y) \end{vmatrix}^2 d\alpha(y) \right] d\alpha(x)$$
$$= \int_a^b f^2(x)d\alpha(x)\int_a^b g^2(y)d\alpha(y) - \left[\int_a^b f(x)g(x)d\alpha(x)\right]^2$$

which implies that

$$\left(\int_a^b f(x)g(x)d\alpha(x)\right)^2 \le \left(\int_a^b f^2(x)d\alpha(x)\right)\left(\int_a^b g^2(x)d\alpha(x)\right).$$

Remark: (1) Here is another proof: Let $A = \int_a^b f^2(x)d\alpha(x)$, $B = \int_a^b f(x)g(x)d\alpha(x)$, and $C = \int_a^b g^2(x)d\alpha(x)$. From the fact,

$$0 \le \int_a^b [f(x)z + g(x)]^2 dx \text{ for any real } z$$
$$= Az^2 + 2Bz + C.$$

It implies that

$$B^2 \le AC.$$

That is,

$$\left(\int_a^b f(x)g(x)d\alpha(x)\right)^2 \le \left(\int_a^b f^2(x)d\alpha(x)\right)\left(\int_a^b g^2(x)d\alpha(x)\right).$$

Note: (1) The reader may recall the **inner product** in **Linear Algebra**. We often consider **Riemann Integral** by defining

$$< f,g >:= \int_a^b f(x)g(x)dx$$

where f and g are real continuous functions defined on $[a,b]$. This definition is a real case. For complex case, we need to preserve its positive definite. So, we define

$$< f,g >:= \int_a^b f(x)\bar{g}(x)dx$$

where f and g are complex continuous functions defined on $[a,b]$, and $\bar{g}$ means its conjugate. In addition, in this sense, we have the **triangular inequality**:

$$\|f-g\| \le \|f-h\| + \|f-h\|, \text{ where } \|f\| = \sqrt{< f,f >}.$$

(2) Suppose that $f \in R(\alpha)$ on $[a,b]$ where $\alpha \nearrow$ on $[a,b]$ and given $\varepsilon > 0$, then there exists a continuous function g on $[a,b]$ such that

$$\|f-g\| < \varepsilon.$$

Proof: Let $K = \sup_{x\in[a,b]}|f(x)|$, and given $\varepsilon > 0$, we want to show that

$$\|f-g\| < \varepsilon.$$

Since $f \in R(\alpha)$ on $[a,b]$ where $\alpha \nearrow$ on $[a,b]$, given $(1 >)\varepsilon' > 0$, there is a partition $P = \{x_0 = a,\ldots,x_n = b\}$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{j=1}^n [M_j(f) - m_j(f)]\Delta\alpha_j < (\varepsilon')^2. \quad 1$$

Write $P = A \cup B$, where $A = \{x_j : M_j(f) - m_j(f) < \varepsilon'\}$ and $B = \{x_j : M_j(f) - m_j(f) \ge \varepsilon'\}$, then

$$\varepsilon \sum_B \Delta\alpha_j \le \sum_B [M_j(f) - m_j(f)]\Delta\alpha_j < (\varepsilon')^2 \text{ by (1)}$$

which implies that

$$\sum_B \Delta\alpha_j < \varepsilon'. \qquad (2)$$

For this partition P, we define the function g as follows.

$$g(t) = \frac{x_j - t}{x_j - x_{j-1}} f(x_{j-1}) + \frac{t - x_{j-1}}{x_j - x_{j-1}} f(x_j), \text{ where } x_{j-1} \le t \le x_j.$$

So, it is clear that g is continuous on $[a,b]$. In every subinterval $[x_{j-1}, x_j]$

$$\begin{aligned} |f(t) - g(t)| &= \left| \frac{x_j - t}{x_j - x_{j-1}}[f(t) - f(x_{j-1})] + \frac{t - x_{j-1}}{x_j - x_{j-1}}[f(t) - f(x_j)] \right| \\ &\le |f(t) - f(x_{j-1})| + |f(t) - f(x_j)| \\ &\le 2[M_j(f) - m_j(f)] \qquad (3) \end{aligned}$$

Consider

$$\begin{aligned} \sum_A \int_{[x_{j-1},x_j]} |f(t) - g(t)|^2 d\alpha &\le \sum_A 4[M_j(f) - m_j(f)]^2 \Delta\alpha_j \text{ by (3)} \\ &\le 4\sum_A \varepsilon'[M_j(f) - m_j(f)]\Delta\alpha_j \text{ by definition of } A \\ &\le 4\varepsilon' \sum_A \Delta\alpha_j \text{ by } \varepsilon' < 1 \\ &= 4\varepsilon'[\alpha(b) - \alpha(a)] \end{aligned}$$

and

$$\begin{aligned} \sum_B \int_{[x_{j-1},x_j]} |f(t) - g(t)|^2 d\alpha &\le \sum_B 4K^2 \Delta\alpha_j \\ &\le 4K^2\varepsilon' \text{ by (2).} \end{aligned}$$

Hence,

$$\begin{aligned} \int_a^b |f(t) - g(t)|^2 d\alpha &\le 4\varepsilon'[\alpha(b) - \alpha(a)] + 4K^2\varepsilon' \\ &< \varepsilon^2 \end{aligned}$$

if we choose ε' is small enough so that $4\varepsilon'[\alpha(b) - \alpha(a)] + 4K^2\varepsilon' < \varepsilon^2$. That is, we have proved that

$$\|f - g\| < \varepsilon.$$

P.S.: The exercise tells us a Riemann-Stieltjes integrable function can be approximated (approached) by continuous functions.

(3)There is another important result called **Holder's inequality**. It is useful in Analysis and more general than **Cauchy-Schwarz inequality**. In fact, it is the case $p = q = 2$ in **Holder's inequality.** We consider the following results.

Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove that the following statements.

(a) If $u \ge 0$ and $v \ge 0$, then

$$uv \le \frac{u^p}{p} + \frac{v^p}{q}.$$

Equality holds if and only if $u^p = v^q$.

Proof: Let $f(u) = \frac{u^p}{p} + \frac{v^p}{q} - uv$ be a function defined on $[0, +\infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 0$, $q > 0$ and $v \ge 0$, then $f'(u) = u^{p-1} - v$. So, we know that

$$f'(u) < 0 \text{ if } u \in \left(0, v^{\frac{1}{p-1}}\right) \text{ and } f'(u) > 0 \text{ if } u \in \left(v^{\frac{1}{p-1}}, +\infty\right)$$

which implies that, by $f\left(v^{\frac{1}{p-1}}\right) = 0$, $f(u) \ge 0$. Hence, we know that $f(u) \ge 0$ for all $u \ge 0$. That is, $uv \le \frac{u^p}{p} + \frac{v^p}{q}$. In addition, $f(u) = 0$ if and only if $u = v^{\frac{1}{p-1}}$ if and only if $u^p = v^q$. So, Equality holds if and only if $u^p = v^q$.

Note: (1) Here is another good proof by using **Young's Inequality**, let $f(x)$ be an strictly increasing and continuous function defined on $\{x : x \ge 0\}$, with $f(0) = 0$. Then we have, let $a > 0$ and $b > 0$,

$$ab \le \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx, \text{ where } f^{-1} \text{ is the inverse function of } f.$$

And the equality holds if and only if $f(a) = b$.

Proof: The proof is easy by drawing the function f on $x - y$ plane. So, we omit it.

So, by **Young's Inequality,** let $f(x) = x^\alpha$, where $\alpha > 0$, we have the **Holder's inequality.**

(2) The reader should be noted that there are many proofs of (a), for example, using the concept of convex function, or using $A.P. \ge G.P.$ along with continuity.

(b) If $f, g \in R(\alpha)$ on $[a,b]$ where $\alpha \nearrow$ on $[a,b]$, $f, g \ge 0$ on $[a,b]$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg d\alpha \le 1.$$

Proof: By **Holder's inequality,** we have

$$fg \le \frac{f^p}{p} + \frac{g^q}{q}$$

which implies that, by $\alpha \nearrow$ on $[a,b]$, and $\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha$,

$$\int_a^b fg d\alpha \le \int_a^b \frac{f^p}{p} d\alpha + \int_a^b \frac{g^q}{q} d\alpha = \frac{1}{p} + \frac{1}{q} = 1.$$

(c) If f and g are complex functions in $R(\alpha)$, where $\alpha \nearrow$ on $[a,b]$, then

$$\left|\int_a^b fg d\alpha\right| \le \left\{\int_a^b |f|^p d\alpha\right\}^{1/p} \left\{\int_a^b |g|^q d\alpha\right\}^{1/q}.$$ *

Proof: First, we note that

$$\left|\int_a^b fg d\alpha\right| \le \int_a^b |fg| d\alpha.$$

Also,

$$\int_a^b |f|^p d\alpha = M^p \Rightarrow \int_a^b \left(\frac{|f|}{M}\right)^p d\alpha = 1$$

and

$$\int_a^b |g|^q d\alpha = N^q \Rightarrow \int_a^b \left(\frac{|g|}{N}\right)^q d\alpha = 1.$$

Then we have by (b),

$$\int_a^b \frac{|f|}{M}\frac{|g|}{N} d\alpha \leq 1$$

which implies that, by (*)

$$\left|\int_a^b fg d\alpha\right| \leq MN = \left\{\int_a^b |f|^p d\alpha\right\}^{1/p}\left\{\int_a^b |g|^q d\alpha\right\}^{1/q}.$$

(d) Show that **Holder's inequality** is also true for the "improper" integrals.

Proof: It is clear by (c), so we omit the proof.

7.17 Assume that $f \in R(\alpha), g \in R(\alpha)$, and $f \cdot g \in R(\alpha)$ on $[a,b]$. Show that

$$\frac{1}{2}\int_a^b\left[\int_a^b (f(y) - f(x))(g(y) - g(x))d\alpha(y)\right]d\alpha(x)$$
$$= (\alpha(b) - \alpha(a))\int_a^b f(x)g(x)d\alpha(x) - \left(\int_a^b f(x)d\alpha(x)\right)\left(\int_a^b g(x)d\alpha(x)\right).$$

If $\alpha \nearrow$ on $[a,b]$, deduce the inequality

$$\left(\int_a^b f(x)d\alpha(x)\right)\left(\int_a^b g(x)d\alpha(x)\right) \leq (\alpha(b) - \alpha(a))\int_a^b f(x)g(x)d\alpha(x)$$

when both f and g are increasing (or both are decreasing) on $[a,b]$. Show that the reverse inequality holds if f increases and g decreases on $[a,b]$.

Proof: Since

$$\frac{1}{2}\int_a^b\left[\int_a^b (f(y) - f(x))(g(y) - g(x))d\alpha(y)\right]d\alpha(x)$$
$$= \frac{1}{2}\int_a^b\left[\int_a^b f(y)g(y) - f(y)g(x) - f(x)g(y) + f(x)g(x)d\alpha(y)\right]d\alpha(x)$$
$$= (\alpha(b) - \alpha(a))\int_a^b f(y)g(y)d\alpha(y) - \left(\int_a^b f(x)d\alpha(x)\right)\left(\int_a^b g(x)d\alpha(x)\right)$$

which implies that, (let α, f, and $g \nearrow$ on $[a,b]$),

$$0 \leq \frac{1}{2}\int_a^b\left[\int_a^b (f(y) - f(x))(g(y) - g(x))d\alpha(y)\right]d\alpha(x)$$

and (let α, and $f \nearrow$ on $[a,b]$, $g \searrow$ on $[a,b]$),

$$0 \geq \frac{1}{2}\int_a^b\left[\int_a^b (f(y) - f(x))(g(y) - g(x))d\alpha(y)\right]d\alpha(x),$$

we know that, (let α, f, and $g \nearrow$ on $[a,b]$)

$$\left(\int_a^b f(x)d\alpha(x)\right)\left(\int_a^b g(x)d\alpha(x)\right) \leq (\alpha(b) - \alpha(a))\int_a^b f(x)g(x)d\alpha(x)$$

and (let α, and $f \nearrow$ on $[a,b]$, $g \searrow$ on $[a,b]$)

$$\left(\int_a^b f(x)d\alpha(x)\right)\left(\int_a^b g(x)d\alpha(x)\right) \geq (\alpha(b) - \alpha(a))\int_a^b f(x)g(x)d\alpha(x).$$

7.18 Assume $f \in R(\alpha)$ on $[a,b]$. Use Exercise 7.4 to prove that the limit

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$$

exists and has the value $\int_a^b f(x)dx$. Deduce that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \frac{\pi}{4},\ \lim_{n\to\infty} \sum_{k=1}^{n} (n^2+k^2)^{-1/2} = \log\left(1+\sqrt{2}\right).$$

Proof: Since $f \in R(\alpha)$ on $[a,b]$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $\|P\| < \delta$, we have

$$\left|S(P,f) - \int_a^b f(x)dx\right| < \varepsilon.$$

For this δ, we choose n large enough so that $\frac{b-a}{n} < \delta$, that is, as $n \geq N$, we have $\frac{b-a}{n} < \delta$. So,

$$\left|S(P,f) - \int_a^b f(x)dx\right| < \varepsilon$$

which implies that

$$\left|\frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) - \int_a^b f(x)dx\right| < \varepsilon.$$

That is,

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$$

exists and has the value $\int_a^b f(x)dx$.

Since $\sum_{k=1}^{n} \frac{n}{k^2+n^2} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\left(\frac{k}{n}\right)^2+1}$, we know that by above result,

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\left(\frac{k}{n}\right)^2+1} \\ &= \int_0^1 \frac{dx}{1+x^2} \\ &= \arctan 1 - \arctan 0 \\ &= \pi/4.\end{aligned}$$

Since $\sum_{k=1}^{n} (n^2+k^2)^{-1/2} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\left[1+\left(\frac{k}{n}\right)^2\right]^{1/2}}$, we know that by above result,

$$\lim_{n\to\infty}\sum_{k=1}^{n}(n^2+k^2)^{-1/2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\left[1+\left(\frac{k}{n}\right)^2\right]^{1/2}}$$
$$= \int_0^1 \frac{dx}{(1+x^2)^{1/2}}$$
$$= \int_0^{\pi/4} \sec\theta d\theta, \text{ let } x = \tan\theta$$
$$= \int_0^{\pi/4} \sec\theta \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta} d\theta$$
$$= \int_1^{1+\sqrt{2}} \frac{du}{u}, \text{ let } \sec\theta+\tan\theta = u$$
$$= \log\left(1+\sqrt{2}\right).$$

7.19 Define

$$f(x) = \left(\int_0^x e^{-t^2}dt\right)^2,\ g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}dt.$$

(a) Show that $g'(x)+f'(x) = 0$ for all x and deduce that $f(x)+g(x) = \pi/4$.
Proof: Since

$$f'(x) = 2\left(\int_0^x e^{-t^2}dt\right)e^{-x^2}$$

and note that if $h(x,t) = \frac{e^{-x^2(t^2+1)}}{t^2+1}$, we know that h is continuous on $[0,a]\times[0,1]$ for any real $a > 0$, and $h_x = -2xe^{-x^2(t^2+1)}$ is continuous on $[0,a]\times[0,1]$ for any real $a > 0$,

$$g'(x) = \int_0^1 h_x dt$$
$$= \int_0^1 -2xe^{-x^2(t^2+1)}dt$$
$$= -2e^{-x^2}\int_0^1 xe^{-(xt)^2}dt$$
$$= -2e^{-x^2}\int_0^x e^{-u^2}du,$$

we know that $g'(x)+f'(x) = 0$ for all x. Hence, we have $f(x)+g(x) = C$ for all x, constant. Since $C = f(0)+g(0) = \int_0^1 \frac{dt}{1+t^2} = \pi/4$, $f(x)+g(x) = \pi/4$.

Remark: The reader should think it twice on how to find the auxiliary function g.

(b) Use (a) to prove that

$$\lim_{x\to\infty}\int_0^x e^{-t^2}dt = \frac{1}{2}\sqrt{\pi}.$$

Proof: Note that

$$\left|h(x,t) = \frac{e^{-x^2(t^2+1)}}{t^2+1}\right| \le |e^{-x^2(t^2+1)}| \le \frac{1}{x^2(t^2+1)} \text{ for all } x > 0;$$

we know that

$$\left|\int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}dt\right| \le \left|\frac{1}{x^2}\int_0^1 \frac{dt}{1+t^2}\right| \to 0 \text{ as } x\to\infty.$$

So, by (a), we get

$$\lim_{x\to\infty} f(x) = \pi/4$$

which implies that

$$\lim_{x\to\infty} \int_0^x e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}$$

since $\lim_{x\to\infty} \int_0^x e^{-t^2} dt$ exists by $\int_0^x e^{-t^2} dt \le \int_0^x \frac{dt}{1+t^2} = \arctan x \to \pi/2$ as $x \to \infty$.

Remark: (1) There are many methods to show this. But here is an elementary proof with help of **Taylor series and Wallis formula**. We prove it as follows. In addition, the reader will learn some beautiful and useful methods in the future. For example, use the application of **Gamma function**, and so on.

Proof: Note that two inequalities,

$$1 + x^2 \le e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \text{ for all } x$$

and

$$\sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \le \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2} \text{ if } |x| < 1$$

which implies that

$$1 - x^2 \le e^{-x^2} \text{ if } 0 \le x \le 1 \Rightarrow (1-x^2)^n \le e^{-nx^2} \tag{1}$$

and

$$e^{-x^2} \le \frac{1}{1+x^2} \text{ if } x \le 0 \Rightarrow e^{-nx^2} \le \left(\frac{1}{1+x^2}\right)^n. \tag{2}$$

So, we have, by (1) and (2),

$$\int_0^1 (1-x^2)^n dx \le \int_0^1 e^{-nx^2} dx \le \int_0^\infty e^{-nx^2} dx \le \int_0^\infty \left(\frac{1}{1+x^2}\right)^n dx. \tag{3}$$

Note that

$$\int_0^\infty e^{-nx^2} dx = \frac{1}{\sqrt{n}} \int_0^\infty e^{-x^2} dx := \frac{K}{\sqrt{n}}.$$

Also,

$$\int_0^1 (1-x^2)^n dx = \int_0^{\pi/2} \sin^{2n+1} t dt = \frac{2 \cdot 4 \cdot 6 \cdots (2n-2)(2n)}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

and

$$\int_0^\infty \left(\frac{1}{1+x^2}\right)^n dx = \int_0^{\pi/2} \sin^{2n-2} t dt = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} \frac{\pi}{2},$$

so

$$\sqrt{n} \frac{2 \cdot 4 \cdot 6 \cdots (2n-2)(2n)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \le K \le \sqrt{n} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)} \frac{\pi}{2}$$

which implies that

$$\frac{n}{2n+1} \frac{[2 \cdot 4 \cdot 6 \cdots (2n-2)(2n)]^2}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]^2 (2n+1)} \le K^2 \le \frac{n}{2n-1} \frac{[1 \cdot 3 \cdot 5 \cdots (2n-3)]^2 (2n-1)}{[2 \cdot 4 \cdot 6 \cdots (2n-2)]^2} \left(\frac{\pi}{2}\right)^2 \tag{4}$$

By **Wallis formula**, we know that, by (4)

$$K = \frac{\sqrt{\pi}}{2}.$$

That is, we have proved that **Euler-Possion Integral**

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

Note: (**Wallis formula**)

$$\lim_{n\to\infty} \frac{[2 \cdot 4 \cdot 6 \cdots (2n-2)(2n)]^2}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]^2(2n+1)} = \frac{\pi}{2}.$$

Proof: As $0 \le x \le \pi/2$, we have

$$\sin^{2n+1}t \le \sin^{2n}t \le \sin^{2n-1}t, \text{ where } n \in N.$$

So, we know that

$$\int_0^{\pi/2} \sin^{2n+1}tdt \le \int_0^{\pi/2} \sin^{2n}tdt \le \int_0^{\pi/2} \sin^{2n-1}tdt$$

which implies that

$$\frac{(2n)(2n-2)\cdots 4 \cdot 2}{(2n+1)(2n-1)\cdots 3 \cdot 1} \le \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{(2n)(2n-2)\cdots 4 \cdot 2}\frac{\pi}{2} \le \frac{(2n-2)(2n-4)\cdots 4 \cdot 2}{(2n-1)(2n-3)\cdots 3 \cdot 1}.$$

So,

$$\left[\frac{(2n)(2n-2)\cdots 4 \cdot 2}{(2n-1)(2n-3)\cdots 3 \cdot 1}\right]^2 \frac{1}{2n+1} \le \frac{\pi}{2} \le \left[\frac{(2n)(2n-2)\cdots 4 \cdot 2}{(2n-1)(2n-3)\cdots 3 \cdot 1}\right]^2 \frac{1}{2n}.$$

Hence, from

$$\left[\frac{(2n)(2n-2)\cdots 4 \cdot 2}{(2n-1)(2n-3)\cdots 3 \cdot 1}\right]^2 \left(\frac{1}{2n} - \frac{1}{2n+1}\right) \le \frac{1}{2n}\frac{\pi}{2} \to 0,$$

we know that

$$\lim_{n\to\infty} \frac{[2 \cdot 4 \cdot 6 \cdots (2n-2)(2n)]^2}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]^2(2n+1)} = \frac{\pi}{2}.$$

(2) Here is another exercise from **Hadamard**'s result. We Write it as follows. Let $f \in C^k(R)$ with $f(0) = 0$. Prove that there exists an unique function $g \in C^{k-1}(R)$ such that $f = xg(x)$ on R.

Proof: Consider

$$\begin{aligned} f(x) &= f(x) - f(0) \\ &= \int_0^1 df(xt) \\ &= \int_0^1 xf'(xt)dt \\ &= x\int_0^1 f'(xt)dt; \end{aligned}$$

we know that if $g(x) := \int_0^1 f'(xt)dt$, then we have prove it.

Note: In fact, we can do this job by rountine work. Define

$$g(x) = \begin{cases} \frac{f(x)}{x} \text{ if } x \ne 0 \\ 0 \text{ if } x = 0. \end{cases}$$

However, it is too long to write. The trouble is to make sure that $g \in C^{k-1}(R)$.

7.20 Assume $g \in R$ on $[a,b]$ and define $f(x) = \int_a^x g(t)dt$ if $t \in [a,b]$. Prove that the integral $\int_a^x |g(t)|dt$ gives the total variation of f on $[a,x]$.

Proof: Since $\int_a^x |g(t)|dt$ exists, given $\varepsilon > 0$, there exists a partition $P_1 = \{x_0 = a, \ldots, x_n = x\}$ such that

$$L(P,|g|) > \int_a^x |g(t)|dt - \varepsilon. \qquad 1$$

So, for this P_1, we have

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} g(t)dt \right| = \sum_{k=1}^{n} |c_k(x_k - x_{k-1})| \text{ by } \textbf{Mean Value Theorem} \qquad 2$$

where $\inf_{x\in[x_{k-1},x_k]} |g(x)| \leq c_k \leq \sup_{x\in[x_{k-1},x_k]} |g(x)|$.

Hence, we know that, by (1) and (2),

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| > \int_a^x |g(t)|dt - \varepsilon$$

which implies that

$$V_f(a,b) \geq \int_a^x |g(t)|dt$$

since ε is arbitrary.

Conversely, since $\int_a^x |g(t)|dt$ exists, given $\varepsilon > 0$, there exists a partition P_2 such that

$$U(P_2,|g|) < \int_a^x |g(t)|dt + \varepsilon/2. \qquad 3$$

Also, for the same ε, there exists a partition $P_3 = \{t_0 = a, \ldots, t_m = x\}$ such that

$$V_f(a,b) - \varepsilon/2 < \sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|. \qquad 4$$

Let $P = P_2 \cup P_3 = \{s_0 = a, \ldots, s_p = x\}$, then by (3) and (4), we have

$$U(P,|g|) < \int_a^x |g(t)|dt + \varepsilon/2$$

and

$$\begin{aligned} V_f(a,b) - \varepsilon/2 &< \sum_{k=1}^{p} |f(s_k) - f(s_{k-1})| \\ &= \sum_{k=1}^{p} \left| \int_{s_{k-1}}^{s_k} g(t)dt \right| \\ &= \sum_{k=1}^{p} |\tilde{c}_k(x_k - x_{k-1})| \\ &\leq U(P,|g|) \end{aligned}$$

which imply that

$$V_f(a,b) \leq \int_a^x |g(t)|dt$$

since ε is arbitrary.

Therefore, from above discussion, we have proved that

$$V_f(a,b) = \int_a^x |g(t)|dt.$$

7.21 If $f = (f_1, \ldots, f_n)$ be a vector-valued function with a continuous derivative f' on

$[a,b]$. Prove that the curve described by f has length

$$\Lambda_f(a,b) = \int_a^b \|f'(t)\|dt.$$

Proof: Since $f' = (f_1',\ldots,f_n')$ is continuous on $[a,b]$, we know that $\left[\sum_{j=1}^n (f_j')^2(t)\right]^{1/2} = \|f'(t)\|$ is uniformly continuous on $[a,b]$. So, given $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that as $|x-y| < \delta_1$, where $x,y \in [a,b]$, we have

$$|\|f'(x)\| - \|f'(y)\|| < \frac{\varepsilon}{3(b-a)}. \quad (1)$$

Since $\|f'(t)\| \in R$ on $[a,b]$, for the same ε, there exists $\delta_2 > 0$ such that as $\|P_1\| < \delta_2$, where $P_1 = \{x_0 = a,\ldots,x_n = b\}$ we have

$$\left|S(P_1,\|f'\|) - \int_a^b \|f'(t)\|dt\right| < \varepsilon/3, \text{ where } S(P_1,\|f'\|) = \sum_{j=1}^n \|f'(t_j)\|\Delta x_j \quad (2)$$

and $\Lambda_f(a,b)$ exists by **Theorem 6.17**, for the same ε, there exists a partition $P_2 = \{s_0 = a,\ldots,s_m = b\}$ such that

$$\begin{aligned}\Lambda_f(a,b) - \varepsilon/3 &< \sum_{k=1}^m \|f(s_k) - f(s_{k-1})\| \\ &= \sum_{k=1}^m \left\{\sum_{j=1}^n [(f_j)(s_k) - (f_j)(s_{k-1})]^2\right\}^{1/2}. \quad (3)\end{aligned}$$

Let $\delta = \min(\delta_1,\delta_2)$ and $P \subseteq P_2$ so that $\|P\| < \delta$, where $P = \{y_0 = a,\ldots,y_q = b\}$ then by (1)-(3), we have

(i) As $|x-y| < \delta$, where $x,y \in [a,b]$, we have

$$|\|f'(x)\| - \|f'(y)\|| < \frac{\varepsilon}{3(b-a)}. \quad (4)$$

(ii) As $\|P\| < \delta$, we have

$$\left|S(P,\|f'\|) - \int_a^b \|f'(t)\|dt\right| < \varepsilon/3, \text{ where } S(P,\|f'\|) = \sum_{j=1}^q \|f'(\tilde{t}_j)\|\Delta y_j \quad (5)$$

(iii) As $\|P\| < \delta$, we have

$$\begin{aligned}\Lambda_f(a,b) - \varepsilon/3 &\le \sum_{k=1}^m \left\{\sum_{j=1}^n [(f_j)(s_k) - (f_j)(s_{k-1})]^2\right\}^{1/2} \\ &\le \sum_{k=1}^q \left\{\sum_{j=1}^n [(f_j)(y_k) - (f_j)(y_{k-1})]^2\right\}^{1/2} \\ &= \sum_{k=1}^q \left\{\sum_{j=1}^n [f_j'(z_k)]^2\right\}\Delta y_j, \text{ by } \textbf{Mean Value Theorem} \\ &= \sum_{k=1}^q \|f'(z_k)\|\Delta y_j \\ &\le \Lambda_f(a,b) \quad (6)\end{aligned}$$

By (ii) and (iii), we have

$$\begin{aligned}\left|\sum_{k=1}^{q} g(z_k)\Delta y_j - S(P,g)\right| &= \left|\sum_{k=1}^{q} g(z_k)\Delta y_j - \sum_{j=1}^{q} g(\tilde{t}_j)\Delta y_j\right| \\ &\leq \sum_{k=1}^{q} |g(z_k) - g(\tilde{t}_j)|\Delta y_j \\ &< \sum_{k=1}^{q} \frac{\varepsilon}{3(b-a)}\Delta y_j \\ &= \varepsilon/3. \end{aligned} \quad 7$$

Hence, (5)-(7) implies that

$$\left|\int_a^b \|f'(t)\| dt - \Lambda_f(a,b)\right| < \varepsilon.$$

Since ε is arbitrary, we have proved that

$$\Lambda_f(a,b) = \int_a^b \|f'(t)\| dt.$$

7.22 If $f^{(n+1)}$ is continuous on $[a,x]$, define

$$I_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt.$$

(a) Show that

$$I_{k-1}(x) - I_k(x) = \frac{f^{(k)}(a)(x-a)^k}{k!},\ k = 1,2,\ldots,n.$$

Proof: Since, for $k = 1,2,\ldots,n$,

$$\begin{aligned} I_k(x) &= \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)dt \\ &= \frac{1}{k!}\int_a^x (x-t)^k df^{(k)}(t) \\ &= \frac{1}{k!}\left[(x-t)^k f^{(k)}(t)\Big|_a^x + k\int_a^x (x-t)^{k-1} f^{(k)}(t)dt\right] \\ &= -\frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{(k-1)!}\int_a^x (x-t)^{k-1} f^{(k)}(t)dt \\ &= -\frac{f^{(k)}(a)(x-a)^k}{k!} + I_{k-1}(x), \end{aligned}$$

we know that

$$I_{k-1}(x) - I_k(x) = \frac{f^{(k)}(a)(x-a)^k}{k!},\ \text{for } k = 1,2,\ldots,n.$$

(b) Use (a) to express the remainder in Taylor's formula (Theorem 5.19) as an integral.

Proof: Since $f(x) - f(a) = I_0(x)$, we know that

$$\begin{aligned} f(x) &= f(a) + I_0(x) \\ &= f(a) + \sum_{k=1}^{n}[I_{k-1}(x) - I_k(x)] + I_n(x) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt \text{ by (a).} \end{aligned}$$

So, by **Taylor's formula**, we know that

$$R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt, \text{ for some } c \in (a,x).$$

where $R_n(x)$ is the remainder term.

Remark: 1. The reader should be noted that with help of **Mean Value Theorem**, we have

$$\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}. \quad *$$

2. Use **Integration by parts** repeatedly; we can show (*). Of course, there is other proofs such as **Mathematical Induction**.

Proof: Since

$$\int uv^{(n+1)}dt = uv^{(n)} - u'v^{(n-1)} + u''v^{(n-2)} - \ldots + (-1)^n u^{(n)}v + (-1)^{(n+1)}\int u^{(n+1)}vdt,$$

letting $v(t) = (x-t)^n$ and $u(t) = f(t)$, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt.$$

Note: The reader should give it a try to show it. Since it is not hard, we omit the detail.

3. The remainder term as an integral is useful; the reader should see the textbook in **Ch9, pp242-244**.

4. There is a good exercise related with an application of Taylor's Remainder. We write it as a reference.

Let $u''(t) + f(t)u(t) = 0$, where $f(t)$ is continuous and non-negative on $[0,c]$ If u is defined and not a zero function on $[0,c]$ and

$$\int_a^b (b-t)(a-t)f(t) < b-a \text{ for all } a,b \in [0,c], \text{ where } a < b. \quad *$$

Then u at most has one zero on $[0,c]$.

Proof: First, we note that u has at most finitely many zeros in the interval $[0,c]$ by **uniqueness theorem on O.D.E.** So, let $u(a) = u(b) = 0$, where $a,b \in [0,c]$ with $a < b$, and no point $y \in (a,b)$ such that $u(y) = 0$. Consider $[a,b]$ and by **Taylor's Theorem with Remainder Term as an integral**, we have

$$\begin{aligned} u(x) &= u(a) + u'(a)(x-a) + \int_a^x (x-t)u''(t)dt \\ &= u'(a)(x-a) + \int_a^x (x-t)u''(t)dt \\ &= u'(a)(x-a) - \int_a^x (x-t)u(t)f(t)dt. \quad ** \end{aligned}$$

Note that $u(x)$ is positive on (a,b) (Or, $u(x)$ is negative on (a,b)) So, we have

$$|u(x)| \leq |u'(a)|(x-a). \qquad ***$$

By (**),

$$0 = u(b) = u'(a)(b-a) - \int_a^b (b-t)u(t)f(t)dt$$

which implies that

$$u'(a)(b-a) = \int_a^b (b-t)u(t)f(t)$$

which implies that by (***), and note that $u'(a) \neq 0$,

$$b-a \leq \int_a^b (b-t)(t-a)f(t)dt$$

which contradicts to (*). So, u at most has one zero on $[0,c]$.

Note: (i) In particular, let $f(t) = e^{-t}$, we have (*) holds.

Proof: Since

$$\int_a^b (b-t)(t-a)e^{-t}dt = e^{-a}(-2+b-a) + e^{-b}(2+b-a)$$

by **integration by parts** twice, we have, (let $b-a = x$),

$$\begin{aligned} &e^{-a}(-2+b-a) + e^{-b}(2+b-a) - (b-a) \\ &= e^{-a}(-2+x) + e^{-x-a}(2+x) - x \\ &= x(e^{-a}-1) + e^{-a-x}(-2e^{x} + (x+2)) \\ &< 0 \text{ since } a < b \text{ and } e^{x} > 1+x. \end{aligned}$$

(ii) In the proof of exercise, we use the **uniqueness theorem**: If $p(x)$ and $q(x)$ are continuous on $[0,a]$, then

$$y'' + p(x)y' + q(x)y = 0, \text{ where } y(0) = y_0, \text{ and } y'(0) = y_0'$$

has one and only one solution. In particular, if $y(0) = y'(0) = 0$, then $y = 0$ on $[0,a]$ is the only solution. We do NOT give a proof; the reader can see the book, **Theory of Ordinary Differential Equation by Ince, section 3.32, or Theory of Ordinary Differential Equation by Coddington and Levison, Chapter 6**.

However, we need use the **uniqueness theorem** to show that u (in the exercise) has at most finitely many zeros in $[0,c]$.

Proof: Let $S = \{x : u(x) = 0, x \in [0,c]\}$. If $\#(S) = \infty$, then by **Bolzano-Weierstrass Theorem**, S has an accumulation point p in $[0,c]$. Then $u(p) = 0$ by continuity of u. In addition, let $r_n \to p$, and $u(r_n) = 0$, then

$$u'(p) = \lim_{x \to p} \frac{u(x)-u(p)}{x-p} = \lim_{n \to \infty} \frac{u(r_n)-u(p)}{r_n - p} = 0.$$

(Note that if p is the endpoint of $[0,c]$, we may consider $x \to p^+$ or $x \to p^-$). So, by **uniqueness theorem,** we then have $u = 0$ on $[0,c]$ which contradicts to the hypothesis, u is not a zero function on $[0,c]$. So, $\#(S) < \infty$.

7.23 Let f be continuous on $[0,a]$. If $x \in [0,a]$, define $f_0(x) = f(x)$ and let

$$f_{n+1}(x) = \frac{1}{n!}\int_0^x (x-t)^n f(t)dt, \; n = 0,1,2,\ldots$$

(a) Show that the nth derivative of f_n exists and equals f.

Proof: Consider, by **Chain Rule**,

$$f_n' = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt = f_{n-1} \text{ for all } n \in N,$$

we have

$$f_n^{(n)} = f.$$

That is, nth derivative of f_n exists and equals f.

Remark: (1) There is another proof by **Mathematical Induction** and **Integration by parts**. It is not hard; we omit the proof.

(2) The reader should note that the exercise tells us that given any continuous function f on $[a,b]$, there exists a function g_n on $[a,b]$ such that $g_n^{(n)} = f$, where $n \in N$. In fact, the function

$$g_n = \frac{1}{n!}\int_a^x (x-t)^n f(t)dt, n = 0,1,2,\ldots$$

(3) The reader should compare the exercise with 7.22. At the same time, look at two integrands in both exercises.

(b) Prove the following **theorem of M. Fekete**: The number of changes in sign of f in $[0,a]$ is not less than the number of changes in sign in the ordered set of numbers

$$f(a), f_1(a), \ldots, f_n(a).$$

Hint: Use **mathematical induction**.

Proof: Let $T(f)$ denote the number of changes in sign of f on $[0,a]$ and $S_n(f)$ the number of changes in sign in the ordered set of numbers

$$f(a), f_1(a), \ldots, f_n(a).$$

We prove $T(f) \geq S_n(f)$ for each n by **Mathematical Induction** as follows. Note that $S_n(f) \leq n$.

As $n = 1$, if $S_1(f) = 0$, then there is nothing to prove it. If $S_1(f) = 1$, it means that $f(a)f_1(a) < 0$. Without loss of generality, we may assume that $f(a) > 0$, so $f_1(a) < 0$ which implies that

$$0 > f_1(a) = \frac{1}{0!}\int_0^a f(t)dt$$

which implies that there exists a point $y \in [0,a)$ such that $f(y) < 0$. Hence, $T(f) \geq S_1(f)$ holds for any continuous functions defined on $[0,a]$.

Assume that $n = k$ holds for any continuous functions defined on $[0,a]$, As $n = k+1$,. we consider the ordered set of numbers

$$f(a), f_1(a), \ldots, f_k(a), f_{k+1}(a).$$

Note that

$$f_{n+1}(a) = (f_1)_n(a) \text{ for all } n \in N,$$

so by induction hypothesis,

$$T(f_1) \geq S_k(f_1)$$

Suppose $S_k(f_1) = p$, and $f_1(0) = 0$, then $f_1' = f$ at least has p zeros by **Rolle's Theorem**. Hence,

$$T(f) \geq T(f_1) \geq S_k(f_1) = p \qquad *$$

We consider two cases as follows.

(i) $f(a)f_1(a) \geq 0$:With help of (*),

$$T(f) \geq S_k(f_1) = S_{k+1}(f).$$

(ii) $f(a)f_1(a) < 0$:Claim that

$$T(f) > S_k(f_1) = p$$

as follows. Suppose **NOT**, it means that $T(f) = T(f_1) = p$ by (*). Say

$$f(a_1) = f(a_2) = \ldots = f(a_p) = 0, \text{ where } 0 < a_1 < a_2 < \ldots < a_p < 1.$$

and

$$f_1(b_1) = f_1(b_2) = \ldots = f_1(b_p) = 0, \text{ where } 0 < b_1 < b_2 < \ldots < b_p < 1.$$

By $f(a)f_1(a) < 0,$ we know that

$$f(x)f_1(x) < 0 \text{ where } x \in (0,c),\ c = \min(a_1,b_1)$$

which is impossible since

$$\begin{aligned} f(x)f_1(x) &= f(x)[f_1(x) - f_1(0)] \text{ by } f_1(0) = 0 \\ &= f(x)f_1'(y), \text{ where } y \in (0,x) \subseteq (0,c) \\ &= f(x)f(y) \\ &> 0 \text{ since } f(x) \text{ and } f(y) \text{ both positive or negative.} \end{aligned}$$

So, we obtain that $T(f) > S_k(f_1) = p.$ That is, $T(f) \geq S_k(f_1) + 1 = S_{k+1}(f).$

From above results, we have proved it by **Mathmatical Induction**.

(c) Use (b) to prove the following theorem of **Feje'r**: The number of changes in sign of f in $[0,a]$ is not less than the number of changes in sign in the ordered set

$$f(0), \int_0^a f(t)dt, \int_0^a tf(t)dt, \ldots, \int_0^a t^n f(t)dt.$$

Proof: Let $g(x) = f(a-x),$ then, define $g_0(x) = g(x),$ and for $n = 0,1,2,\ldots,$

$$\begin{aligned} g_{n+1}(a) &= \frac{1}{n!}\int_0^a (a-t)^n g(t)dt \\ &= \frac{1}{n!}\int_0^a u^n f(u)du \text{ by change of variable } (u = a-t). \end{aligned}$$

So, by (b), the number of changes in sign of g in $[0,a]$ is not less than the number of changes in sign in the ordered set

$$g(a), g_1(a), \ldots, g_{n+1}(a).$$

That is, the number of changes in sign of g in $[0,a]$ is not less than the number of changes in sign in the ordered set

$$f(0), \int_0^a f(t)dt, \int_0^a tf(t)dt, \ldots, \int_0^a t^n f(t)dt.$$

Note that the number of changes in sign of g in $[0,a]$ equals the number of changes in sign of f in $[0,a],$ so we have proved the **Feje'r Theorem.**

7.24 Let f be a positive continuous function in $[a,b].$ Let M denote the maximum value of f on $[a,b].$ Show that

$$\lim_{n\to\infty}\left(\int_a^b f(x)^n dx\right)^{1/n} = M.$$

Proof: Since f is a positive continuous function in $[a,b],$ there exists a point $c \in [a,b]$ such that $f(c) = M = \sup_{x\in[a,b]} f(x) > 0.$ Then given $(M >)\varepsilon > 0,$ there is a $\delta > 0$ such that as $x \in B(c,\delta) \cap [a,b] := I,$ we have

$$(0 <)M - \varepsilon < f(x) < M + \varepsilon.$$

Hence, we have

$$|I|^{1/n}(M-\varepsilon) \le \left(\int_I f^n(x)dx\right)^{1/n} \le \left(\int_a^b f(x)^n dx\right)^{1/n} \le (b-a)^{1/n}M$$

which implies that

$$M-\varepsilon \le \lim_{n\to\infty}\inf\left(\int_a^b f(x)^n dx\right)^{1/n} \le M.$$

So, $\lim_{n\to\infty}\inf\left(\int_a^b f(x)^n dx\right)^{1/n} = M$ since ε is arbitrary. Similarly, we can show that $\lim_{n\to\infty}\sup\left(\int_a^b f(x)^n dx\right)^{1/n} = M.$ So, we have proved that $\lim_{n\to\infty}\left(\int_a^b f(x)^n dx\right)^{1/n} = M.$

Remark: There is good exercise; we write it as a reference. Let $f(x)$ and $g(x)$ are continuous and non-negative function defined on $[a,b]$. Then

$$\lim_{n\to\infty}\left(\int_a^b f(x)^n g(x)dx\right)^{1/n} = \max_{x\in[a,b]} f(x).$$

Since the proof is similar, we omit it. (The reader may let $\alpha(x) = \int_a^x g(t)dt$).

7.25 A function f of two real variables is defined for each point (x,y) in the unit square $0 \le x \le 1,\ 0 \le y \le 1$ as follows:

$$f(x,y) = \begin{cases} 1 \text{ if } x \text{ is rational,} \\ 2y \text{ if } x \text{ is irrational.} \end{cases}$$

(a) Compute $\overline{\int_0^1} f(x,y)dx$ and $\underline{\int_0^1} f(x,y)dx$ in terms of y.

Proof: Consider two cases for **upper and lower Riemann-Stieltjes integrals** as follows.

(i) As $y \in [0,1/2]$: Given any partition $P = \{x_0 = 0,\ldots,x_n = 1\}$, we have

$$\sup_{x\in[x_{j-1},x_j]} f(x,y) = 1, \text{ and } \inf_{x\in[x_{j-1},x_j]} f(x,y) = 2y.$$

Hence, $\overline{\int_0^1} f(x,y)dx = 1$, and $\underline{\int_0^1} f(x,y)dx = 2y$.

(ii) As $y \in (1/2,1]$: Given any partition $P = \{x_0 = 0,\ldots,x_n = 1\}$, we have

$$\sup_{x\in[x_{j-1},x_j]} f(x,y) = 2y, \text{ and } \inf_{x\in[x_{j-1},x_j]} f(x,y) = 1.$$

Hence, $\overline{\int_0^1} f(x,y)dx = 2y$, and $\underline{\int_0^1} f(x,y)dx = 1$.

(b) Show that $\int_0^1 f(x,y)dy$ exists for each fixed x and compute $\int_0^t f(x,y)dy$ in terms of x and t for $0 \le x \le 1,\ 0 \le t \le 1$.

Proof: If $x \in Q \cap [0,1]$, then $f(x,y) = 1$. And if $x \in Q^c \cap [0,1]$, then $f(x,y) = 2y$. So, for each fixed x, we have

$$\int_0^1 f(x,y)dy = \int_0^1 1dy = 1 \text{ if } x \in Q \cap [0,1]$$

and

$$\int_0^1 f(x,y)dy = \int_0^1 2ydy = 1 \text{ if } x \in Q^c \cap [0,1].$$

In addition,

$$\int_0^t f(x,y)dy = \int_0^t 1dy = t \text{ if } x \in Q \cap [0,1]$$

and

$$\int_0^t f(x,y)dy = \int_0^t 2ydy = t^2 \text{ if } x \in Q^c \cap [0,1].$$

(c) Let $F(x) = \int_0^1 f(x,y)dy$. Show that $\int_0^1 F(x)dx$ exists and find its value.

Proof: By (b), we have

$$F(x) = 1 \text{ on } [0,1].$$

So, $\int_0^1 F(x)dx$ exists and

$$\int_0^1 F(x)dx = 1.$$

7.26 Let f be defined on $[0,1]$ as follows: $f(0) = 0$; if $2^{-n-1} < x \leq 2^{-n}$, then $f(x) = 2^{-n}$ for $n = 0,1,2,\ldots$

(a) Give two reasons why $\int_0^1 f(x)dx$ exists.

Proof: (i) $f(x)$ is monotonic decreasing on $[0,1]$. (ii) $\{x : f \text{ is discontinuous at } x\}$ has measure zero.

Remark: We compute the value of the integral as follows.

Solution: Consider the interval $I_n = [2^{-n},1]$ where $n \in N$, then we have $f \in R$ on I_n for each n, and

$$\begin{aligned}
\int_{2^{-n}}^1 f(x)dx &= \sum_{k=1}^n \int_{2^{-k}}^{2^{-k+1}} f(x)dx \\
&= \sum_{k=1}^n 2^{-k+1} \int_{2^{-k}}^{2^{-k+1}} dx \\
&= \sum_{k=1}^n (2^{-k+1})(2^{-k}) \\
&= 2\frac{\frac{1}{4}\left[1-\left(\frac{1}{4}\right)^n\right]}{1-\frac{1}{4}} \\
&= \frac{2}{3}\left[1-\left(\frac{1}{4}\right)^n\right] \to \frac{2}{3} \text{ as } n \to \infty.
\end{aligned}$$

So, th integral $\int_0^1 f(x)dx = \frac{2}{3}$.

Note: In the remark, we use the following fact. If $f \in R$ on $[a,b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_{a_n}^b f(x)dx$$

where $\{a_n\}$ is a sequence with $a_n \to a$, and $a_n \geq a$ for all n.

Proof: Since $a_n \to a$, given $\varepsilon > 0$, there is a positive integer N such that as $n \geq N$, we have

$$|a_n - a| < \varepsilon/M, \text{ where } M = \sup_{x\in[a,b]} |f(x)|$$

So,

$$\left|\int_a^b f(x)dx - \int_{a_n}^b f(x)dx\right| = \left|\int_a^{a_n} f(x)dx\right|$$
$$\leq M|a_n - a|$$
$$< \varepsilon.$$

That is, $\int_a^b f(x)dx = \lim_{n\to\infty} \int_{a_n}^b f(x)dx$.

(b) Let $F(x) = \int_0^x f(t)dt$. Show that for $0 < x \leq 1$ we have

$$F(x) = xA(x) - \frac{1}{3}A(x)^2,$$

where $A(x) = 2^{-[-\log x/\log 2]}$ and where $[y]$ is the greatest integer in y.

Proof: First, we note that $F(x) = \int_0^1 f(t)dt - \int_x^1 f(t)dt = \frac{2}{3} - \int_x^1 f(t)dt$. So, it suffices to consider the value of the integral

$$\int_x^1 f(t)dt.$$

Given any $x \in [0,1]$, then there exists a positive N such that $2^{-N-1} < x \leq 2^{-N}$. So,

$$\int_x^1 f(t)dt = \int_{2^{-N}}^1 f(t)dt + \int_x^{2^{-N}} f(t)dt$$
$$= \frac{2}{3}\left[1 - \left(\frac{1}{4}\right)^N\right] + 2^{-N}(2^{-N} - x) \text{ by Remark in (a)}$$
$$= \frac{2}{3} + \frac{1}{3}\left(\frac{1}{4}\right)^N - \left(\frac{1}{2}\right)^N x.$$

So,

$$F(x) = \left(\frac{1}{2}\right)^N x - \frac{1}{3}\left(\frac{1}{4}\right)^N$$
$$= 2^{-N}x - \frac{1}{3}2^{-2N}$$
$$= xA(x) - \frac{1}{3}A(x)^2$$

where $A(x) = 2^{-N}$. Note that $2^{-N-1} < x \leq 2^{-N}$, we have

$$N = \left[\log_{1/2} x\right], \text{ where } [y] \text{ is the Gauss symbol.}$$

Hence,

$$A(x) = 2^{-[-\log x/\log 2]}.$$

Remark: (1) The reader should give it a try to show it directly by considering $[0,x]$, where $0 \leq x \leq 1$.

(2) Here is a good exercise. We write it as a reference. Suppose that f is defined on $[0,1]$ by the following

$$f(x) = \begin{cases} \frac{1}{2^n} \text{ if } x = \frac{j}{2^n} \text{ where } j \text{ is an odd integer and } 0 < j < 2^n,\ n = 1,2,\ldots, \\ 0 \text{ otherwise.} \end{cases}$$

Show that $f \in R$ on $[0,1]$ and has the value of the integral 0.

Proof: In order to show this, we consider the Riemann's condition with respect to $\alpha(x) = x$ as follows. Given a partition
$P = \left\{x_0 = 0 = \frac{0}{2^n}, x_1 = \frac{1}{2^n}, x_2 = \frac{2}{2^n}, \ldots, x_j = \frac{j}{2^n}, \ldots, x_{2^n} = \frac{2^n}{2^n} = 1\right\}$, then the upper sum

$$\begin{aligned} U(P,f) &= \sum_{k=1}^{2^n} M_k \Delta x_k \\ &= \frac{1}{2^n} \sum_{k=1}^{2^n} M_k \\ &= \frac{1}{2^n}\left(\frac{2}{2^n} + n - 1\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

So, f satisfies the Riemann's condition on $[0,1]$.

Note: (1) The reader should give it a try to show that the set of discontinuities of f has measure zero. Thus by **Theorem 7.48 (Lebesgue's Criterion for Riemann Integral)**, we know that $f \in R$ on $[0,1]$. In addition, by the fact, the lower Riemann integral equals the Riemann integral, we know that its integral is zero.

(2) For the existence of Riemann integral, we summarize to be the theorem: Let f be a bounded function on $[a,b]$. Then the following statements are equivalent:
(i) $f \in R$ on $[a,b]$.
(ii) f satisfies Riemann's condition on $[a,b]$.
(iii) $\int_a^{\bar{b}} f(x)dx = \int_{a_-}^{b} f(x)dx$
(iv) the set of discontinuities of f on I has measure zero.

P.S.: The reader should see the textbook, **pp 391**; we have the general discussion.

7.27 Assume f has a derivative which is monotonic decreasing and satisfies $f'(x) \geq m > 0$ for all x in $[a,b]$. Prove that

$$\left|\int_a^b \cos f(x)dx\right| \leq \frac{2}{m}.$$

Hint: Multiply and divide the integrand by $f'(x)$ and use **Theorem 7.37**(ii).

Proof: Since $f'(x) \geq m > 0$, and $\frac{1}{f'}$ is monotonic increasing on $[a,b]$, we consider

$$\begin{aligned} \int_a^b \cos f(x)dx &= \int_a^b \frac{\cos f(x)}{f'(x)} f'(x)dx \\ &= \frac{1}{f'(b)} \int_c^b [\cos f(x)]f'(x)dx, \text{ by } \textbf{Theorem 7.37}\text{(ii)} \\ &= \frac{1}{f'(b)} \int_{f(c)}^{f(b)} \cos u du, \text{ by } \textbf{Change of Variable} \\ &= \frac{\sin f(b) - \sin f(c)}{f'(b)} \end{aligned}$$

which implies that

$$\left|\int_a^b \cos f(x)dx\right| \leq \frac{2}{m}.$$

7.28 Given a decreasing seq uence of real numbers $\{G(n)\}$ such that $G(n) \to 0$ as $n \to \infty$. Define a function f on $[0,1]$ in terms of $\{G(n)\}$ as follows: $f(0) = 1$; if x is irrational, then $f(x) = 0$; if x is rational m/n (in lowest terms), then $f(m/n) = G(n)$. Compute the oscillation $\omega_f(x)$ at each x in $[0,1]$ and show that $f \in R$ on $[0,1]$.

Proof: Let $x_0 \in Q^c \cap [0,1]$. Since $\lim_{n\to\infty} G(n) = 0$, given $\varepsilon > 0$, there exists a positive integer K such that as $n \geq K$, we have $|G(n)| < \varepsilon$. So, there exists a finite number of positive integers n such that $G(n) \geq \varepsilon$. Denote $S = \{x : |f(x)| \geq \varepsilon\}$, then $\#(S) < \infty$. Choose a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta)(\subseteq [0,1])$ does **NOT** contain all points of S. Note that $f(x_0) = 0$. Hence, we know that f is continuous at x_0. That is, $\omega_f(x) = 0$ for all $x \in Q^c \cap [0,1]$.

Let $x_0 = 0$, then it is clear that $\omega_f(0) = 1(= f(0)) > 0$. So, f is not continuous at $0. \in Q \cap [0,1]$.

Let $x_0 \in Q \cap (0,1]$, say $x_0 = \frac{M}{N}$ (in lowest terms). Since $\{G(n)\}$ is monotonic decreasing, there exists a finite number of positive integers n such that $G(n) \geq G(N)$. Denote $T = \{x : |f(x)| \geq G(N)\}$, then $\#(T) < \infty$. Choose a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \cap [0,1]$ does **NOT** contain all points of T. Let $h \in (0,\delta)$, then

$$\sup\{f(x) - f(y) : x, y \in (x_0 - h, x_0 + h) \cap [0,1]\} = f(x_0) = G(N).$$

So, $\omega_f(x_0) = G(N)$. That is, $\omega_f(x) = f(x)$ for all $x \in Q \cap (0,1]$.

Remark: (1) If we have proved f is continuous on $Q^c \cap [0,1]$, then f is automatically Riemann integrable on $[0,1]$ since $D(\subseteq Q \cap [0,1] \subseteq Q)$, the set of discontinuities of f has measure zero.

(2) Here is a good exercise. We write it as a reference. Given a function f defined on (a,b), then the set of continuities of f on (a,b) is G_δ set.

Proof: Let C denote the set of continuities of f on (a,b), then

$$C = \{x : \omega_{f(x)} = 0\}$$
$$= \cap_{k=1}^{\infty} \{x : \omega_f(x) < 1/k\}$$

and $\{x : \omega_{f(x)} < 1/k\}$ is open. We know that C is a G_δ set.

Note: (i) We call S a G_δ set if $S = \cap_{n=1}^{\infty} O_n$, where O_n is open for each n.

(ii) Given $y \in \{x \in (a,b) : \omega_f(x) < 1/k\} := I$, then $\omega_f(y) < 1/k$. Hence, there exists a $d > 0$, such that

$$\Omega_f(B(y,d)) < 1/k, \text{ where } B(y,d) \subseteq (a,b)$$

For $z \in B(y,d)$, consider a smaller δ so that $B(z,\delta) \subseteq B(y,d)$. Hence,

$$\Omega_f(B(z,\delta)) < 1/k$$

which implies that

$$\omega_f(z) < 1/k.$$

Hence, $B(y,d) \subseteq I$. That is, y is an interior point of I. That is, I is open since every point of I is interior.

7.29 Let f be defined as in Exercise 7.28 with $G(n) = 1/n$. Let $g(x) = 1$ if $0 < x \leq 1$, $g(0) = 0$. Show that the composite function h defined by $h(x) = g[f(x)]$ is not Riemann-integrable on $[0,1]$, although both $f \in R$ and $g \in R$ on $[0,1]$.

Proof: By **Exercise 7.28,** we know that

$$h(x) = \begin{cases} 0 \text{ if } x \in Q^c \cap [0,1] \\ 1 \text{ if } x \in Q \cap [0,1] \end{cases}$$

which is discontinuous everywhere on $[0,1]$. Hence, the function h (**Dirichlet Function**) is not Riemann-integrable on $[0,1]$.

7.30 Use Lebesgue's theorem to prove Theorem 7.49.

(a) If f is of bounded variation on $[a,b]$, then $f \in R$ on $[a,b]$.

Proof: Since f is of bounded variation on $[a,b]$, by **Theorem 6.13 (Jordan Theorem)**, $f = f_1 - f_2$, where f_1 and f_2 are increasing on $[a,b]$. Let D_i denote the set of discontinuities of f_i on $[a,b]$, $i = 1,2$. Hence, D, the set of discontinuities of f on $[a,b]$ is

$$D \subseteq D_1 \cup D_2.$$

Since $|D_1| = |D_2| = 0$, we know that $|D| = 0$. In addition, f is of bounded on $[a,b]$ since f is bounded variation on $[a,b]$. So, by **Theorem 7.48**, $f \in R$ on $[a,b]$.

(b) If $f \in R$ on $[a,b]$, then $f \in R$ on $[c,d]$ for every subinterval $[c,d] \subseteq [a,b]$, $|f| \in R$ and $f^2 \in R$ on $[a,b]$. Also, $f \cdot g \in R$ on $[a,b]$ whenever $g \in R$ on $[a,b]$.

Proof: (i) Let $D_{[a,b]}$ and $D_{[c,d]}$ denote the set of discontinuities of f on $[a,b]$ and $[c,d]$, respectively. Then

$$D_{[c,d]} \subseteq D_{[a,b]}.$$

Since $f \in R$ on $[a,b]$, and use **Theorem 7.48**, $|D_{[a,b]}| = 0$ which implies that $|D_{[c,d]}| = 0$. In addition, since f is bounded on $[a,b]$, f is automatically is bounded on $[c,d]$ for every compact subinterval $[c,d]$. So, by **Theorem 7.48,** $f \in R$ on $[c,d]$.

(ii) Let D_f and $D_{|f|}$ denote the set of discontinuities of f and $|f|$ on $[a,b]$, respectively, then

$$D_{|f|} \subseteq D_f.$$

Since $f \in R$ on $[a,b]$, and use **Theorem 7.48**, $|D_f| = 0$ which implies that $|D_{|f|}| = 0$. In addition, since f is bounded on $[a,b]$, it is clear that $|f|$ is bounded on $[a,b]$. So, by **Theorem 7.48,** $|f| \in R$ on $[a,b]$.

(iii) Let D_f and D_{f^2} denote the set of discontinuities of f and f^2 on $[a,b]$, respectively, then

$$D_{f^2} \subseteq D_f.$$

Since $f \in R$ on $[a,b]$, and use **Theorem 7.48**, $|D_f| = 0$ which implies that $\left|D_{f^2}\right| = 0$. In addition, since f is bounded on $[a,b]$, it is clear that f^2 is bounded on $[a,b]$. So, by **Theorem 7.48,** $f^2 \in R$ on $[a,b]$.

(iv) Let D_f and D_g denote the set of discontinuities of f and g on $[a,b]$, respectively. Let D_{fg} denote the set of discontinuities of fg on $[a,b]$, then

$$D_{fg} \subseteq D_f \cup D_g$$

Since $f, g \in R$ on $[a,b]$, and use **Theorem 7.48**, $|D_f| = |D_g| = 0$ which implies that $|D_{fg}| = 0$. In addition, since f and g are bounded on $[a,b]$, it is clear that fg is bounded on $[a,b]$. So, by **Theorem 7.48,** $fg \in R$ on $[a,b]$.

(c) If $f \in R$ and $g \in R$ on $[a,b]$, then $f/g \in R$ on $[a,b]$ whenever g is bounded away from 0.

Proof: Let D_f and D_g denote the set of discontinuities of f and g on $[a,b]$, respectively. Since g is bounded away from 0, we know that f/g is well-defined and f/g is also bounded on $[a,b]$. Consider $D_{f/g}$, the set of discontinuities of f/g on $[a,b]$ is

$$D_{f/g} \subseteq D_f \cup D_g.$$

Since $f \in R$ and $g \in R$ on $[a,b]$, by **Theorem 7.48**, $|D_f| = |D_g| = 0$ which implies that $|D_{f/g}| = 0$. Since f/g is bounded on $[a,b]$ with $|D_{f/g}| = 0$, $f/g \in R$ on $[a,b]$ by **Theorem 7.48**.

Remark: The condition that the function g is bounded away from 0 **CANNOT** omit. For example, say $g(x) = x$ on $(0,1]$ and $g(0) = 1$. Then it is clear that $g \in R$ on $[0,1]$, but

$1/g \notin R$ on $[0,1]$. In addition, the reader should note that when we ask whether a function is Riemann-integrable or not, we always assume that f is **BOUNDED** on a **COMPACT INTERVAL** $[a,b]$.

(d) If f and g are bounded functions having the same discontinuities on $[a,b]$, then $f \in R$ on $[a,b]$ if, and only if, $g \in R$ on $[a,b]$.

Proof: ($\Rightarrow$) Suppose that $f \in R$ on $[a,b]$, then D_f, the set of discontinuities of f on $[a,b]$ has measure zero by **Theorem 7.48**. From hypothesis, $D_f = D_g$, the set of discontinuities of g on $[a,b]$, we know that $g \in R$ on $[a,b]$ by **Theorem 7.48**.

($\Leftarrow$) If we change the roles of f and g, we have proved it.

(e) Let $g \in R$ on $[a,b]$ and assume that $m \leq g(x) \leq M$ for all $x \in [a,b]$. If f is continuous on $[m,M]$, the composite function h defined by $h(x) = f[g(x)]$ is Riemann-integrable on $[a,b]$.

Proof: Note that h is bounded on $[a,b]$. Let D_h and D_g denote the set of dsicontinuities of h and g on $[a,b]$, respectively. Then

$$D_h \subseteq D_g.$$

Since $g \in R$ on $[a,b]$, then $|D_g| = 0$ by **Theorem 7.48**. Hence, $|D_h| = 0$ which implies that $h \in R$ on $[a,b]$ by **Theorem 7.48**.

Remark: (1) There has a more general theorem related with **Riemann-Stieltjes Integral**. We write it as a reference.

(**Theorem**)Suppose $g \in R(\alpha)$ on $[a,b]$, $m \leq g(x) \leq M$ for all $x \in [a,b]$. If f is continuous on $[m,M]$, the composite function h defined by $h(x) = f[g(x)] \in R(\alpha)$ on $[a,b]$.

Proof: It suffices to consider the case that α is increasing on $[a,b]$. If $\alpha(a) = \alpha(b)$, there is nothing to prove it. So, we assume that $\alpha(a) < \alpha(b)$. In addition, let $K = \sup_{x\in[a,b]}|h(x)|$. We claim that h satisfies Riemann condition with respect to α on $[a,b]$. That is, given $\varepsilon > 0$, we want to find a partition P such that

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{k=1}^{n}[M_k(h) - m_k(h)]\Delta\alpha_k < \varepsilon.$$

Since f is uniformly continuous on $[m,M]$, for this $\varepsilon > 0$, there is a $\left(\frac{\varepsilon}{2(K+1)} >\right)\delta > 0$ such that as $|x-y| < \delta$ where $x,y \in [m,M]$, we have

$$|f(x) - f(y)| < \frac{\varepsilon}{2[\alpha(b) - \alpha(a)]}. \quad 1$$

Since $g \in R(\alpha)$ on $[a,b]$, for this $\delta > 0$, there exists a partition P such that

$$U(P,g,\alpha) - L(P,g,\alpha) = \sum_{k=1}^{n}[M_k(g) - m_k(g)]\Delta\alpha_k < \delta^2. \quad 2$$

Let $P = A \cup B$, where $A = \{x_j : M_k(g) - m_k(g) < \delta\}$ and $B = \{x_j : M_k(g) - m_k(g) \geq \delta\}$, then

$$\delta\sum_B \Delta\alpha_k \leq \sum_B[M_k(h) - m_k(h)]\Delta\alpha_k \leq \delta^2 \text{ by (2)}$$

which implies that

$$\sum_B \Delta\alpha_k \leq \delta.$$

So, we have

$$U(A,h,\alpha) - L(A,h,\alpha) \le \sum_A [M_k(h) - m_k(h)]\Delta\alpha_k$$
$$\le \varepsilon/2 \text{ by (1)}$$

and

$$U(B,h,\alpha) - L(B,h,\alpha) \le \sum_B [M_k(h) - m_k(h)]\Delta\alpha_k$$
$$\le 2K\sum_B \Delta\alpha_k$$
$$\le 2K\delta$$
$$< \varepsilon/2.$$

It implies that

$$U(P,h,\alpha) - L(P,h,\alpha) = U(A,h,\alpha) - L(A,h,\alpha) + U(B,h,\alpha) - L(B,h,\alpha)$$
$$< \varepsilon.$$

That is, we have proved that h satisfies the Riemann condition with respect to α on $[a,b]$. So, $h \in R(\alpha)$ on $[a,b]$.

Note: We mention that if we change the roles of f and g, then the conclusion does **NOT** hold. Since the counterexample is constructed by some conclusions that we will learn in Real Analysis, we do **NOT** give it a proof. Let C be the standard Cantor set in $[0,1]$ and C' the Cantor set with positive measure in $[0,1]$. Use similar method on defining Cantor Lebesgue Function, then there is a continuous function $f : [0,1] \to [0,1]$ such that $f(C) = C'$. And Choose $g = X_C$ on $[0,1]$. Then

$$h = g \circ f = X_{C'}$$

which is **NOT** Riemann integrable on $[0,1]$.

(2) The reader should note the followings. Since these proofs use the **exercise 7.30** and **Theorem 7.49**, we omit it.

(i) If $f \in R$ on $[a,b]$, then $|f| \in R$ on $[a,b]$, and $f^r \in R$ on $[a,b]$, where $r \in [0,\infty)$.

(ii) If $|f| \in R$ on $[a,b]$, it does **NOT** implies $f \in R$ on $[a,b]$. And if $f^2 \in R$ on $[a,b]$, it does **NOT** implies $f \in R$ on $[a,b]$. For example,

$$f = \begin{cases} 1 \text{ if } x \in Q \cap [a,b] \\ -1 \in Q^c \cap [a,b] \end{cases}$$

(iii) If $f^3 \in R$ on $[a,b]$, then $f \in R$ on $[a,b]$.

7.31 Use Lebesgue's theorem to prove that if $f \in R$ and $g \in R$ on $[a,b]$ and if $f(x) \ge m > 0$ for all x in $[a,b]$, then the function h defined by

$$h(x) = f(x)^{g(x)}$$

is Riemann-integrable on $[a,b]$.

Proof: Consider

$$h(x) = \exp(h\log f),$$

then by **Theorem 7.49**,

$$f \in R \Rightarrow \log f \in R \Rightarrow h\log f \in R \Rightarrow \exp(h\log f) = h \in R.$$

7.32 Let $I = [0,1]$ and let $A_1 = I - (\frac{1}{3}, \frac{2}{3})$ be the subset of I obtained by removing those points which lie in the open middle third of I; that is, $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Let A_2 be the subset of A_1 obtained by removing he open middle third of $[0, \frac{1}{3}]$ and of $[\frac{2}{3}, 1]$. Continue this process and define $A_3, A_4, \ldots$. The set $C = \cap_{n=1}^{\infty} A_n$ is called the **Cantor set**. Prove that

(a) C is compact set having measure zero.

Proof: Write $C = \cap_{n=1}^{\infty} A_n$. Note that every A_n is closed, so C is closed. Since $A_1 (\neq \phi)$ is closed and bounded, A_1 is compact and $C \subseteq A_1$, we know that C is compact by **Theorem 3.39**.

In addition, it is clear that $|A_n| = (\frac{2}{3})^n$ for each n. Hence, $|C| \leq \lim_{n\to\infty} |A_n| = 0$, which implies that C has a measure zero.

(b) $x \in C$ if, and only if, $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where each a_n is either 0 or 2.

Proof: ($\Rightarrow$)Let $x \in C = \cap_{n=1}^{\infty} A_n$, then $x \in A_n$ for all n. Consider the followings.

(i) Since $x \in A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, then it implies that $a_1 = 0$ or 2.

(ii) Since $x \in A_2 = ([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}]) \cup ([\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{9}{9}])$, then it implies that $a_2 = 0$ or 2.

Inductively, we have $a_n = 0$ or 2. So, $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where each a_n is either 0 or 2.

($\Leftarrow$) If $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where each a_n is either 0 or 2, then it is clear that $x \in A_n$ for each n. Hence, $x \in C$.

(c) C is uncountable.

Proof: Suppose that C is countable, write $C = \{x_1, x_2, \ldots\}$. We consider unique ternary expansion: if $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, then $x := (a_1, \ldots, a_n, \ldots)$. From this definition, by (b), we have

$$x_k = (x_{k1}, x_{k2}, \ldots x_{kk}, \ldots) \text{ where each component is 0 or 2.}$$

Choose $y = (y_1, y_2, \ldots)$ where

$$y_j = \begin{cases} 2 \text{ if } x_{jj} = 0 \\ 0 \text{ if } x_{jj} = 2. \end{cases}$$

By (b), $y \in C$. It implies that $y = x_k$ for some k which contradicts to the choice of y. Hence, C is uncountable.

Remark: (1) In fact, $C = C'$ means that C is a perfect set. Hence, C is uncountable. The reader can see the book, **Principles of Mathematical Analysis by Walter Rudin, pp 41-42.**

(2) Let $C = \left\{x : x = \sum_{n=1}^{\infty} a_n 3^{-n}, \text{ where each } a_n \text{ is either 0 or 2}\right\}$. Define a new function $\phi : C \to [0, 1]$ by

$$\phi(x) = \sum_{n=1}^{\infty} \frac{(a_n)/2}{2^n},$$

then it is clear that ϕ is 1-1 and onto. So, C is equivalent to $[0, 1]$. That is, C is uncountable.

(d) Let $f(x) = 1$ if $x \in C$, $f(x) = 0$ if $x \notin C$. Prove that $f \in R$ on $[0, 1]$.

Proof: In order to show that $f \in R$ on $[0, 1]$, it suffices to show that, by **Theorem 7.48**, f is continuous on $[0, 1] - C$ since it implies that $D \subseteq C$, where D is the set of discontinuities of f on $[0, 1]$.

Let $x_0 \in [0, 1] - C$, and note that $C = C'$, so there is a $\delta > 0$ such that

$(x_0-\delta, x_0+\delta) \cap C = \phi$, where $(x_0-\delta, x_0+\delta) \subseteq [0,1]$. Then given $\varepsilon > 0$, there is a $\delta > 0$ such that as $x \in (x_0-\delta, x_0+\delta)$, we have

$$|f(x) - f(x_0)| = 0 < \varepsilon.$$

Remark: (1) $C = C'$:Given $x \in C = \bigcap_{n=1}^{\infty} A_n$, and note that every endpoints of A_n belong to C. So, x is an accumulation point of the set $\{y : y \text{ is the endpoints of } A_n\}$. So, $C \subseteq C'$. In addition, $C' \subseteq C$ since C is closed. Hence, $C = C'$.

(2) In fact, we have

$$f \text{ is continuous on } [0,1] - C \text{ and } f \text{ is not continuous on } C.$$

Proof: In (d), we have proved that f is continuous on $[0,1] - C$, so it remains to show that f is not continuous on C. Let $x_0 \in C$, if f is continuous at x_0, then given $\varepsilon = 1/2$, there is a $\delta > 0$ such that as $x \in (x_0-\delta, x_0+\delta) \cap [0,1]$, we have

$$|f(x) - f(x_0)| < 1/2$$

which is absurb since we can choose $y \in (x_0-\delta, x_0+\delta) \cap [0,1]$ and $y \notin C$ by the fact C does **NOT** contain an open interval since C has measure zero. So, we have proved that f is not continuous on C.

Note: In a metric space M, a set $S(\subseteq M)$ is called nonwhere dense if $int(cl(S)) = \phi$. Hence, we know that C is a nonwhere dense set.

Supplement on Cantor set.

From the exercise 7.32, we have learned what the **Cantor set** is. We write some conclusions as a reference.

(1) The Cantor set C is compact and perfect.
(2) The Cantor set C is uncountable. In fact, $\#(C) = \#(R)$.
(3) The Cantor set C has measure zero.
(4) The Cantor set C is nonwhere dense.
(5) Every point x in C can be expressed as $x = \sum_{n=1}^{\infty} a_n 3^{-n}$, where each a_n is either 0 or 2.
(6) $X_C : [0,1] \to \{0,1\}$ the characteristic function of C on $[0,1]$ is Riemann integrable.

The reader should be noted that Cantor set C in the exercise is 1 $-$dimensional case. We can use the same method to construct a n $-$dimensional Cantor set in the set $\{(x_1,\ldots,x_n) : 0 \leq x_j \leq 1, j = 1,2,..n\}$. In addition, there are many researches on Cantor set. For example, we will learn so called **Space-Filling Curve** on the textbook, **Ch9, pp 224-225.**

In addition, there is an important function called **Cantor-Lebesgue Function** related with Cantor set. The reader can see the book, **Measure and Integral (An Introduction to Real Analysis) written by Richard L. Wheeden and Antoni Zygmund**, **pp 35.**

7.33 The exercise outlines a proof (due to **Ivan Niven**) that π^2 is irrational. Let $f(x) = x^n(1-x)^n/n!$. Prove that:

(a) $0 < f(x) < 1/n!$ if $0 < x < 1$.

Proof: It is clear.

(b) Each kth derivative $f^{(k)}(0)$ and $f^{(k)}(1)$ is an integer.

Proof: By **Leibnitz Rule**,

$$f^{(k)}(x) = \frac{1}{n!}\sum_{j=0}^{k}\binom{k}{j}\{[n\cdot\cdot\cdot(n-j+1)]x^{n-j}\}\left\{(-1)^{k-j}[n\cdot\cdot\cdot(n-k+j+1)](1-x)^{n-k+j}\right\}$$

which implies that

$$f^{(k)}(0) = \begin{cases} 0 \text{ if } k < n \\ 1 \text{ if } k = n \\ \binom{k}{n}(-1)^{k-n}[n\cdot\cdot\cdot(2n-k+1)] \text{ if } k > n \end{cases}.$$

So, $f^{(k)}(0) \in Z$ for each $k \in N$. Similarly, $f^{(k)}(1) \in Z$ for each $k \in Z$.

Now assume that $\pi^2 = a/b$, where a and b are positive integers, and let

$$F(x) = b^n\sum_{k=0}^{n}(-1)^k f^{(2k)}(x)\pi^{2n-2k}.$$

Prove that:

(c) $F(0)$ and $F(1)$ are integers.

Proof: By (b), it is clear.

(d) $\pi^2 a^n f(x)\sin\pi x = \frac{d}{dx}\{F'(x)\sin\pi x - \pi F(x)\cos\pi x\}$

Proof: Note that

$$\begin{aligned} F''(x) + \pi^2 F(x) &= b^n\sum_{k=0}^{n}(-1)^k f^{(2k+2)}(x)\pi^{2n-2k} + \pi^2 b^n\sum_{k=0}^{n}(-1)^k f^{(2k)}(x)\pi^{2n-2k} \\ &= b^n\sum_{k=0}^{n-1}(-1)^k f^{(2k+2)}(x)\pi^{2n-2k} + b^n(-1)^n f^{(2n+2)}(x) \\ &\quad + \pi^2 b^n\sum_{k=1}^{n}(-1)^k f^{(2k)}(x)\pi^{2n-2k} + \pi^2 b^n f(x)\pi^{2n} \\ &= b^n\sum_{k=0}^{n-1}\left[(-1)^k f^{(2k+2)}(x)\pi^{2n-2k}\right] + \left[(-1)^{k+1} f^{(2k+2)}(x)\pi^{2n-2k}\right] \\ &\quad + b^n(-1)^n f^{(2n+2)}(x) + \pi^2 b^n f(x)\pi^{2n} \\ &= \pi^2 a^n f(x) \text{ since } f \text{ is a polynomial of degree } 2n. \end{aligned}$$

So,

$$\begin{aligned} &\frac{d}{dx}\{F'(x)\sin\pi x - \pi F(x)\cos\pi x\} \\ &= (\sin\pi x)[F''(x) + \pi^2 F(x)] \\ &= \pi^2 a^n f(x)\sin\pi x. \end{aligned}$$

(e) $F(1) + F(0) = \pi a^n\int_0^1 f(x)\sin\pi x dx$.

Proof: By (d), we have

$$\begin{aligned} \pi^2 a^n\int_0^1 f(x)\sin\pi x dx &= F'(x)\sin\pi x - \pi F(x)\cos\pi x|_0^1 \\ &= [F'(1)\sin\pi - \pi F(1)\cos\pi] - [F'(0)\sin 0 - \pi F(0)\cos 0] \\ &= \pi[F(1) + F(0)] \end{aligned}$$

which implies that

$$F(1) + F(0) = \pi a^n \int_0^1 f(x) \sin \pi x dx.$$

(f) Use (a) in (e) to deduce that $0 < F(1) + F(0) < 1$ if n is sufficiently large. This contradicts (c) and show that π^2 (and hence π) is irrational.

Proof: By (a), and $\sin x \in [0,1]$ on $[0,\pi]$, we have

$$0 < \pi a^n \int_0^1 f(x) \sin \pi x dx \leq \frac{\pi a^n}{n!} \int_0^1 \sin \pi x dx = \frac{2a^n}{n!} \to 0 \text{ as } n \to \infty.$$

So, as n is sufficiently large, we have, by (d),

$$0 < F(1) + F(0) < 1$$

which contradicts (c). So, we have proved that π^2 (and hence π) is irrational.

Remark: The reader should know that π is a transcendental number. (Also, so is e). It is well-known that a transcendental number must be an irrational number.

In 1900, **David Hilbert** asked **23** problems, the **7th** problem is that, if $\alpha(\neq 0,1)$ is an algebraic number and β is an algebraic number but not rational, then is it true that α^β is a transcendental number. The problem is completely solved by **Israil Moiseevic Gelfand** in 1934. There are many open problem now on algebraic and transcendental numbers. For example, It is an open problem: Is the **Euler Constant**

$$\gamma = \lim\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right)$$

a transcendental number.

7.34 Given a real-valued function α, continuous on the interval $[a,b]$ and having a finite bounded derivative α' on (a,b). Let f be defined and bounded on $[a,b]$ and assume that both integrals

$$\int_a^b f(x) d\alpha(x) \text{ and } \int_a^b f(x)\alpha'(x) dx$$

exists. Prove that these integrals are equal. (It is not assumed that α' is continuous.)

Proof: Since both integrals exist, given $\varepsilon > 0$, there exists a partition $P = \{x_0 = a, \ldots, x_n = b\}$ such that

$$\left| S(P,f,\alpha) - \int_a^b f(x) d\alpha(x) \right| < \varepsilon/2$$

where

$$S(P,f,\alpha) = \sum_{j=1}^n f(t_j)\Delta\alpha_j \text{ for } t_j \in [x_{j-1},x_j]$$

$$= \sum_{j=1}^n f(t_j)\alpha'(s_j)\Delta x_j \text{ by Mean Value Theorem, where } s_j \in (x_{j-1},x_j) \qquad *$$

and

$$\left| S(P,f\alpha') - \int_a^b f(x)\alpha'(x) dx \right| < \varepsilon/2$$

where

$$S(P,f\alpha') = \sum_{j=1}^n f(t_j)\alpha'(t_j)\Delta x_j \text{ for } t_j \in [x_{j-1},x_j] \qquad **$$

So, let $t_j = s_j$, then we have

$$S(P,f,\alpha) = S(P,f\alpha').$$

Hence,

$$\left|\int_a^b f(x)d\alpha(x) - \int_a^b f(x)\alpha'(x)dx\right|$$
$$\leq \left|S(P,f,\alpha) - \int_a^b f(x)d\alpha(x)\right| + \left|S(P,f\alpha') - \int_a^b f(x)\alpha'(x)dx\right|$$
$$< \varepsilon.$$

So, we have proved that both integrals are equal.

7.35 Prove the following theorem, which implies that a function with a positive integral must itself be positive on some interval. Assume that $f \in R$ on $[a,b]$ and that $0 \leq f(x) \leq M$ on $[a,b]$, where $M > 0$. Let $I = \int_a^b f(x)dx$, let $h = \frac{1}{2}I/(M+b-a)$, and assume that $I > 0$. Then the set $T = \{x : f(x) \geq h\}$ contains a finite number of intervals, the sum of whose lengths is at least h.

Hint. Let P be a partition of $[a,b]$ such that every Riemann sum $S(P,f) = \sum_{k=1}^n f(t_k)\Delta x_k$ satisfies $S(P,f) > I/2$. Split $S(P,f)$ into two parts, $S(P,f) = \sum_{k\in A} + \sum_{k\in B}$, where

$$A = \{k : [x_{k-1},x_k] \subseteq T\}, \text{ and } B = \{k : k \notin A\}.$$

If $k \in A$, use the inequality $f(t_k) \leq M$; if $k \in B$, choose t_k so that $f(t_k) < h$. Deduce that $\sum_{k\in A} \Delta x_k > h$.

Proof: It is clear by Hint, so we omit the proof.

Remark: There is another proof about that a function with a positive integral must itself be positive on some interval.

Proof: Suppose **NOT**, it means that in every subinterval, there is a point p such that $f(p) \leq 0$. So,

$$L(P,f) = \sum_{j=1}^n m_j\Delta x_j \leq 0 \text{ since } m_j \leq 0$$

for any partition P. Then it implies that

$$\sup_P L(P,f) = \int_a^b f(x)dx \leq 0$$

which contradicts to a function with a positive integral. Hence, we have proved that a function with a positive integral must itself be positive on some interval.

Supplement on integration of vector-valued functions.

(**Definition**) Given $f_1,\ldots,f_n$ real valued functions defined on $[a,b]$, and let $\mathbf{f} = (f_1,\ldots,f_n) : [a,b] \to R^n$. If $\alpha \nearrow$ on $[a,b]$. We say that $\mathbf{f} \in R(\alpha)$ on $[a,b]$ means that $f_j \in R(\alpha)$ on $[a,b]$ for $j = 1,2,\ldots,n$. If this is the case, we define

$$\int_a^b \mathbf{f}d\alpha = \left(\int_a^b f_1 d\alpha,\ldots,\int_a^b f_n d\alpha\right).$$

From the definition, the reader should find that the definition is **NOT** stranger for us. When we talk $\mathbf{f} = (f_1,\ldots,f_n) \in R(\alpha)$ on $[a,b]$, it suffices to consider each $f_j \in R(\alpha)$ on $[a,b]$ for $j = 1,2,\ldots,n$.

For example, if $\mathbf{f} \in R(\alpha)$ on $[a,b]$ where $\alpha \nearrow$ on $[a,b]$, then $\|\mathbf{f}\| \in R(\alpha)$ on $[a,b]$.

Proof: Since $\mathbf{f} \in R(\alpha)$ on $[a,b]$, we know that $f_j \in R(\alpha)$ on $[a,b]$ for $j = 1,2,\ldots,n$. Hence,

$$\sum_{k=1}^{n} f_j^2 \in R(\alpha) \text{ on } [a,b]$$

which implies that, by **Remark (1) in Exercise 7.30**,

$$\|\mathbf{f}\| = \left(\sum_{k=1}^{n} f_j^2 \in R(\alpha) \text{ on } [a,b]\right)^{1/2} \in R(\alpha) \text{ on } [a,b].$$

Remark: In the case above, we have

$$\left\| \int_a^b \mathbf{f} d\alpha \right\| \leq \int_a^b \|\mathbf{f}\| d\alpha.$$

Proof: Consider

$$\begin{aligned}\|\mathbf{y}\|^2 &= < \int_a^b f_1 d\alpha, \ldots, \int_a^b f_n d\alpha, \int_a^b f_1 d\alpha, \ldots, \int_a^b f_n d\alpha > \\ &= \sum_{j=1}^{n} \left(\int_a^b f_j d\alpha\right)\left(\int_a^b f_j d\alpha\right),\end{aligned}$$

which implies that, (let $y_j = \int_a^b f_j d\alpha$, $\mathbf{y} = (y_1, \ldots, y_n)$),

$$\begin{aligned}\|\mathbf{y}\|^2 &= \sum_{j=1}^{n} y_j \left(\int_a^b f_j d\alpha\right) \\ &= \sum_{j=1}^{n} \int_a^b f_j y_j d\alpha \\ &= \int_a^b \left(\sum_{j=1}^{n} f_j y_j\right) d\alpha \\ &\leq \int_a^b \|\mathbf{f}\| \|\mathbf{y}\| d\alpha \\ &= \|\mathbf{y}\| \int_a^b \|\mathbf{f}\| d\alpha\end{aligned}$$

which implies that

$$\|\mathbf{y}\| \leq \int_a^b \|\mathbf{f}\| d\alpha.$$

Note: The equality holds if, and only if, $\mathbf{f}(t) = k(t)\mathbf{y}$.

Existence theorems for integral and differential equations

The following exercises illustrate how the fixed-point theorem for contractions. (Theorem 4.48) is used to prove existence theorems for solutions of certain integral and differential equations. We denote by $C[a,b]$ the metric space of all real continuous functions on $[a,b]$ with the metric

$$d(f,g) = \|f - g\| = \max_{x \in [a,b]} |f(x) - g(x)|,$$

and recall the $C[a,b]$ is a complete metric space.

7.36 Given a function g in $C[a,b]$, and a function K is continuous on the rectangle $Q = [a,b] \times [a,b]$, consider the function T defined on $C[a,b]$ by the equation

$$T(\varphi)(x) = g(x) + \lambda \int_a^b K(x,t)\varphi(t) dt,$$

where λ is a given constant.

(a) Prove that T maps $C[a,b]$ into itself.

Proof: Since K is continuous on the rectangle $Q = [a,b] \times [a,b]$, and $\varphi(x) \in C[a,b]$, we know that

$$\int_a^b K(x,t)\varphi(t)dt \in C[a,b].$$

Hence, we prove that $T(\varphi)(x) \in C[a,b]$. That is, T maps $C[a,b]$ into itself.

(b) If $|K(x,y)| \leq M$ on Q, where $M > 0$, and if $|\lambda| < M^{-1}(b-a)^{-1}$, prove that T is a contraction of $C[a,b]$ and hence has a fixed point φ which is a solution of the integral equation $\varphi(x) = g(x) + \lambda \int_a^b K(x,t)\varphi(t)dt$.

Proof: Consider

$$\begin{aligned} \|T(\varphi_1)(x) - T(\varphi_2)(x)\| &= \left\| \lambda \int_a^b K(x,t)[\varphi_1(t) - \varphi 2(t)]dt \right\| \\ &\leq |\lambda| \int_a^b |K(x,t)[\varphi_1(t) - \varphi 2(t)]|dt \\ &\leq |\lambda| M \int_a^b |\varphi_1(t) - \varphi 2(t)|dt \\ &\leq |\lambda| M(b-a) \|\varphi_1(t) - \varphi 2(t)\|. \end{aligned} \quad *$$

Since $|\lambda| < M^{-1}(b-a)^{-1}$, then there exists c such that $|\lambda| < c < M^{-1}(b-a)^{-1}$. Hence, by (*), we know that

$$\|T(\varphi_1)(x) - T(\varphi_2)(x)\| < \gamma \|\varphi_1(t) - \varphi 2(t)\|$$

where $0 < cM(b-a) := \gamma < 1$. So, T is a contraction of $C[a,b]$ and hence has a fixed point φ which is a solution of the integral equation $\varphi(x) = g(x) + \lambda \int_a^b K(x,t)\varphi(t)dt$.

7.37 Assume f is continuous on a rectangle $Q = [a-h, a+h] \times [b-k, b+k]$, where $h > 0, k > 0$.

(a) Let φ be a function, continuous on $[a-h, a+h]$, such that $(x, \varphi(x)) \in Q$ for all x in $[a-h, a+h]$. If $0 < c \leq h$, prove that φ satisfies the differential equation $y' = f(x,y)$ on $(a-c, a+c)$ and the initial condition $\varphi(a) = b$ if, and only if, φ satisfies the integral equation

$$\varphi(x) = b + \int_a^x f(t, \varphi(t))dt \text{ on } (a-c, a+c).$$

Proof: ($\Rightarrow$)Since $\varphi'(t) = f(t, \varphi(t))$ on $(a-c, a+c)$ and $\varphi(a) = b$, we have, $x \in (a-c, a+c)$

$$\begin{aligned} \varphi(x) &= \varphi(a) + \int_a^x \varphi'(t)dt \\ &= \varphi(a) + \int_a^x f(t, \varphi(t))dt \text{ on } (a-c, a+c). \end{aligned}$$

($\Leftarrow$)Assume

$$\varphi(x) = b + \int_a^x f(t, \varphi(t))dt \text{ on } (a-c, a+c),$$

then

$$\varphi'(x) = f(t, \varphi(x)) \text{ on } (a-c, a+c).$$

(b) Assume that $|f(x,y)| \leq M$ on Q, where $M > 0$, and let $c = \min\{h, k/M\}$. Let S

denote the metric subspace of $C[a-c,a+c]$ consisting of all φ such that $|\varphi(x)-b| \leq Mc$ on $[a-c,a+c]$. Prove that S is closed subspace of $C[a-c,a+c]$ and hence that S is itself a complete metric space.

Proof: Since $C[a-c,a+c]$ is complete, if we can show that S is closed, then S is complete. Hence, it remains to show that S is closed.

Given $f \in S'$, then there exists a sequence of functions $\{f_n\}$ such that $f_n \to f$ under the sup norm $\|.\|$. So, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$\max_{x\in[a-c,a+c]} |f_n(x)-f(x)| < \varepsilon.$$

Consider

$$\begin{aligned}|f(x)-b| &\leq |f(x)-f_N(x)| + |f_N(x)-b| \\ &\leq \|f(x)-f_N(x)\| + \|f_N(x)-b\| \\ &< \varepsilon + Mc\end{aligned}$$

which implies that

$$|f(x)-b| \leq Mc \text{ for all } x$$

since ε is arbitrary. So, $f \in S$. It means that S is closed.

(c) Prove that the function T defined on S by the equation

$$T(\varphi)(x) = b + \int_a^x f(t,\varphi(t))dt$$

maps S into itself.

Proof: Since

$$\begin{aligned}|T(\varphi)(x)-b| &= \left|\int_a^x f(t,\varphi(t))dt\right| \\ &\leq \int_a^x |f(t,\varphi(t))|dt \\ &\leq (x-a)M \\ &\leq Mc\end{aligned}$$

we know that $T(\varphi)(x) \in S$. That is, T maps S into itself.

(d) Now assume that f satisfies a Lipschitz condition of the form

$$|f(x,y)-f(x,z)| \leq A|y-z|$$

for every pair of points (x,y) and (x,z) in Q, where $A > 0$. Prove that T is a contraction of S if $h < 1/A$. Deduce that for $h < 1/A$ the differential equation $y' = f(x,y)$ has exactly one solution $y = \varphi(x)$ on $(a-c,a+c)$ such that $\varphi(a) = b$.

Proof: Note that $h < 1/A$, there exists λ such that $h < \lambda < 1/A$. Since

$$\begin{aligned}&\|T(\varphi_1)(x) - T(\varphi_2)(x)\| \\ &\leq \int_a^x |f(t,\varphi_1(t)) - f(t,\varphi_2(t))|dt \\ &\leq A\int_a^x |\varphi_1(t)-\varphi_2(t)|dt \text{ by } |f(x,y)-f(x,z)| \leq A|y-z| \\ &\leq A(x-a)\|\varphi_1(t)-\varphi_2(t)\| \\ &\leq Ah\|\varphi_1(t)-\varphi_2(t)\| \\ &< \gamma\|\varphi_1(t)-\varphi_2(t)\|\end{aligned}$$

where $0 < \lambda A := \gamma < 1$. Hence, T is a contraction of S. It implies that there exists one and only one $\varphi \in S$ such that

$$\varphi(x) = b + \int_a^x f(t, \varphi(t))dt$$

which implies that

$$\varphi'(x) = f(x, \varphi(x)).$$

That is, the differential equation $y' = f(x,y)$ has exactly one solution $y = \varphi(x)$ on $(a-c, a+c)$ such that $\varphi(a) = b$.

Supplement on Riemann Integrals

1. The reader should be noted that the metric space $(R([a,b]), d)$ is **NOT** complete, where

$$d(f,g) = \int_a^b |f(x) - g(x)|dx.$$

We do **NOT** give it a proof. The reader can see the book, **Measure and Integral (An Introduction to Real Analysis) by Richard L. Wheeden and Antoni Zygmund, Ch5**.

2. The reader may recall the **Mean Value Theorem**: Let f be a continuous function on $[a,b]$. Then

$$\int_a^b f(x)dx = f(x_0)(b-a)$$

where $x_0 \in [a,b]$. In fact, the point x_0 can be chosen to be interior of $[a,b]$. That is, $x_0 \in (a,b)$.

Proof: Let $M = \sup_{x\in[a,b]} f(x)$, and $m = \inf_{x\in[a,b]} f(x)$. If $M = m$, then it is clear. So, we may assume that $M \neq m$ as follows. Suppose **NOT**, it means that $x_0 = a$ or b. Note that,

$$f(x_1) = m \leq f(x_0) := r \leq M = f(x_2)$$

by continuity of f on $[a,b]$. Then we claim that

$$f(x_0) = m \text{ or } M.$$

If **NOT**, i.e.,

$$f(x_1) < r < f(x_2)$$

it means that there exists a point $p \in (x_1, x_2)$ such that $f(p) = r$ by **Intermediate Value Theorem**. It contradicts to $p = a$ or b. So, we have proved the claim. If $f(a) = m$, then

$$\int_a^b f(x)dx = m(b-a) \Rightarrow 0 = \int_a^b [f(x) - m]dx$$

which implies that, by $f(x) - m \geq 0$ on $[a,b]$,

$$f(x) = m \text{ forall } x \in [a,b].$$

So, it is impopssible. Similarly for other cases.

Remark: (1) The reader can give it a try to consider the Riemann-Stieltjes Integral as follows. Let α be a continuous and increasing function on $[a,b]$. If f is continuous on $[a,b]$, then

$$\int_a^b f(x)d\alpha(x) = f(c)[\alpha(b) - \alpha(a)]$$

where $c \in (a,b)$.

Note: We do **NOT** omit the continuity of α on $[a,b]$ since

$$f(x) = x \text{ on } [0,1];\ \alpha(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x \in (0,1] \end{cases}.$$

(2) The reader can see the textbook, **exercise 14.13 pp 404.**

Exercise: Show that

$$\frac{\pi}{2} < \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \frac{1}{2}\sin^2 x}} < \frac{\pi}{\sqrt{2}}.$$

Proof: It is clear by the choice of $x_0 \in (0, \pi/2)$.

3. Application on Integration by parts for Riemann-integrable function. It is well-known that

$$\int f(x)dx = xf(x) - \int x df(x). \qquad *$$

If $f(x)$ has the inverse function $g(y) = x$, then (*) implies that

$$\int f(x)dx = xf(x) - \int g(y)dy.$$

For example,

$$\int \arcsin x dx = x \arcsin x - \int \sin y dy.$$

4. Here is an observation on Series, Differentiation and Integration. We write it as a table to make the reader think it twice.

$$\begin{pmatrix}\text{Series}\end{pmatrix} : \quad \text{Summation by parts} \quad \text{Cesaro Sum} \quad ?$$

$$\begin{pmatrix}\text{Differentiation}\end{pmatrix} : \quad (fg)' = f'g + fg' \quad \text{Mean Value Theorem} \quad \text{Chain Rule}$$

$$\begin{pmatrix}\text{Integration}\end{pmatrix} : \quad \text{Integration by parts} \quad \text{Mean Value Theorem} \quad \text{Change of Variable} \quad .$$