Functions of Bounded Variation and Rectifiable Curves

Functions of bounded variation

6.1 Determine which of the follwoing functions are of bounded variation on $[0,1]$.
(a) $f(x) = x^2 \sin(1/x)$ if $x \neq 0$, $f(0) = 0$.
(b) $f(x) = \sqrt{x} \sin(1/x)$ if $x \neq 0$, $f(0) = 0$.

Proof: (a) Since

$$f'(x) = 2x \sin(1/x) - \cos(1/x) \text{ for } x \in (0,1] \text{ and } f'(0) = 0,$$

we know that $f'(x)$ is bounded on $[0,1]$, in fact, $|f'(x)| \leq 3$ on $[0,1]$. Hence, f is of bounded variation on $[0,1]$.

(b) First, we choose $n+1$ be an even integer so that $\frac{1}{\frac{\pi}{2}(n+1)} < 1$, and thus consider a partition $P = \left\{0 = x_0,\ x_1 = \frac{1}{\frac{\pi}{2}},\ x_2 = \frac{1}{2\frac{\pi}{2}}, \ldots, x_n = \frac{1}{n\frac{\pi}{2}},\ x_{n+1} = \frac{1}{(n+1)\frac{\pi}{2}},\ x_{n+2} = 1\right\}$, then we have

$$\sum_{k=1}^{n+2} |\Delta f_k| \geq 2\sqrt{\frac{2}{\pi}} \left(\sum_{k=1}^{n} \sqrt{1/k} \right).$$

Since $\sum \sqrt{1/k}$ diverges to $+\infty$, we know that f is not of bounded variation on $[0,1]$.

6.2 A function f, defined on $[a,b]$, is said to satisfy a uniform Lipschitz condition of order $\alpha > 0$ on $[a,b]$ if there exists a constant $M > 0$ such that $|f(x) - f(y)| < M|x-y|^\alpha$ for all x and y in $[a,b]$. (Compare with Exercise 5.1.)

(a) If f is such a function, show that $\alpha > 1$ implies f is constant on $[a,b]$, whereas $\alpha = 1$ implies f is of bounded variation $[a,b]$.

Proof: As $\alpha > 1$, we consider, for $x \neq y$, where $x, y \in [a,b]$,

$$0 \leq \frac{|f(x) - f(y)|}{|x-y|} < M|x-y|^{\alpha-1}.$$

Hence, $f'(x)$ exists on $[a,b]$, and we have $f'(x) = 0$ on $[a,b]$. So, we know that f is constant.

As $\alpha = 1$, consider any partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$, we have

$$\sum_{k=1}^{n} |\Delta f_k| \leq M \sum_{k=1}^{n} |x_{k+1} - x_k| = M(b-a).$$

That is, f is of bounded variation on $[a,b]$.

(b) Give an example of a function f satisfying a uniform Lipschitz condition of order $\alpha < 1$ on $[a,b]$ such that f is not of bounded variation on $[a,b]$.

Proof: First, note that x^α satisfies uniform Lipschitz condition of order α, where $0 < \alpha < 1$. Choosing $\beta > 1$ such that $\alpha\beta < 1$ and let $M = \sum_{k=1}^{\infty} \frac{1}{k^\beta}$ since the series converges. So, we have $1 = \frac{1}{M} \sum_{k=1}^{\infty} \frac{1}{k^\beta}$.

Define a function f as follows. We partition $[0,1]$ into infinitely many subsintervals. Consider

$$x_0 = 0,\ x_1 - x_0 = \frac{1}{M}\frac{1}{1^\beta},\ x_2 - x_1 = \frac{1}{M}\frac{1}{2^\beta}, \ldots, x_n - x_{n-1} = \frac{1}{M}\frac{1}{n^\beta}, \ldots.$$

And in every subinterval $[x_i, x_{i+1}]$, where $i = 0, 1, \ldots,$ we define

$$f(x) = \left(\left|x - \frac{x_i + x_{i+1}}{2}\right|\right)^{\alpha},$$

then f is a continuous function and is not bounded variation on $[0,1]$ since $\sum_{k=1}^{\infty}\left(\frac{1}{2M}\frac{1}{k^{\beta}}\right)^{\alpha}$ diverges.

In order to show that f satisfies uniform Lipschitz condition of order $\alpha,$ we consider three cases.

(1) If $x,y \in [x_i, x_{i+1}]$, and $x,y \in [x_i, \frac{x_i+x_{i+1}}{2}]$ or $x,y \in [\frac{x_i+x_{i+1}}{2}, x_{i+1}]$, then

$$|f(x) - f(y)| = |x^{\alpha} - y^{\alpha}| \leq |x-y|^{\alpha}.$$

(2) If $x,y \in [x_i, x_{i+1}]$, and $x \in [x_i, \frac{x_i+x_{i+1}}{2}]$ or $y \in [\frac{x_i+x_{i+1}}{2}, x_{i+1}]$, then there is a $z \in [x_i, \frac{x_i+x_{i+1}}{2}]$ such that $f(y) = f(z)$. So,

$$|f(x) - f(y)| = |f(x) - f(z)| \leq |x^{\alpha} - z^{\alpha}| \leq |x-z|^{\alpha} \leq |x-y|^{\alpha}.$$

(3) If $x \in [x_i, x_{i+1}]$ and $y \in [x_j, x_{j+1}]$, where $i > j$.

If $x \in [x_i, \frac{x_i+x_{i+1}}{2}]$, then there is a $z \in [x_i, \frac{x_i+x_{i+1}}{2}]$ such that $f(y) = f(z)$. So,

$$|f(x) - f(y)| = |f(x) - f(z)| \leq |x^{\alpha} - z^{\alpha}| \leq |x-z|^{\alpha} \leq |x-y|^{\alpha}.$$

Similarly for $x \in [\frac{x_i+x_{i+1}}{2}, x_{i+1}]$.

Remark: Here is another example. Since it will use **Fourier Theory**, we do not give a proof. We just write it down as a reference.

$$f(t) = \sum_{k=1}^{\infty} \frac{\cos(3^k t)}{3^{k\alpha}}.$$

(c) Give an example of a function f which is of bounded variation on $[a,b]$ but which satisfies no uniform Lipschitz condition on $[a,b]$.

Proof: Since a function satisfies uniform Lipschitz condition of order $\alpha > 0,$ it must be continuous. So, we consider

$$f(x) = \begin{cases} x \text{ if } x \in [a,b) \\ b+1 \text{ if } x = b. \end{cases}$$

Trivially, f is not continuous but increasing. So, the function is desired.

Remark: Here is a good problem, we write it as follows. If f satisfies

$$|f(x) - f(y)| \leq K|x-y|^{1/2} \text{ for } x \in [0,1], \text{ where } f(0) = 0.$$

define

$$g(x) = \begin{cases} \frac{f(x)}{x^{1/3}} \text{ if } x \in (0,1] \\ 0 \text{ if } x = 0. \end{cases}$$

Then g satisfies uniform Lipschitz condition of order 1/6.

Proof: Note that if one of x, and y is zero, the result is trivial. So, we may consider $0 < y < x \leq 1$ as follows. Consider

$$\begin{aligned} |g(x) - g(y)| &= \left|\frac{f(x)}{x^{1/3}} - \frac{f(y)}{y^{1/3}}\right| \\ &= \left|\frac{f(x)}{x^{1/3}} - \frac{f(y)}{x^{1/3}} + \frac{f(y)}{x^{1/3}} - \frac{f(y)}{y^{1/3}}\right| \\ &\leq \left|\frac{f(x)}{x^{1/3}} - \frac{f(y)}{x^{1/3}}\right| + \left|\frac{f(y)}{x^{1/3}} - \frac{f(y)}{y^{1/3}}\right|. \quad * \end{aligned}$$

For the part

$$\left|\frac{f(x)}{x^{1/3}}-\frac{f(y)}{x^{1/3}}\right| = \frac{1}{x^{1/3}}|f(x)-f(y)|$$
$$\le \frac{K}{x^{1/3}}|x-y|^{1/2} \text{ by hypothesis}$$
$$\le K|x-y|^{1/2}|x-y|^{-1/3} \text{ since } x \ge x-y > 0$$
$$= K|x-y|^{1/6}. \qquad \text{A}$$

For another part $\left|\frac{f(y)}{x^{1/3}}-\frac{f(y)}{y^{1/3}}\right|$, we consider two cases.

(1) $x \ge 2y$ which implies that $x > x-y \ge y > 0$,

$$\left|\frac{f(y)}{x^{1/3}}-\frac{f(y)}{y^{1/3}}\right| = |f(y)|\left|\frac{x^{1/3}-y^{1/3}}{(xy)^{1/3}}\right|$$
$$\le |f(y)|\left|\frac{(x-y)^{1/3}}{(xy)^{1/3}}\right| \text{ since } |x^{1/3}-y^{1/3}| \le |x-y|^{1/3} \text{ for all } x,y \ge 0$$
$$\le |f(y)|\left|\frac{x^{1/3}}{(xy)^{1/3}}\right| \text{ since } (x-y)^{1/3} \le x^{1/3}$$
$$\le |f(y)|\left|\frac{1}{y^{1/3}}\right|$$
$$\le K\frac{|y|^{1/2}}{|y|^{1/3}} \text{ by hypothesis}$$
$$\le K|y|^{1/6}$$
$$\le K|x-y|^{1/6} \text{ since } y \le x-y. \qquad \text{B}$$

(2) $x < 2y$ which implies that $x > y > x-y > 0$,

$$\left|\frac{f(y)}{x^{1/3}}-\frac{f(y)}{y^{1/3}}\right| = |f(y)|\left|\frac{x^{1/3}-y^{1/3}}{(xy)^{1/3}}\right|$$
$$\le |f(y)|\left|\frac{(x-y)^{1/3}}{(xy)^{1/3}}\right| \text{ since } |x^{1/3}-y^{1/3}| \le |x-y|^{1/3} \text{ for all } x,y \ge 0$$
$$\le |f(y)|\left|\frac{(x-y)^{1/3}}{y^{2/3}}\right| \text{ since } x > y$$
$$\le K|y|^{1/2}\left|\frac{(x-y)^{1/3}}{y^{2/3}}\right| \text{ by hypothesis}$$
$$\le K|y|^{-1/6}|x-y|^{1/3}$$
$$\le K|x-y|^{-1/6}|x-y|^{1/3} \text{ since } y > x-y$$
$$= K|x-y|^{1/6}. \qquad \text{C}$$

So, by (A)-(C), (*) tells that g satisfies uniform Lipschitz condition of order 1/6.

Note: Here is a general result. Let $0 \le \beta < \alpha < 2\beta$. If f satisfies

$$|f(x)-f(y)| \le K|x-y|^{\alpha} \text{ for } x \in [0,1], \text{ where } f(0) = 0.$$

define

$$g(x) = \begin{cases} \frac{f(x)}{x^{\beta}} \text{ if } x \in (0,1] \\ 0 \text{ if } x = 0. \end{cases}$$

Then g satisfies uniform Lipschitz condition of order $\alpha - \beta$. The proof is similar, so we omit it.

6.3 Show that a polynomial f is of bounded variation on every compact interval $[a,b]$. Describe a method for finding the total variation of f on $[a,b]$ if the zeros of the derivative f' are known.

Proof: If f is a constant, then the total variation of f on $[a,b]$ is zero. So, we may assume that f is a polynomial of degree $n \geq 1$, and consider $f'(x) = 0$ by two cases as follows.

(1) If there is no point such that $f'(x) = 0$, then by **Intermediate Value Theorem of Differentiability**, we know that $f'(x) > 0$ on $[a,b]$, or $f'(x) < 0$ on $[a,b]$. So, it implies that f is monotonic. Hence, the total variation of f on $[a,b]$ is $|f(b) - f(a)|$.

(2) If there are m points such that $f'(x) = 0$, say $a = x_0 \leq x_1 < x_2 <\ldots< x_m \leq b = x_{m+1}$, where $1 \leq m \leq n$, then we know the monotone property of function f. So, the total variation of f on $[a,b]$ is

$$\sum_{i=1}^{m+1} |f(x_i) - f(x_{i-1})|.$$

Remark: Here is another proof. Let f be a polynomial on $[a,b]$, then we know that f' is bounded on $[a,b]$ since f' is also polynomial which implies that it is continuous. Hence, we know that f is of bounded variation on $[a,b]$.

6.4 A nonempty set S of real-valued functions defined on an interval $[a,b]$ is called a linear space of functions if it has the following two properties:

(a) If $f \in S$, then $cf \in S$ for every real number c.

(b) If $f \in S$ and $g \in S$, then $f + g \in S$.

Theorem 6.9 shows that the set V of all functions of bounded variation on $[a,b]$ is a linear space. If S is any linear space which contains all monotonic functions on $[a,b]$, prove that $V \subseteq S$. This can be described by saying that **the functions of bounded variation form the samllest linear space containing all monotonic functions.**

Proof: It is directlt from Theorem 6.9 and some facts in Linear Algebra. We omit the detail.

6.5 Let f be a real-valued function defined on $[0,1]$ such that $f(0) > 0$, $f(x) \neq x$ for all x, and $f(x) \leq f(y)$ whenever $x \leq y$. Let $A = \{x : f(x) > x\}$. Prove that $\sup A \in A$, and that $f(1) > 1$.

Proof: Note that since $f(0) > 0$, A is not empty. Suppose that $\sup A := a \notin A$, i.e., $f(a) < a$ since $f(x) \neq x$ for all x. So, given any $\varepsilon_n > 0$, then there is a $b_n \in A$ such that

$$a - \varepsilon_n < b_n. \quad *$$

In addition,

$$b_n < f(b_n) \text{ since } b_n \in A. \quad **$$

So, by (*) and (**), we have (let $\varepsilon_n \to 0^+$),

$$a \leq f(a^-) \ (< f(a)) \text{ since } f \text{ is monotonic increasing.}$$

which contradicts to $f(a) < a$. Hence, we know that $\sup A \in A$.

Claim that $1 = \sup A$. Suppose **NOT**, that is, $a < 1$. Then we have

$$a < f(a) < f(1) < 1.$$

Since $a = \sup A$, consider $x \in (a, f(a))$, then

$$f(x) < x$$

which implies that

$$f(a^+) \leq a$$

which contradicts to $a < f(a)$. So, we know that $\sup A = 1$. Hence, we have proved that $f(1) > 1$.

Remark: The reader should keep the method in mind if we ask how to show that $f(1) > 1$ directly. The set A is helpful to do this. Or equivalently, let f be strictly increasing on $[0,1]$ with $f(0) > 0$. If $f(1) \leq 1$, then there exists a point $x \in [0,1]$ such that $f(x) = x$.

6.6 If f is defined everywhere in R^1, then f is said to be of bounded variation on $(-\infty, +\infty)$ if f is of bounded variation on every finite interval and if there exists a positive number M such that $V_f(a,b) < M$ for all compact interval $[a,b]$. The total variation of f on $(-\infty, +\infty)$ is then defined to be the sup of all numbers $V_f(a,b)$, $-\infty < a < b < +\infty$, and denoted by $V_f(-\infty, +\infty)$. Similar definitions apply to half open infinite intervals $[a, +\infty)$ and $(-\infty, b]$.

(a) State and prove theorems for the inifnite interval $(-\infty, +\infty)$ analogous to the Theorems 6.7, 6.9, 6.10, 6.11, and 6.12.

(**Theorem 6.7***) Let $f : R \to R$ be of bounded variaton, then f is bounded on R.

Proof: Given any $x \in R$, then $x \in [0,a]$ or $x \in [a,0]$. If $x \in [0,a]$, then f is bounded on $[0,a]$ with

$$|f(x)| \leq |f(0)| + V_f(0,a) \leq |f(0)| + V_f(-\infty, +\infty).$$

Similarly for $x \in [a,0]$.

(**Theorem 6.9***) Assume that f, and g be of bounded variaton on R, then so are thier sum, difference, and product. Also, we have

$$V_{f\pm g}(-\infty, +\infty) \leq V_f(-\infty, +\infty) + V_g(-\infty, +\infty)$$

and

$$V_{fg}(-\infty, +\infty) \leq AV_f(-\infty, +\infty) + BV_g(-\infty, +\infty),$$

where $A = \sup_{x \in R}|g(x)|$ and $B = \sup_{x \in R}|f(x)|$.

Proof: For sum and difference, given any compact interval $[a,b]$, we have

$$\begin{aligned} V_{f\pm g}(a,b) &\leq V_f(a,b) + V_g(a,b), \\ &\leq V_f(-\infty, +\infty) + V_g(-\infty, +\infty) \end{aligned}$$

which implies that

$$V_{f\pm g}(-\infty, +\infty) \leq V_f(-\infty, +\infty) + V_g(-\infty, +\infty).$$

For product, given any compact interval $[a,b]$, we have (let $A(a,b) = \sup_{x \in [a,b]}|g(x)|$, and $B(a,b) = \sup_{x \in [a,b]}|f(x)|$),

$$\begin{aligned} V_{fg}(a,b) &\leq A(a,b)V_f(a,b) + B(a,b)V_g(a,b) \\ &\leq AV_f(-\infty, +\infty) + BV_g(-\infty, +\infty) \end{aligned}$$

which implies that

$$V_{fg}(-\infty,+\infty) \le AV_f(-\infty,+\infty) + BV_g(-\infty,+\infty).$$

(**Theorem 6.10***) Let f be of bounded variation on R, and assume that f is bounded away from zero; that is, suppose that there exists a positive number m such that $0 < m \le |f(x)|$ for all $x \in R$. Then $g = 1/f$ is also of bounded variation on R, and

$$V_g(-\infty,+\infty) \le \frac{V_f(-\infty,+\infty)}{m^2}.$$

Proof: Given any compacgt interval $[a,b]$, we have

$$V_g(a,b) \le \frac{V_f(a,b)}{m^2} \le \frac{V_f(-\infty,+\infty)}{m^2}$$

which implies that

$$V_g(-\infty,+\infty) \le \frac{V_f(-\infty,+\infty)}{m^2}.$$

(**Theorem 6.11***) Let f be of bounded variation on R, and assume that $c \in R$. Then f is of bounded variation on $(-\infty,c]$ and on $[c,+\infty)$ and we have

$$V_f(-\infty,+\infty) = V_f(-\infty,c) + V_f(c,+\infty).$$

Proof: Given any a compact interval $[a,b]$ such that $c \in (a,b)$. Then we have

$$V_f(a,b) = V_f(a,c) + V_f(c,b).$$

Since

$$V_f(a,b) \le V_f(-\infty,+\infty)$$

which implies that

$$V_f(a,c) \le V_f(-\infty,+\infty) \text{ and } V_f(c,b) \le V_f(-\infty,+\infty),$$

we know that the existence of $V_f(-\infty,c)$ and $V_f(c,+\infty)$. That is, f is of bounded variation on $(-\infty,c]$ and on $[c,+\infty)$.

Since

$$V_f(a,c) + V_f(c,b) = V_f(a,b) \le V_f(-\infty,+\infty)$$

which implies that

$$V_f(-\infty,c) + V_f(c,+\infty) \le V_f(-\infty,+\infty), \quad *$$

and

$$V_f(a,b) = V_f(a,c) + V_f(c,b) \le V_f(-\infty,c) + V_f(c,+\infty)$$

which implies that

$$V_f(-\infty,+\infty) \le V_f(-\infty,c) + V_f(c,+\infty), \quad **$$

we know that

$$V_f(-\infty,+\infty) = V_f(-\infty,c) + V_f(c,+\infty).$$

(**Theorem 6.12***) Let f be of bounded variation on R. Let $V(x)$ be defined on $(-\infty,x]$ as follows:

$$V(x) = V_f(-\infty,x) \text{ if } x \in R, \text{ and } V(-\infty) = 0.$$

Then (i) V is an increasing function on $(-\infty,+\infty)$ and (ii) $V-f$ is an increasing function on $(-\infty,+\infty)$.

Proof: (i) Let $x < y$, then we have $V(y) - V(x) = V_f(x,y) \ge 0$. So, we know that V is an increasing function on $(-\infty,+\infty)$.

(ii) Let $x < y$, then we have $(V-f)(y) - (V-f)(x) = V_f(x,y) - (f(y) - f(x)) \ge 0$. So, we know that $V-f$ is an increasing function on $(-\infty,+\infty)$.

(b) Show that Theorem 6.5 is true for $(-\infty,+\infty)$ if "monotonic" is replaced by "bounded and monotonic." State and prove a similar modefication of Theorem 6.13.

(**Theorem 6.5***) If f is bounded and monotonic on $(-\infty,+\infty)$, then f is of bounded variation on $(-\infty,+\infty)$.

Proof: Given any compact interval $[a,b]$, then we have $V_f(a,b)$ exists, and we have $V_f(a,b) = |f(b)-f(a)|$, since f is monotonic. In addition, since f is bounded on R, say $|f(x)| \leq M$ for all x, we know that $2M$ is a upper bounded of $V_f(a,b)$ for all a,b. Hence, $V_f(-\infty,+\infty)$ exists. That is, f is of bounded variation on R.

(**Theorem 6.13***) Let f be defined on $(-\infty,+\infty)$, then f is of bounded variation on $(-\infty,+\infty)$ if, and only if, f can be expressed as the difference of two increasing and bounded functions.

Proof: Suppose that f is of bounded variation on $(-\infty,+\infty)$, then by **Theorem 6.12***, we know that

$$f = V-(V-f),$$

where V and $V-f$ are increasing on $(-\infty,+\infty)$. In addition, since f is of bounded variation on R, we know that V and f is bounded on R which implies that $V-f$ is bounded on R. So, we have proved that if f is of bounded variation on $(-\infty,+\infty)$ then f can be expressed as the difference of two increasing and bounded functions.

Suppose that f can be expressed as the difference of two increasing and bounded functions, say $f = f_1 - f_2$, Then by **Theorem 6.9***, and **Theorem 6.5***, we know that f is of bounded variaton on R.

Remark: The representation of a function of bounded variation as a difference of two increasing and bounded functions is by no mean unique. It is clear that **Theorem 6.13*** also holds if "increasing" is replaced by "strictly increasing." For example, $f = (f_1+g)-(f_2+g)$, where g is any strictly increasing and bounded function on R. One of such g is $\arctan x$.

6.7 Assume that f is of bounded variation on $[a,b]$ and let

$$P = \{x_0,x_1,\ldots,x_n\} \in \wp[a,b].$$

As usual, write $\Delta f_k = f(x_k)-f(x_{k-1})$, $k = 1,2,\ldots,n$. Define

$$A(P) = \{k : \Delta f_k > 0\},\ B(P) = \{k : \Delta f_k < 0\}.$$

The numbers

$$p_f(a,b) = \sup\left\{\sum_{k\in A(P)} \Delta f_k : P \in \wp[a,b]\right\}$$

and

$$n_f(a,b) = \sup\left\{\sum_{k\in B(P)} |\Delta f_k| : P \in \wp[a,b]\right\}$$

are called respectively, the positive and negative variations of f on $[a,b]$. For each x in $(a,b]$. Let $V(x) = V_f(a,x)$, $p(x) = p_f(a,x)$, $n(x) = n_f(a,x)$, and let $V(a) = p(a) = n(a) = 0$. Show that we have:

(a) $V(x) = p(x)+n(x)$.

Proof: Given a partition P on $[a,x]$, then we have

$$\sum_{k=1}^{n}|\Delta f_k| = \sum_{k\in A(P)}|\Delta f_k| + \sum_{k\in B(P)}|\Delta f_k|$$
$$= \sum_{k\in A(P)}\Delta f_k + \sum_{k\in B(P)}|\Delta f_k|,$$

which implies that (taking supermum)

$$V(x) = p(x) + n(x). \qquad *$$

Remark: The existence of $p(x)$ and $q(x)$ is clear, so we know that (*) holds by **Theorem 1.15.**

(b) $0 \leq p(x) \leq V(x)$ and $0 \leq n(x) \leq V(x)$.

Proof: Consider $[a,x]$, and since

$$V(x) \geq \sum_{k=1}^{n}|\Delta f_k| \geq \sum_{k\in A(P)}|\Delta f_k|,$$

we know that $0 \leq p(x) \leq V(x)$. Similarly for $0 \leq n(x) \leq V(x)$.

(c) p and n are increasing on $[a,b]$.

Proof: Let x,y in $[a,b]$ with $x < y$, and consider $p(y) - p(x)$ as follows. Since

$$p(y) \geq \sum_{k\in A(P),\,[a,y]}\Delta f_k \geq \sum_{k\in A(P),\,[a,x]}\Delta f_k,$$

we know that

$$p(y) \geq p(x).$$

That is, p is increasing on $[a,b]$. Similarly for n.

(d) $f(x) = f(a) + p(x) - n(x)$. Part (d) gives an alternative proof of Theorem 6.13.

Proof: Consider $[a,x]$, and since

$$f(x) - f(a) = \sum_{k=1}^{n}\Delta f_k = \sum_{k\in A(P)}\Delta f_k + \sum_{k\in B(P)}\Delta f_k$$

which implies that

$$f(x) - f(a) + \sum_{k\in B(P)}|\Delta f_k| = \sum_{k\in A(P)}\Delta f_k$$

which implies that $f(x) = f(a) + p(x) - n(x)$.

(e) $2p(x) = V(x) + f(x) - f(a)$, $2n(x) = V(x) - f(x) + f(a)$.

Proof: By (d) and (a), the statement is obvious.

(f) Every point of continuity of f is also a point of continuity of p and of n.

Proof: By (e) and **Theorem 6.14**, the statement is obvious.

Curves

6.8 Let f and g be complex-valued functions defined as follows:

$$f(t) = e^{2\pi it} \text{ if } t \in [0,1],\quad g(t) = e^{2\pi it} \text{ if } t \in [0,2].$$

(a) Prove that f and g have the same graph but are not equivalent according to defintion in Section 6.12.

Proof: Since $\{f(t) : t \in [0,1]\} = \{g(t) : t \in [0,2]\}$ = the circle of unit disk, we know that f and g have the same graph.

If f and g are equivalent, then there is an 1-1 and onto function $\phi : [0,2] \to [0,1]$ such that

$$f(\phi(t)) = g(t).$$

That is,

$$e^{2\pi i\phi(t)} = \cos 2\pi(\phi(t)) + i\sin 2\pi(\phi(t)) = e^{2\pi it} = \cos 2\pi t + i\sin 2\pi t.$$

In paticular, $\phi(1) := c \in (0,1)$. However,

$$f(c) = \cos 2\pi c + i\sin 2\pi c = g(1) = 1$$

which implies that $c \in Z$, a contradiction.

(b) Prove that the length of g is twice that of f.

Proof: Since

$$\text{the length of } g = \int_0^2 |g'(t)|dt = 4\pi$$

and

$$\text{the length of } f = \int_0^1 |f'(t)|dt = 2\pi,$$

we know that the length of g is twice that of f.

6.9 Let f be rectifiable path of length L defined on $[a,b]$, and assume that f is not constant on any subinterval of $[a,b]$. Let s denote the arc length function given by $s(x) = \Lambda_f(a,x)$ if $a < x \leq b$, $s(a) = 0$.

(a) Prove that s^{-1} exists and is continuous on $[0,L]$.

Proof: By **Theorem 6.19**, we know that $s(x)$ is continuous and strictly increasing on $[0,L]$. So, the inverse function s^{-1} exists since s is an 1-1 and onto function, and by **Theorem 4.29**, we know that s^{-1} is continuous on $[0,L]$.

(b) Define $g(t) = f[s^{-1}(t)]$ if $t \in [0,L]$ and show that g is equivalent to f. Since $f(t) = g[s(t)]$, the function g is said to provide a representation of the graph of f **with arc length as parameter.**

Proof: t is clear by **Theorem 6.20**.

6.10 Let f and g be two real-valued continuous functions of bounded variation defined on $[a,b]$, with $0 < f(x) < g(x)$ for each x in (a,b), $f(a) = g(a)$, $f(b) = g(b)$. Let h be the complex-valued function defined on the interval $[a, 2b-a]$ as follows:

$$\begin{aligned} h(t) &= t + if(t), \text{ if } a \leq t \leq b \\ &= 2b - t + ig(2b-t), \text{ if } b \leq t \leq 2b-a. \end{aligned}$$

(a) Show that h describes a rectifiable curve Γ.

Proof: It is clear that h is continuous on $[a, 2b-a]$. Note that t, f and g are of bounded variation on $[a,b]$, so $\Lambda_h(a, 2b-a)$ exists. That is, h is rectifiable on $[a, 2b-a]$.

(b) Explain, by means of a sketch, the geometric relationship between f, g, and h.

Solution: The reader can give it a draw and see the graph lying on $x-y$ plane is a closed region.

(c) Show that the set of points

$$S = \{(x,y) : a \leq x \leq b, f(x) \leq y \leq g(x)\}$$

in a region in R^2 whose boundary is the curve Γ.

Proof: It can be answered by (b), so we omit it.

(d) Let H be the complex-valued function defined on $[a,2b-a]$ as follows:

$$H(t) = t - \frac{1}{2}i[g(t) - f(t)], \text{ if } a \leq t \leq b$$
$$= 2b - t + \frac{1}{2}i[g(2b-t) - f(2b-t)], \text{ if } b \leq t \leq 2b-a.$$

Show that H describes a rectifiable curve Γ_0 which is the boundary of the region

$$S_0 = \{(x,y) : a \leq x \leq b, f(x) - g(x) \leq 2y \leq g(x) - f(x)\}.$$

Proof: Let $F(t) = \frac{-1}{2}[g(t) - f(t)]$ and $G(t) = \frac{1}{2}[g(t) - f(t)]$ defined on $[a,b]$. It is clear that $F(t)$ and $G(t)$ are of bounded variation and continuous on $[a,b]$ with $0 < F(x) < G(x)$ for each $x \in (a,b)$, $F(b) = G(b) = 0$, and $F(b) = G(b) = 0$. In addition, we have

$$H(t) = t + iF(t), \text{ if } a \leq t \leq b$$
$$= 2b - t + iG(2b-t), \text{ if } b \leq t \leq 2b-a.$$

So, by preceding (a)-(c), we have prove it.

(e) Show that, S_0 has the x –axis as a line of symmetry. (The region S_0 is called the symmetrization of S with respect to x –axis.)

Proof: It is clear since $(x,y) \in S_0 \Leftrightarrow (x,-y) \in S_0$ by the fact

$$f(x) - g(x) \leq 2y \leq g(x) - f(x).$$

(f) Show that the length of Γ_0 does not exceed the length of Γ.

Proof: By (e), the symmetrization of S with respect to x –axis tells that $\Lambda_H(a,b) = \Lambda_H(b,2b-a)$. So, it suffices to show that $\Lambda_h(a,2b-a) \geq 2\Lambda_H(a,b)$. Choosing a partition $P_1 = \{x_0 = a, \ldots, x_n = b\}$ on $[a,b]$ such that

$$2\Lambda_H(a,b) - \varepsilon < 2\Lambda_H(P_1)$$
$$= 2\sum_{i=1}^{n}\left\{(x_i - x_{i-1})^2 + \left[\frac{1}{2}(f-g)(x_i) - \frac{1}{2}(f-g)(x_{i-1})\right]^2\right\}^{1/2}$$
$$= \sum_{i=1}^{n}\left\{4(x_i - x_{i-1})^2 + [(f-g)(x_i) - (f-g)(x_{i-1})]^2\right\}^{1/2} \quad *$$

and note that $b - a = (2b-a) - b$, we use this P_1 to produce a partition $P_2 = P_1 \cup \{x_n = b, x_{n+1} = b + (x_n - x_{n-1}), \ldots, x_{2n} = 2b-a\}$ on $[a,2b-a]$. Then we have

$$\begin{aligned}\Lambda_h(P_2) &= \sum_{i=1}^{2n} \| h(x_i) - h(x_{i-1}) \| \\ &= \sum_{i=1}^{n} \| h(x_i) - h(x_{i-1}) \| + \sum_{i=n+1}^{2n} \| h(x_i) - h(x_{i-1}) \| \\ &= \sum_{i=1}^{n} \left[(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2 \right]^{1/2} + \sum_{i=n+1}^{2n} \left[(x_i - x_{i-1})^2 + (g(x_i) - g(x_{i-1}))^2 \right]^{1/2} \\ &= \sum_{i=1}^{n} \left\{ \left[(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2 \right]^{1/2} + \left[(x_i - x_{i-1})^2 + (g(x_i) - g(x_{i-1}))^2 \right]^{1/2} \right\} \qquad **\end{aligned}$$

From (*) and (**), we know that

$$2\Lambda_H(a,b) - \varepsilon < 2\Lambda_H(P_1) \leq \Lambda_h(P_2) \qquad ***$$

which implies that

$$\Lambda_H(a, 2b-a) = 2\Lambda_H(a,b) \leq \Lambda_h(a, 2b-a).$$

So, we know that the length of Γ_0 does not exceed the length of Γ.

Remark: Define $x_i - x_{i-1} = a_i$, $f(x_i) - f(x_{i-1}) = b_i$, and $g(x_i) - g(x_{i-1}) = c_i$, then we have

$$\left(4a_i^2 + (b_i - c_i)^2\right)^{1/2} \leq (a_i^2 + b_i^2)^{1/2} + (a_i^2 + c_i^2)^{1/2}.$$

Hence we have the result (***).

Proof: It suffices to square both side. We leave it to the reader.

Absolutely continuous functions

A real-valued function f defined on $[a,b]$ is said to be **absolutely continuous** on $[a,b]$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every n **disjoint** open subintervals (a_k, b_k) of $[a,b]$, $n = 1,2,\ldots,$ the sum of whose lengths $\sum_{k=1}^{n}(b_k - a_k)$ is less than δ.

Absolutely continuous functions occur in the Lebesgue theory of integration and differentiation. The following exercises give some of their elementary properties.

6.11 Prove that every absolutely continuous function on $[a,b]$ is continuous and of bounded variation on $[a,b]$.

Proof: Let f be absolutely continuous on $[a,b]$. Then $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every n **disjoint** open subintervals (a_k, b_k) of $[a,b]$, $n = 1,2,\ldots,$ the sum of whose lengths $\sum_{k=1}^{n}(b_k - a_k)$ is less than δ. So, as $|x-y| < \delta$, where $x,y \in [a,b]$, we have

$$|f(x) - f(y)| < \varepsilon.$$

That is, f is uniformly continuous on $[a,b]$. So, f is continuous on $[a,b]$.

In addition, given any $\varepsilon = 1$, there exists a $\delta > 0$ such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals in $[a,b]$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < 1.$$

For this δ, and let K be the smallest positive integer such that $K(\delta/2) \geq b - a$. So, we partition $[a,b]$ into K closed subintervals, i.e.,
$P = \{y_0 = a, y_1 = a + \delta/2, \ldots, y_{K-1} = a + (K-1)(\delta/2), y_K = b\}$. So, it is clear that f is of bounded variation $[y_i, y_{i+1}]$, where $i = 0, 1, \ldots, K$. It implies that f is of bounded variation on $[a,b]$.

Note: **There exists functions which are continuous and of bounded variation but not absolutely continuous**.

Remark: 1. The standard example is called **Cantor-Lebesgue function**. The reader can see this in the book, **Measure and Integral, An Introduction to Real Analysis by Richard L. Wheeden and Antoni Zygmund**, pp 35 and pp 115.

2. If we wrtie "absolutely continuous" by **ABC**, "continuous" by **C,** and "bounded variation" by **B,** then it is clear that by preceding result, **ABC** implies **B** and **C**, and **B** and **C** do **NOT** imply **ABC**.

6.12 Prove that f is absolutely continuous if it satisfies a uniform Lipschitz condition of order 1 on $[a,b]$. (See Exercise 6.2)

Proof: Let f satisfy a uniform Lipschitz condition of order 1 on $[a,b]$, i.e., $|f(x) - f(y)| \leq M|x - y|$ where $x, y \in [a,b]$. Then given $\varepsilon > 0$, there is a $\delta = \varepsilon/M$ such that as $\sum_{k=1}^{n} (b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open subintervals on $[a,b]$, $k = 1, .., n$, we have

$$\begin{aligned}
\sum_{k=1}^{n} |f(b_k) - f(a_k)| &\leq \sum_{k=1}^{n} M|b_k - a_k| \\
&= \sum_{k=1}^{n} M(b_k - a_k) \\
&< M\delta \\
&= \varepsilon.
\end{aligned}$$

Hence, f is absolutely continuous on $[a,b]$.

6.13 If f and g are absolutely continunous on $[a,b]$, prove that each of the following is also: $|f|$, cf (c constant), $f + g$, $f \cdot g$; also f/g if g is bounded away from zero.

Proof: (1) ($|f|$ is absolutely continuous on $[a,b]$): Given $\varepsilon > 0$, we want to find a $\delta > 0$, such that as $\sum_{k=1}^{n} (b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} ||f(b_k)| - |f(a_k)|| < \varepsilon. \qquad 1*$$

Since f is absolutely continunous on $[a,b]$, for this ε, there is a $\delta > 0$ such that as $\sum_{k=1}^{n} (b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

which implies that (1*) holds by the following

$$\sum_{k=1}^{n} ||f(b_k)| - |f(a_k)|| \leq \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon.$$

So, we know that $|f|$ is absolutely continuous on $[a,b]$.

(2) (cf is absolutely continuous on $[a,b]$): If $c = 0$, it is clear. So, we may assume that $c \neq 0$. Given $\varepsilon > 0$, we want to find a $\delta > 0$, such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |cf(b_k) - cf(a_k)| < \varepsilon. \qquad 2^*$$

Since f is absolutely continunous on $[a,b]$, for this ε, there is a $\delta > 0$ such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon/|c|$$

which implies that (2*) holds by the following

$$\sum_{k=1}^{n} |cf(b_k) - cf(a_k)| = |c| \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon.$$

So, we know that cf is absolutely continuous on $[a,b]$.

(3) ($f+g$ is absolutely continuous on $[a,b]$): Given $\varepsilon > 0$, we want to find a $\delta > 0$, such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |(f+g)(b_k) - (f+g)(a_k)| < \varepsilon. \qquad 3^*$$

Since f and g are absolutely continunous on $[a,b]$, for this ε, there is a $\delta > 0$ such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon/2 \text{ and } \sum_{k=1}^{n} |g(b_k) - g(a_k)| < \varepsilon/2$$

which implies that (3*) holds by the following

$$\begin{aligned} &\sum_{k=1}^{n} |(f+g)(b_k) - (f+g)(a_k)| \\ &= \sum_{k=1}^{n} |f(b_k) - f(a_k) + g(b_k) - g(a_k)| \\ &\leq \sum_{k=1}^{n} |f(b_k) - f(a_k)| + \sum_{k=1}^{n} |g(b_k) - g(a_k)| \\ &< \varepsilon. \end{aligned}$$

So, we know that $f+g$ is absolutely continuous on $[a,b]$.

(4) ($f \cdot g$ is absolutely continuous on $[a,b]$.): Let $M_f = \sup_{x \in [a,b]} |f(x)|$ and $M_g = \sup_{x \in [a,b]} |g(x)|$. Given $\varepsilon > 0$, we want to find a $\delta > 0$, such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n}|(f+g)(b_k)-(f+g)(a_k)|<\varepsilon. \qquad 4^*$$

Since f and g are absolutely continunous on $[a,b]$, for this ε, there is a $\delta>0$ such that as $\sum_{k=1}^{n}(b_k-a_k)<\delta$, where $(a_k,b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n}|f(b_k)-f(a_k)|<\frac{\varepsilon}{2(M_g+1)} \text{ and } \sum_{k=1}^{n}|g(b_k)-g(a_k)|<\frac{\varepsilon}{2(M_f+1)}$$

which implies that (4*) holds by the following

$$\begin{aligned}
&\sum_{k=1}^{n}|(f\cdot g)(b_k)-(f\cdot g)(a_k)|\\
&=\sum_{k=1}^{n}|f(b_k)(g(b_k)-g(a_k))+g(a_k)(f(b_k)-f(a_k))|\\
&\le M_f\sum_{k=1}^{n}|g(b_k)-g(a_k)|+M_g\sum_{k=1}^{n}|f(b_k)-f(a_k)|\\
&<\frac{\varepsilon M_f}{2(M_f+1)}+\frac{\varepsilon M_g}{2(M_g+1)}\\
&<\varepsilon.
\end{aligned}$$

Remark: The part shows that f^n is absolutely continuous on $[a,b]$, where $n\in N$, if f is absolutely continuous on $[a,b]$.

(5) (f/g is absolutely continuous on $[a,b]$): By (4) it suffices to show that $1/g$ is absolutely continuous on $[a,b]$. Since g is bounded away from zero, say $0<m\le g(x)$ for all $x\in[a,b]$. Given $\varepsilon>0$, we want to find a $\delta>0$, such that as $\sum_{k=1}^{n}(b_k-a_k)<\delta$, where $(a_k,b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n}|(1/g)(b_k)-(1/g)(a_k)|<\varepsilon. \qquad 5^*$$

Since g is absolutely continunous on $[a,b]$, for this ε, there is a $\delta>0$ such that as $\sum_{k=1}^{n}(b_k-a_k)<\delta$, where $(a_k,b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n}|g(b_k)-g(a_k)|<m^2\varepsilon$$

which implies that (4*) holds by the following

$$\begin{aligned}
&\sum_{k=1}^{n}|(1/g)(b_k)-(1/g)(a_k)|\\
&=\sum_{k=1}^{n}\left|\frac{g(b_k)-g(a_k)}{g(b_k)g(a_k)}\right|\\
&\le\frac{1}{m^2}\sum_{k=1}^{n}|g(b_k)-g(a_k)|\\
&<\varepsilon.
\end{aligned}$$