Derivatives

Real-valued functions

In each following exercise assume, where mecessary, a knowledge of the formulas for differentiating the elementary trigonometric, exponential, and logarithmic functions.

5.1 Assume that f is said to satisfy a **Lipschitz condition of order α at c** if there exists a positive number M (which may depend on c) and 1 −ball $B(c)$ such that

$$|f(x) - f(c)| < M|x - c|^{\alpha}$$

whenever $x \in B(c)$, $x \neq c$.

(a) Show that a function which satisfies a Lipschitz condition of order α is continuous at c if $\alpha > 0$, and has a derivative at c if $\alpha > 1$.

Proof: 1. As $\alpha > 0$, given $\varepsilon > 0$, there is a $\delta \leq (\varepsilon/M)^{1/\alpha}$ such that as $x \in (c - \delta, c + \delta) \subseteq B(c)$, we have

$$|f(x) - f(c)| < M|x - c|^{\alpha} \leq M\delta^{\alpha} = \varepsilon.$$

So, we know that f is continuous at c.

2. As $\alpha > 1$, consider $x \in B(c)$, and $x \neq c$, we have

$$\left|\frac{f(x) - f(c)}{x - c}\right| \leq M|x - c|^{\alpha - 1} \to 0 \text{ as } x \to c.$$

So, we know that f has a derivative at c with $f'(c) = 0$.

Remark: It should be note that (a) also holds if we consider the higher dimension.

(b) Given an example of a function satisfying a Lipschitz condition of order 1 at c for which $f'(c)$ does not exist.

Solution: Consider

$$||x| - |c|| \leq |x - c|,$$

we know that $|x|$ is a function satisfying a Lipschitz condition of order 1 at 0 for which $f'(0)$ does not exist.

5.2 In each of the following cases, determine the intervals in which the function f is increasing or decreasing and find the maxima and minima (if any) in the set where each f is defined.

(a) $f(x) = x^3 + ax + b$, $x \in R$.

Solution: Since $f'(x) = 3x^2 + a$ on R, we consider two cases: (i) $a \geq 0$, and (ii) $a < 0$.

(i) As $a \geq 0$, we know that f is increasig on R by $f' \geq 0$ on R. In addition, if f has a local extremum at some point c, then $f'(c) = 0$. It implies that $a = 0$ and $c = 0$. That is, $f(x) = x^3 + b$ has a local extremum at 0. It is impossible since x^3 does not. So, we know that f has no maximum and minimum.

(ii) As $a < 0$, since $f' = 3x^2 + a = 3\left(x - \sqrt{-a/3}\right)\left(x + \sqrt{-a/3}\right)$, we know that

	$(-\infty, -\sqrt{-a/3}]$	$[-\sqrt{-a/3}, \sqrt{-a/3}]$	$[\sqrt{-a/3}, +\infty)$
$f'(x)$:	≥ 0	≤ 0	≥ 0

which implies that

	$(-\infty, -\sqrt{-a/3}]$	$[-\sqrt{-a/3}, \sqrt{-a/3}]$	$[\sqrt{-a/3}, +\infty)$
$f(x)$:	↗	↘	↗

.   *

Hence, f is increasing on $(-\infty, -\sqrt{-a/3}\,]$ and $[\sqrt{-a/3}, +\infty)$, and decreasing on $\left[-\sqrt{-a/3}, \sqrt{-a/3}\,\right]$. In addition, if f has a local extremum at some point c, then $f'(c) = 0$. It implies that $c = \pm\sqrt{-a/3}$. With help of (*), we know that $f(x)$ has a local maximum $f\left(-\sqrt{-a/3}\right)$ and a local minimum $f\left(\sqrt{-a/3}\right)$.

(b) $f(x) = \log(x^2 - 9)$, $|x| > 3$.

Solution: Since $f'(x) = \frac{2x}{x^2-9}$, $|x| > 3$, we know that

$$f'(x) : \begin{array}{ll} (-\infty, -3) & (3, +\infty) \\ < 0 & > 0 \end{array}$$

which implies that

$$f(x) : \begin{array}{ll} (-\infty, -3) & (3, +\infty) \\ \searrow & \nearrow \end{array} .$$

Hence, f is increasing on $(3, +\infty)$, and decreasing on $(-\infty, -3)$. It is clear that f cannot have local extremum.

(c) $f(x) = x^{2/3}(x-1)^4$, $0 \le x \le 1$.

Solution: Since $f'(x) = \frac{2(x-1)^3}{3x^{1/3}}(7x-1)$, $0 \le x \le 1$, we know that

$$f'(x) : \begin{array}{ll} [0, 1/7] & [1/7, 1] \\ \ge 0 & \le 0 \end{array}$$

which implies that

$$f(x) : \begin{array}{ll} [0, 1/7] & [1/7, 1] \\ \nearrow & \searrow \end{array} . \qquad **$$

Hence, we know that f is increasing on $[0, 1/7]$, and decreasing on $[1/7, 1]$. In addition, if f has a local extremum at some interior point c, then $f'(c) = 0$. It implies that $c = 1/7$. With help of (**), we know that f has a local maximum $f(1/7)$, and two local minima $f(0)$, and $f(1)$.

Remark: f has the absolute maximum $f(1/7)$, and the absolute minima $f(0) = f(1) = 0$.

(d) $f(x) = (\sin x)/x$ if $x \ne 0$, $f(0) = 1$, $0 \le x \le \pi/2$.

Sulotion: Since $f'(x) = \cos x \frac{x - \tan x}{x^2}$ as $0 < x \le \pi/2$, and $f'_+(0) = 0$, in addition, $f'(x) \to 0$ as $x \to 0^+$ by **L-Hospital Rule**, we know that

$$f'(x) : \begin{array}{l} [0, \pi/2] \\ \le 0 \end{array}$$

which implies that

$$f(x) : \begin{array}{l} [0, \pi/2] \\ \searrow \end{array} . \qquad (***)$$

Hence, we know that f is decreasing on $[0, \pi/2]$. In addition, note that there is no interior point c such that $f'(c) = 0$. With help of (***), we know that f has local maximum $f(0)$, and local minimum $f(\pi/2)$.

Remark: 1. Here is a proof on $f'_+(0)$: Since

$$\lim_{x\to 0^+}\frac{\sin x-1}{x}=\lim_{x\to 0^+}\frac{-2(\sin x/2)^2}{x}=0,$$

we know that $f'_+(0)=0$.

2. f has the absolute maximum $f(0)$, and the absolute minimum $f(\pi/2)$.

5.3 Find a polynomial f of lowest possible degree such that

$$f(x_1)=a_1,\ f(x_2)=a_2,\ f'(x_1)=b_1,\ f'(x_2)=b_2$$

where $x_1\neq x_2$ and $a_1,\ a_2,\ b_1,\ b_2$ are given real numbers.

Proof: It is easy to know that the lowest degree is at most 3 since there are 4 unknows. The degree is depends on the values of $a_1,\ a_2,\ b_1,\ b_2$.

5.4 Define f as follows: $f(x)=e^{-1/x^2}$ if $x\neq 0$, $f(0)=0$. Show that

(a) f is continuous for all x.

Proof: In order to show f is continuous on R, it suffices to show f is continuous at 0. Since

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\left(\frac{1}{e}\right)^{1/x^2}=0=f(0),$$

we know that f is continuous at 0.

(b) $f^{(n)}$ is continuous for all x, and $f^{(n)}(0)=0$, $(n=1,2,\ldots)$

Proof: In order to show $f^{(n)}$ is continuous on R, it suffices to show $f^{(n)}$ is continuous at 0. Note that

$$\lim_{x\to\pm\infty}\frac{p(x)}{e^x}=0,\ \text{where } p(x) \text{ is any real polynomial.} \qquad *$$

Claim that for $x\neq 0$, we have $f^{(n)}(x)=e^{-1/x^2}P_{3n}(1/x)$, where $P_{3n}(t)$ is a real polynomial of degree $3n$ for all $n=1,2,\ldots$. As $n=0$, $f^{(0)}(x)=f(x)=e^{-1/x^2}=e^{-1/x^2}P_0(1/x)$, where $P_0(1/x)$ is a constant function 1. So, as $n=0$, it holds. Suppose that $n=k$ holds, i.e., $f^{(k)}(x)=e^{-1/x^2}P_{3k}(1/x)$, where $P_{3k}(t)$ is a real polynomial of degree $3k$. Consider $n=k+1$, we have

$$\begin{aligned} f^{(k+1)}(x) &= \left(f^{(k)}(x)\right)' \qquad ** \\ &= \left(e^{-1/x^2}P_{3k}(1/x)\right)' \text{ by induction hypothesis} \\ &= e^{-1/x^2}\left\{\left[2\left(\frac{1}{x}\right)^3P_{3k}\left(\frac{1}{x}\right)\right]-\left[\left(\frac{1}{x}\right)^2P'_{3k}\left(\frac{1}{x}\right)\right]\right\}. \end{aligned}$$

Since $[2t^3P_{3k}(t)]-[t^2P'_{3k}(t)]$ is a real polynomial of degree $3k+3$, we define $[2t^3P_{3k}(t)]-[t^2P'_{3k}(t)]=P_{3k+3}(t)$, and thus we have by (**)

$$f^{(k+1)}(x)=e^{-1/x^2}P_{3k+3}(1/x).$$

So, as $n=k+1$, it holds. Therefore, by **Mathematical Induction**, we have proved the claim.

Use the claim to show that $f^{(n)}(0)=0$, $(n=1,2,\ldots)$ as follows. As $n=0$, it is trivial by hypothesis. Suppose that $n=k$ holds, i.e., $f^{(k)}(0)=0$. Then as $n=k+1$, we have

$$\begin{aligned}\frac{f^{(k)}(x)-f^{(k)}(0)}{x-0} &= \frac{f^{(k)}(x)}{x} \text{ by induction hypothesis}\\ &= \frac{e^{-1/x^2}P_{3k}(1/x)}{x}\\ &= \frac{tP_{3k}(t)}{e^{t^2}} \text{ (let } t = 1/x)\\ &= \left(\frac{tP_{3k}(t)}{e^t}\right)\left(\frac{e^t}{e^{t^2}}\right) \to 0 \text{ as } t \to \pm\infty \ (\Leftrightarrow x \to 0) \text{ by (*).}\end{aligned}$$

Hence, $f^{(k+1)}(0) = 0.$ So, by **Mathematical Induction,** we have proved that $f^{(n)}(0) = 0,$ $(n = 1,2,\ldots).$

Since

$$\begin{aligned}\lim_{x\to 0} f^{(n)}(x) &= \lim_{x\to 0} e^{-1/x^2}P_{3n}(1/x)\\ &= \lim_{x\to 0}\frac{P_{3n}(1/x)}{e^{1/x^2}}\\ &= \lim_{t\to\pm\infty}\left(\frac{P_{3n}(t)}{e^t}\right)\left(\frac{e^t}{e^{t^2}}\right)\\ &= 0 \text{ by (*)}\\ &= f^{(n)}(0),\end{aligned}$$

we know that $f^{(n)}(x)$ is continuous at 0.

Remark: 1. Here is a proof on (*). Let $P(x)$ be a real polynomial of degree $n,$ and choose an even number $2N > n.$ We consider a **Taylor Expansion with Remainder** as follows. Since for any $x,$ we have

$$e^x = \sum_{k=0}^{2N+1}\frac{1}{k!}x^k + \frac{e^{\xi(x,0)}}{(2N+2)!}x^{2N+2} \geq \sum_{k=0}^{2N+1}\frac{1}{k!}x^k,$$

then

$$0 \leq \left|\frac{P(x)}{e^x}\right| \leq \left|\frac{P(x)}{\sum_{k=0}^{2N+1}\frac{1}{k!}x^k}\right| \to 0 \text{ as } x \to \pm\infty$$

since $\deg(P(x)) = n < \deg\left(\sum_{k=0}^{2N+1}\frac{1}{k!}x^k\right) = 2N+1.$ By **Sandwich Theorem**, we have proved

$$\lim_{x\to\pm\infty}\frac{P(x)}{e^x} = 0.$$

2. Here is another proof on $f^{(n)}(0) = 0,$ $(n = 1,2,\ldots).$ By Exercise 5.15, it suffices to show that $\lim_{x\to 0} f^{(n)}(x) = 0.$ For the part, we have proved in this exercise. So, we omit the proof. Exercise 5.15 tells us that we need not make sure that the derivative of f at 0. The reader should compare with Exercise 5.15 and Exercise 5.5.

3. In the future, we will encounter the exercise in Charpter 9. The Exercises tells us one important thing that **the Taylor's series about** 0 **generated by** f **converges everywhere on** $R,$ **but it represents** f **only at the origin.**

5.5

Define $f,$ $g,$ and h as follows: $f(0) = g(0) = h(0) = 0$ and, if $x \neq 0,$ $f(x) = \sin(1/x),$ $g(x) = x\sin(1/x),$ $h(x) = x^2\sin(1/x).$ Show that

(a) $f'(x) = -1/x^2\cos(1/x),$ if $x \neq 0;$ $f'(0)$ does not exist.

Proof: Trivially, $f'(x) = -1/x^2 \cos(1/x)$, if $x \neq 0$. Let $\left\{x_n = \frac{1}{\pi\left(2n+\frac{1}{2}\right)}\right\}$, and thus consider

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{\sin(1/x_n)}{x_n} = \pi\left(2n + \frac{1}{2}\right) \to \infty \text{ as } n \to \infty.$$

Hence, we know that $f'(0)$ does not exist.

(b) $g'(x) = \sin(1/x) - 1/x\cos(1/x)$, if $x \neq 0$; $g'(0)$ does not exist.

Proof: Trivially, $g'(x) = \sin(1/x) - 1/x\cos(1/x)$, if $x \neq 0$. Let $\left\{x_n = \frac{1}{\pi\left(2n+\frac{1}{2}\right)}\right\}$, and $\{y_n = \frac{1}{2n\pi}\}$, we know that

$$\frac{g(x_n) - g(0)}{x_n - 0} = \sin\left(\frac{1}{x_n}\right) = 1 \text{ for all } n$$

and

$$\frac{g(y_n) - g(0)}{y_n - 0} = \sin\left(\frac{1}{y_n}\right) = 0 \text{ for all } n.$$

Hence, we know that $g'(0)$ does not exist.

(c) $h'(x) = 2x\sin(1/x) - \cos(1/x)$, if $x \neq 0$; $h'(0) = 0$; $\lim_{x\to 0} h'(x)$ does not exist.

Proof: Trivially, $h'(x) = 2x\sin(1/x) - \cos(1/x)$, if $x \neq 0$. Consider

$$\left|\frac{h(x) - h(0)}{x - 0}\right| = |x\sin(1/x)| \leq |x| \to 0 \text{ as } x \to 0,$$

so we know that $h'(0) = 0$. In addition, let $\left\{x_n = \frac{1}{\pi\left(2n+\frac{1}{2}\right)}\right\}$, and $\{y_n = \frac{1}{2n\pi}\}$, we have

$$h'(x_n) = \frac{2}{\pi(2n + \frac{1}{2})} \text{ and } h'(y_n) = -1 \text{ for all } n.$$

Hence, we know that $\lim_{x\to 0} h'(x)$ does not exist.

5.6 Derive Leibnitz's formula for the nth derivative of the product h of two functions f and g :

$$h^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}(x), \text{ where } \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof: We prove it by mathematical Induction. As $n = 1$, it is clear since $h' = f'g + g'f$. Suppose that $n = k$ holds, i.e., $h^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}(x)$. Consider $n = k + 1$, we have

$$h^{(k+1)} = (h^{(k)})' = \left[\sum_{j=0}^{k}\binom{k}{j}f^{(j)}g^{(k-j)}(x)\right]'$$

$$= \sum_{j=0}^{k}\binom{k}{j}[f^{(j)}g^{(k-j)}(x)]'$$

$$= \sum_{j=0}^{k}\binom{k}{j}\{[f^{(j+1)}g^{(k-j)}] + [f^{(j)}g^{(k-j+1)}]\}$$

$$= \begin{matrix}\sum_{j=0}^{k-1}\binom{k}{j}[f^{(j+1)}g^{(k-j)}] + f^{(k+1)}g^{(0)} \\ +\sum_{j=1}^{k}\binom{k}{j}[f^{(j)}g^{(k-j+1)}] + f^{(0)}g^{(k+1)}\end{matrix}$$

$$= \begin{matrix}\sum_{j=0}^{k-1}\binom{k}{j}[f^{(j+1)}g^{(k-j)}] + f^{(k+1)}g^{(0)} \\ +\sum_{j=0}^{k-1}\binom{k}{j+1}[f^{(j+1)}g^{(k-j)}] + f^{(0)}g^{(k+1)}\end{matrix}$$

$$= \sum_{j=0}^{k-1}\left[\binom{k}{j} + \binom{k}{j+1}\right][f^{(j+1)}g^{(k-j)}] + f^{(k+1)}g^{(0)} + f^{(0)}g^{(k+1)}$$

$$= \sum_{j=0}^{k-1}\binom{k+1}{j+1}[f^{(j+1)}g^{(k-j)}] + f^{(k+1)}g^{(0)} + f^{(0)}g^{(k+1)}$$

$$= \sum_{j=0}^{k+1}\binom{k}{j}[f^{(j+1)}g^{(k-j)}].$$

So, as $n = k+1$, it holds. Hence, by **Mathematical Induction**, we have proved the **Leibnitz** formula.

Remark: We use the famous formula called **Pascal Theorem:**$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$, where $0 \leq k < n$.

5.7 Let f and g be two functions defined and having finite third-order derivatives $f'''(x)$ and $g'''(x)$ for all x in R. If $f(x)g(x) = 1$ for all x, show that the relations in (a), (b), (c), and (d) holds at those points where the denominators are not zero:

(a) $f'(x)/f(x) + g'(x)/g(x) = 0$.

Proof: Since $f(x)g(x) = 1$ for all x, we have $f'g + g'f = 0$ for all x. By hypothesis, we have

$$\frac{f'g + g'f}{fg} = 0 \text{ for those points where the denominators are not zero}$$

which implies that

$$f'(x)/f(x) + g'(x)/g(x) = 0.$$

(b) $f''(x)/f'(x) - 2f'(x)/f(x) - g''(x)/g'(x) = 0$.

Proof: Since $f'g + g'f = 0$ for all x, we have $(f'g + g'f)' = f''g + 2f'g' + g''f = 0$. By hypothesis, we have

$$\begin{aligned} 0 &= \frac{f''g + 2f'g' + g''f}{f'g} \\ &= \frac{f''}{f'} + 2\frac{g'}{g} + \frac{g''}{g\left(\frac{f'}{f}\right)} \\ &= \frac{f''}{f'} - 2\frac{f'}{f} - \frac{g''}{g'} \text{ by (a).} \end{aligned}$$

(c) $\frac{f'''(x)}{f'(x)} - 3\frac{f'(x)g''(x)}{f(x)g'(x)} - 3\frac{f''(x)}{f(x)} - \frac{g'''(x)}{g'(x)} = 0.$

Proof: By (b), we have $(f''g + 2f'g' + g''f)' = 0 = f'''g + 3f''g + 3f'g'' + fg'''$. By hypothesis, we have

$$\begin{aligned} 0 &= \frac{f'''g + 3f''g' + 3f'g'' + fg'''}{f'g} \\ &= \frac{f'''}{f'} + 3\frac{f''g'}{f'g} + 3\frac{g''}{g} + \frac{fg'''}{f'g} \\ &= \frac{f'''}{f'} + 3f''\left(\frac{g'}{f'g}\right) + 3g''\left(\frac{1}{g}\right) + g'''\left(\frac{f}{f'g}\right) \\ &= \frac{f'''}{f'} - 3\frac{f''}{f} - 3\frac{f'g''}{fg'} - \frac{g'''}{g'} \text{ by (a).} \end{aligned}$$

(d) $\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2.$

Proof: By (c), we have $\frac{f'''}{f'} - \frac{g'''}{g'} = 3\left(\frac{f''}{f} + \frac{f'g''}{fg'}\right)$. Since

$$\begin{aligned} \frac{f''}{f} + \frac{f'g''}{fg'} &= \left(\frac{f'}{f}\right)\left(\frac{f''}{f'}\right) + \left(\frac{f'}{f}\right)\left(\frac{g''}{g'}\right) \\ &= \left(\frac{f'}{f}\right)\left[\left(\frac{f''}{f'}\right) + \left(\frac{g''}{g'}\right)\right] \\ &= \frac{1}{2}\left[\left(\frac{f''}{f'}\right) - \left(\frac{g''}{g'}\right)\right]\left[\left(\frac{f''}{f'}\right) + \left(\frac{g''}{g'}\right)\right] \text{ by (b)} \\ &= \frac{1}{2}\left[\left(\frac{f''}{f'}\right)^2 - \left(\frac{g''}{g'}\right)^2\right], \end{aligned}$$

we know that $\frac{f'''}{f'} - \frac{g'''}{g'} = \frac{3}{2}\left[\left(\frac{f''}{f'}\right)^2 - \left(\frac{g''}{g'}\right)^2\right]$ which implies that
$\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2.$

Note. The expression which appears on the left side of (d) is called the **Schwarzian derivative of *f* at *x*.**

(e) Show that f and g have the same Schwarzian derivative if

$$g(x) = [af(x) + b]/(cf(x) + d), \text{ where } ad - bc \neq 0.$$

Hint. If $c \neq 0$, write $(af + b)/(cf + d) = (a/c) + (bc - ad)/[c(cf + d)]$, and apply part (d).

Proof: If $c = 0$, we have $g = \frac{a}{d}f + \frac{b}{d}$. So, we have

$$\frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2$$
$$= \frac{\frac{a}{d}f'''(x)}{\frac{a}{d}f'(x)} - \frac{3}{2}\left(\frac{\frac{a}{d}f''(x)}{\frac{a}{d}f'(x)}\right)^2$$
$$= \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

So, f and g have the same Schwarzian derivative.

If $c \neq 0$, write $g = (af+b)/(cf+d) = (a/c) + (bc-ad)/[c(cf+d)]$, then

$$(cg-a)\left[\left(\frac{1}{bc-ad}\right)(cf+d)\right] = 1 \text{ since } ad-bc \neq 0.$$

Let $G = cg-a$, and $F = (\frac{1}{bc-ad})(cf+d)$, then $GF = 1$. It implies that by (d),

$$\frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = \frac{G'''}{G'} - \frac{3}{2}\left(\frac{G''}{G'}\right)^2$$

which implies that

$$\frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = \frac{(\frac{c}{bc-ad})f'''}{(\frac{c}{bc-ad})f'} - \frac{3}{2}\left[\frac{(\frac{c}{bc-ad})f''}{(\frac{c}{bc-ad})f'}\right]^2$$
$$= \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$
$$= \frac{G'''}{G'} - \frac{3}{2}\left(\frac{G''}{G'}\right)^2$$
$$= \frac{cg'''}{cg'} - \frac{3}{2}\left(\frac{cg''}{cg'}\right)^2$$
$$= \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2.$$

So, f and g have the same Schwarzian derivative.

5.8 Let f_1, f_2, g_1, g_2 be functions having derivatives in (a,b). Define F by means of the determinant

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}, \text{ if } x \in (a,b).$$

(a) Show that $F'(x)$ exists for each x in (a,b) and that

$$F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}.$$

Proof: Since $F(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix} = f_1g_2 - f_2g_1$, we have

$$F' = f_1'g_2 + f_1g_2' - f_2'g_1 - f_2g_1'$$
$$= (f_1'g_2 - f_2'g_1) + (f_1g_2' - f_2g_1')$$
$$= \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}.$$

(b) State and prove a more general result for nth order determinants.

Proof: Claim that if

$$F(x) = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix},$$

then

$$F'(x) = \begin{vmatrix} f'_{11} & f'_{12} & \cdots & f'_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f'_{21} & f'_{22} & \cdots & f'_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix} + \ldots + \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f'_{n1} & f'_{n2} & \cdots & f'_{nn} \end{vmatrix}.$$

We prove it by Mahematial Induction. As $n = 2,$ it has proved in (a). Suppose that $n = k$ holds, consider $n = k + 1,$

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1k+1} \\ f_{21} & f_{22} & \cdots & f_{2k+1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{k+11} & f_{k+12} & \cdots & f_{k+1k+1} \end{vmatrix}'$$

$$= \left\{ (-1)^{(k+1)+1} f_{k+11} \begin{vmatrix} f_{12} & \cdots & f_{1k+1} \\ \cdots & \cdots & \cdots \\ f_{k2} & \cdots & f_{kk+1} \end{vmatrix} + \ldots + (-1)^{(k+1)+(k+1)} f_{k+1k+1} \begin{vmatrix} f_{11} & \cdots & f_{1k} \\ \cdots & \cdots & \cdots \\ f_{k1} & \cdots & f_{kk} \end{vmatrix} \right\}'$$

$= \ldots$ (The reader can write it down by induction hypothesis).

Hence, by **Mathematical Induction**, we have proved it.

Remark: The reader should keep it in mind since it is useful in Analysis. For example, we have the following Theorem.

(**Theorem**) Suppose that $f, g,$ and h are continuous on $[a, b],$ and differentiable on $(a, b).$ Then there is a $\xi \in (a, b)$ such that

$$\begin{vmatrix} f'(\xi) & g'(\xi) & h'(\xi) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Proof: Let

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix},$$

then it is clear that $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b) since the operations on determinant involving addition, substraction, and multiplication without division. Consider

$$F(a) = F(b) = 0,$$

then by **Rolle's Theorem**, we know that

$$F'(\xi) = 0, \text{ where } \xi \in (a,b),$$

which implies that

$$\begin{vmatrix} f'(\xi) & g'(\xi) & h'(\xi) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0. \qquad *$$

(**Application- Generalized Mean Value Theorem**) Suppose that f and g are continuous on $[a,b]$, and differentiable on (a,b). Then there is a $\xi \in (a,b)$ such that

$$[f(b) - f(a)]g'(\xi) = f'(\xi)[g(b) - g(a)].$$

Proof: Let $h(x) = 1$, and thus by (*), we have

$$\begin{vmatrix} f'(\xi) & g'(\xi) & 0 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = 0,$$

which implies that

$$-\begin{vmatrix} f'(\xi) & g'(\xi) \\ f(b) & g(b) \end{vmatrix} + \begin{vmatrix} f'(\xi) & g'(\xi) \\ f(a) & g(a) \end{vmatrix} = 0$$

which implies that

$$[f(b) - f(a)]g'(\xi) = f'(\xi)[g(b) - g(a)].$$

Note: Use the similar method, we can show **Mean Value Theorem** by letting $g(x) = x$, and $h(x) = 1$. And from this viewpoint, we know that **Rolle's Theorem, Mean Value Theorem, and Generalized Mean Value Theorem** are equivalent.

5.9 Given n functions $f_1, \ldots, f_n$, each having nth order derivatives in (a,b). A function W, called the **Wronskian** of $f_1, \ldots, f_n$, is defined as follows: For each x in (a,b), $W(x)$ is the value of the determinant of order n whose element in the kth row and mth column is $f_m^{(k-1)}(x)$, where $k = 1,2,..,n$ and $m = 1,2,\ldots,n$. [The expression $f_m^{(0)}(x)$ is written for $f_m(x)$.]

(a) Show that $W'(x)$ can be obtained by replacing the last row of the determinant defining $W(x)$ by the nth derivatives $f_1^{(n)}(x), \ldots, f_n^{(n)}(x)$.

Proof: Write

$$W(x) = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix},$$

and note that if any two rows are the same, its determinant is 0; hence, by Exercise 5.8-(b), we know that

$$W'(x) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ \dots & \dots & \dots & \dots \\ f_1^{(n-2)} & f_2^{(n-2)} & \dots & f_n^{(n-2)} \\ f_1^{(n)} & f_2^{(n)} & \dots & f_n^{(n)} \end{vmatrix}.$$

(b) Assuming the existence of n constants $c_1,\dots,c_n$, not all zero, such that $c_1 f_1(x) + \dots c_n f_n(x) = 0$ for every x in (a,b), show that $W(x) = 0$ for each x in (a,b).

Proof: Since $c_1 f_1(x) + \dots c_n f_n(x) = 0$ for every x in (a,b), where $c_1,\dots,c_n$, not all zero. Without loss of generality, we may assume $c_1 \neq 0$, we know that $c_1 f_1^{(k)}(x) + \dots c_n f_n^{(k)}(x) = 0$ for every x in (a,b), where $0 \le k \le n$. Hence, we have

$$W(x) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \dots & \dots & \dots & \dots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$
$$= 0$$

since the first column is a linear combination of other columns.

Note. A set of functions satisfing such a relation is said to be a l**inearly dependent** set on (a,b).

(c) The vanishing of the Wronskian throughout (a,b) is necessary, but not sufficient. for linear dependence of $f_1,\dots,f_n$. Show that in the case of two functions, if the Wronskian vanishes throughout (a,b) and if one of the functions does not vanish in (a,b), then they form a linearly dependent set in (a,b).

Proof: Let f and g be continuous and differentiable on (a,b). Suppose that $f(x) \neq 0$ for all $x \in (a,b)$. Since the Wronskian of f and g is 0, for all $x \in (a,b)$, we have

$$fg' - f'g = 0 \text{ for all } x \in (a,b). \qquad *$$

Since $f(x) \neq 0$ for all $x \in (a,b)$, we have by (*),

$$\frac{fg' - f'g}{f^2} = 0 \Rightarrow \left(\frac{g}{f}\right)' = 0 \text{ for all } x \in (a,b).$$

Hence, there is a constant c such that $g = cf$ for all $x \in (a,b)$. Hence, $\{f,g\}$ forms a linearly dependent set.

Remark: This exercise in (b) is a impotant theorem on **O.D.E**. We often write (b) in other form as follows.

(**Theorem**) Let $f_1,..,f_n$ be continuous and differentiable on an interval I. If $W(f_1,\dots,f_n)(t_0) \neq 0$ for some $t_0 \in I$, then $\{f_1,..,f_n\}$ is **linearly independent** on I

Note: If $\{f_1,..,f_n\}$ is **linearly independent** on I, It is **NOT** necessary that $W(f_1,\dots,f_n)(t_0) \neq 0$ for some $t_0 \in I$. For example, $f(t) = t^2|t|$, and $g(t) = t^3$. It is easy to check $\{f,g\}$ is linearly independent on $(-1,1)$. And $W(f,g)(t) = 0$ for all $t \in (-1,1)$.

Supplement on Chain Rule and Inverse Function Theorem.

The following theorem is called chain rule, it is well-known that let f be defined on an open interval S, let g be defined on $f(S)$, and consider the composite function $g \circ f$ defined on S by the equation

$$g \circ f(x) = g(f(x)).$$

Assume that there is a point c in S such that $f(c)$ is an interior point of $f(S)$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c, and we have

$$g \circ f'(c) = g'(f(c))f'(c).$$

We do not give a proof, in fact, the proof can be found in this text book. We will give another Theorem called The Converse of Chain Rule as follows.

(**The Converse of Chain Rule**) Suppose that f, g and u are related so that $f(x) = g(u(x))$. If $u(x)$ is continuous at x_0, $f'(x_0)$ exists, $g'(u(x_0))$ exists and not zero. Then $u'(x_0)$ is defined and we have

$$f'(x_0) = g'(u(x_0))u'(x_0).$$

Proof: Since $f'(x_0)$ exists, and $g'(u(x_0))$ exists, then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(|x - x_0|) \quad *$$

and

$$g(u(x)) = g(u(x_0)) + g'(u(x_0))(u(x) - u(x_0)) + o(|u(x) - u(x_0)|). \quad **$$

Since $f(x) = g(u(x))$, and $f(x_0) = g(u(x_0))$, by (*) and (**), we know that

$$u(x) = u(x_0) + \frac{f'(x_0)}{g'(u(x_0))}(x - x_0) + o(|x - x_0|) + o(|u(x) - u(x_0)|). \quad ***$$

Note that since $u(x)$ is continuous at x_0, we know that $o(|u(x) - u(x_0)|) \to 0$ as $x \to x_0$. So, (***) means that $u'(x_0)$ is defined and we have

$$f'(x_0) = g'(u(x_0))u'(x_0).$$

Remark: The condition that $g'(u(x_0))$ is not zero is essential, for example, $g(x) = 1$ on $(-1, 1)$ and $u(x) = |x|$, where $x_0 = 0$.

(**Inverse Function Theroem**) Suppose that f is continuous, strictly monotonic function which has an open interval I for domain and has range J. (It implies that $f(g(x)) = x = g(f(x))$ on its corresponding domain.) Assume that x_0 is a point of J such that $f'(g(x_0))$ is defined and is different from zero. Then $g'(x_0)$ exists, and we have

$$g'(x_0) = \frac{1}{f'(g(x_0))}.$$

Proof: It is a result of **the converse of chain rule** note that

$$f(g(x)) = x.$$

Mean Value Theorem

5.10 Given a function defined and having a finite derivative in (a, b) and such that $\lim_{x \to b^-} f(x) = +\infty$. Prove that $\lim_{x \to b^-} f'(x)$ either fails to exist or is infinite.

Proof: Suppose **NOT**, we have the existence of $\lim_{x \to b^-} f'(x)$, denoted the limit by L. So, given $\varepsilon = 1$, there is a $\delta > 0$ such that as $x \in (b - \delta, b)$ we have

$$|f'(x)| < |L| + 1. \quad *$$

Consider $x, a \in (b - \delta, b)$ with $x > a$, then we have by (*) and **Mean Value Theorem,**

$$\begin{aligned} |f(x) - f(a)| &= |f'(\xi)(x - a)| \text{ where } \xi \in (a, x) \\ &\leq (|L| + 1)|x - a| \end{aligned}$$

which implies that

$$|f(x)| \le |f(a)| + (|L| + 1)\delta$$

which contradicts to $\lim_{x \to b^-} f(x) = +\infty$.

Hence, $\lim_{x \to b^-} f'(x)$ either fails to exist or is infinite.

5.11 Show that the formula in the **Mean Value Theorem** can be written as follows:

$$\frac{f(x+h) - f(x)}{h} = f'(x + \theta h),$$

where $0 < \theta < 1$.

Proof: (**Mean Value Theorem**) Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $\xi \in (a, b)$ such that $f(b) - f(a) = f'(\xi)(b - a)$. Note that $\xi = a + \theta(b - a)$, where $0 < \theta < 1$. So, we have proved the exercise.

Determine θ as a function of x and h, and keep $x \ne 0$ fixed, and find $\lim_{h \to 0} \theta$ in each case.

(a) $f(x) = x^2$.

Proof: Consider

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = 2x + h = 2(x + \theta h) = f'(x + \theta h)$$

which implies that

$$\theta = 1/2.$$

Hence, we know that $\lim_{h \to 0} \theta = 1/2$.

(b) $f(x) = x^3$.

Proof: Consider

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2 = 3(x + \theta h)^2 = f'(x + \theta h)$$

which implies that

$$\theta = \frac{-3x \pm \sqrt{9x^2 + 9xh + 3h^2}}{3h}$$

Since $0 < \theta < 1$, we consider two cases. (i) $x > 0$, (ii) $x < 0$.

(i) As $x > 0$, since

$$0 < \theta = \frac{-3x \pm \sqrt{9x^2 + 9xh + 3h^2}}{3h} < 1,$$

we have

$$\theta = \begin{cases} \frac{-3x + \sqrt{9x^2 + 9xh + 3h^2}}{3h} & \text{if } h > 0, \text{ and } h \text{ is sufficiently close to } 0, \\ \frac{-3x + \sqrt{9x^2 + 9xh + 3h^2}}{3h} & \text{if } h < 0, \text{ and } h \text{ is sufficiently close to } 0. \end{cases}$$

Hence, we know that $\lim_{h \to 0} \theta = 1/2$ by **L-Hospital Rule**.

(ii) As $x < 0$, we have

$$\theta = \begin{cases} \frac{-3x - \sqrt{9x^2 + 9xh + 3h^2}}{3h} & \text{if } h > 0, \text{ and } h \text{ is sufficiently close to } 0, \\ \frac{-3x - \sqrt{9x^2 + 9xh + 3h^2}}{3h} & \text{if } h < 0, \text{ and } h \text{ is sufficiently close to } 0. \end{cases}$$

Hence, we know that $\lim_{h \to 0} \theta = 1/2$ by **L-Hospital Rule.**

From (i) and (ii), we know that as $x \ne 0$, we have $\lim_{h \to 0} \theta = 1/2$.

Remark: For $x = 0,$ we can show that $\lim_{h\to 0}\theta = \frac{\sqrt{3}}{3}$ as follows.

Proof: Since

$$0 < \theta = \frac{\pm\sqrt{3h^2}}{3h} < 1,$$

we have

$$\theta = \begin{cases} \frac{\sqrt{3h^2}}{3h} = \frac{\sqrt{3}\,h}{3h} = \frac{\sqrt{3}}{3} \text{ if } h > 0, \\ \frac{-\sqrt{3h^2}}{3h} = \frac{\sqrt{3}\,h}{3h} = \frac{\sqrt{3}}{3} \text{ if } h < 0. \end{cases}$$

Hence, we know that $\lim_{h\to 0}\theta = \frac{\sqrt{3}}{3}$.

(c) $f(x) = e^x$.

Proof: Consider

$$\frac{f(x+h)-f(x)}{h} = \frac{e^{x+h}-e^x}{h} = e^{x+\theta h} = f'(x+\theta h)$$

which implies that

$$\theta = \frac{\log\frac{e^h-1}{h}}{h}.$$

Hence, we know that $\lim_{h\to 0}\theta = 1/2$ since

$$\begin{aligned} \lim_{h\to 0}\theta &= \lim_{h\to 0}\frac{\log\frac{e^h-1}{h}}{h} \\ &= \lim_{h\to 0}\frac{e^h h - e^h + 1}{h(e^h-1)} \text{ by L-Hospital Rule.} \\ &\quad \text{Note that } e^h = 1 + h + \frac{h^2}{2} + o(h^2) \\ &= \lim_{h\to 0}\frac{\frac{1}{2} + h + o(1)}{1 + \frac{h}{2} + o(h)} \\ &= 1/2. \end{aligned}$$

(d) $f(x) = \log x,\ x > 0.$

Proof: Consider

$$\frac{f(x+h)-f(x)}{h} = \frac{\log(1+\frac{h}{x})}{h} = \frac{1}{x+\theta h}$$

which implies that

$$\theta = \frac{\frac{h}{x} - \log(1+\frac{h}{x})}{\frac{h}{x}(\log(1+\frac{h}{x}))}.$$

Since $\log(1+t) = t - \frac{t^2}{2} + o(t^2),$ we have

$$\begin{aligned}\lim_{h\to 0}\theta &= \lim_{h\to 0}\frac{\frac{h}{x}-\left(\frac{h}{x}-\frac{1}{2}\left(\frac{h}{x}\right)^2+o\left(\left(\frac{h}{x}\right)^2\right)\right)}{\frac{h}{x}\left(\frac{h}{x}-\frac{1}{2}\left(\frac{h}{x}\right)^2+o\left(\left(\frac{h}{x}\right)^2\right)\right)}\\ &= \lim_{h\to 0}\frac{\frac{1}{2}\left(\frac{h}{x}\right)^2+o\left(\left(\frac{h}{x}\right)^2\right)}{\left(\frac{h}{x}\right)^2+\frac{1}{2}\left(\frac{h}{x}\right)^3+o\left(\left(\frac{h}{x}\right)^3\right)}\\ &= \lim_{h\to 0}\frac{\frac{1}{2}+o(1)}{1+\frac{1}{2}\left(\frac{h}{x}\right)+o\left(\frac{h}{x}\right)}\\ &= 1/2.\end{aligned}$$

5.12 Take $f(x) = 3x^4 - 2x^3 - x^2 + 1$ and $g(x) = 4x^3 - 3x^2 - 2x$ in Theorem 5.20. Show that $f'(x)/g'(x)$ is never equal to the quotient $[f(1) - f(0)]/[g(1) - g(0)]$ if $0 < x \le 1$. How do you reconcile this with the equation

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_1)}{g'(x_1)},\ a < x_1 < b,$$

obtainable from Theorem 5.20 when $n = 1$?

Solution: Note that

$$12x^2 - 6x - 2 = 12\left[x - \left(\frac{1}{4} + \sqrt{\frac{11}{48}}\right)\right]\left[x - \left(\frac{1}{4} - \sqrt{\frac{11}{48}}\right)\right],\text{ where } \left(0 < \frac{1}{4} + \sqrt{\frac{11}{48}} < 1\right).$$

So, when we consider

$$\frac{f(1) - f(0)}{g(1) - g(0)} = 0$$

and

$$f'(x) = 12x^3 - 6x^2 - 2x = xg'(x) = x(12x^2 - 6x - 2),$$

we **CANNOT** write $f'(x)/g'(x) = x$. Otherwise, it leads us to get a contradiction.

Remark: It should be careful when we use **Generalized Mean Value Theorem**, we had better not write the above form unless we know that the denominator is not zero.

5.13 In each of the following special cases of Theorem 5.20, take $n = 1$, $c = a$, $x = b$, and show that $x_1 = (a + b)/2$.

(a) $f(x) = \sin x$, $g(x) = \cos x$;

Proof: Since, by **Theorem 5.20**,

$$\begin{aligned}(\sin a - \sin b)[-\sin(x_1)] &= \left[2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\right][-\sin(x_1)]\\ &= (\cos a - \cos b)(\cos x_1)\\ &= \left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\right](\cos x_1),\end{aligned}$$

we find that if we choose $x_1 = (a + b)/2$, then both are equal.

(b) $f(x) = e^x$, $g(x) = e^{-x}$.

Proof: Since, by **Theorem 5.20**,

$$(e^a - e^b)(-e^{-x_1}) = (e^{-a} - e^{-b})(e^{x_1}),$$

we find that if we choose $x_1 = (a + b)/2$, then both are equal.

Can you find a general class of such pairs of functions f and g for which x_1 will always be $(a + b)/2$ and such that both examples (a) and (b) are in this class?

Proof: Look at the **Generalized Mean Value Theorem**, we try to get something from the equality.

$$[f(a)-f(b)]g'\left(\frac{a+b}{2}\right) = [g(a)-g(b)]f'\left(\frac{a+b}{2}\right), \quad *$$

if $f(x)$, and $g(x)$ satisfy following two conditions,

$$\text{(i) } f'(x) = g(-x) \text{ and } g'(x) = -f(-x)$$

and

$$\text{(ii) } [f(a)-f(b)]\left[-f\left(-\frac{a+b}{2}\right)\right] = [g(a)-g(b)]\left[g\left(-\frac{a+b}{2}\right)\right],$$

then we have the equality (*).

5.14 Given a function f defined and having a finite derivative f' in the half-open interval $0 < x \leq 1$ and such that $|f'(x)| < 1$. Define $a_n = f(1/n)$ for $n = 1,2,3,\ldots,$ and show that $\lim_{n\to\infty} a_n$ exists.

Hint. Cauchy condition.

Proof: Consider $n \geq m$, and by **Mean Value Theorem**,

$$|a_n - a_m| = |f(1/n) - f(1/m)| = |f'(p)|\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n} - \frac{1}{m}\right|$$

then $\{a_n\}$ is a Cauchy sequence since $\{1/n\}$ is a Cauchy sequence. Hence, we know that $\lim_{n\to\infty} a_n$ exists.

5.15 Assume that f has a finite derivative at each point of the open interval (a,b). Assume also that $\lim_{x\to c} f'(x)$ exists and is finite for some interior point c. Prove that the value of this limit must be $f'(c)$.

Proof: It can be proved by Exercise 5.16; we omit it.

5.16 Let f be continuous on (a,b) with a finite derivative f' everywhere in (a,b), expect possibly at c. If $\lim_{x\to c} f'(x)$ exists and has the value A, show that $f'(c)$ must also exist and has the value A.

Proof: Consider, for $x \neq c$,

$$\frac{f(x)-f(c)}{x-c} = f'(\xi) \text{ where } \xi \in (x,c) \text{ or } (c,x) \text{ by } \textbf{Mean Value Theorem,} \quad *$$

since $\lim_{x\to c} f'(x)$ exists, given $\varepsilon > 0$, there is a $\delta > 0$ such that as $x \in (c-\delta, c+\delta) - \{c\}$, we have

$$A - \varepsilon < f'(x) < A + \varepsilon.$$

So, if we choose $x \in (c-\delta, c+\delta) - \{c\}$ in (*), we then have

$$A - \varepsilon < \frac{f(x)-f(c)}{x-c} = f'(\xi) < A + \varepsilon.$$

That is, $f'(c)$ exists and equals A.

Remark: (1) Here is another proof by **L-Hospital Rule.** Since it is so obvious that we omit the proof.

(2) We should be noted that Exercise 5.16 implies Exercise 5.15. Both methods mentioned in Exercise 5.16 are suitable for Exercise 5.15.

5.17 Let f be continuous on $[0,1]$, $f(0) = 0$, $f'(x)$ defined for each x in $(0,1)$. Prove that if f' is an increasing function on $(0,1)$, then so is too is the function g defined by the equation $g(x) = f(x)/x$.

Proof: Since f' is an increasing function on $(0,1)$, we know that, for any $x \in (0,1)$

$$f'(x) - \frac{f(x)}{x} = f'(x) - \frac{f(x) - f(0)}{x - 0} = f'(x) - f'(\xi) \geq 0 \text{ where } \xi \in (0, x). \quad *$$

So, let $x > y$, we have

$$\begin{aligned} g(x) - g(y) &= g'(z)(x - y), \text{ where } y < z < x \\ &= \frac{f'(z)z - f(z)}{z^2}(x - y) \\ &\geq 0 \text{ by } (*) \end{aligned}$$

which implies that g is an increasing function on $(0, 1)$.

5.18 Assume f has a finite derivative in (a, b) and is continuous on $[a, b]$ with $f(a) = f(b) = 0$. Prove that for every real λ there is some c in (a, b) such that $f'(c) = \lambda f(c)$.

Hint. Apply Rolle's Theorem to $g(x)f(x)$ for a suitable g depending on λ.

Proof: Consider $g(x) = f(x)e^{-\lambda x}$, then by **Rolle's Theorem**,

$$\begin{aligned} g(a) - g(b) &= g'(c)(a - b), \text{ where } c \in (a, b) \\ &= 0 \end{aligned}$$

which implies that

$$f'(c) = \lambda f(c).$$

Remark: (1) The finding of an auxiliary function usually comes from the equation that we consider. We will give some questions around this to get more.

(2)There are some questions about finding auxiliary functions; we write it as follows.

(i) Show that $e^{\pi} > \pi^{e}$.

Proof: (STUDY) Since $\log x$ is a strictly increasing on $(0, \infty)$, in order to show $e^{\pi} > \pi^{e}$, it suffices to show that

$$\pi \log e = \log e^{\pi} > \log \pi^{e} = e \log \pi$$

which implies that

$$\frac{\log e}{e} > \frac{\log \pi}{\pi}.$$

Consider $f(x) = \frac{\log x}{x} : [e, \infty)$, we have

$$f'(x) = \frac{1 - \log x}{x^2} < 0 \text{ where } x \in (e, \infty).$$

So, we know that $f(x)$ is strictly decreasing on $[e, \infty)$. Hence, $\frac{\log e}{e} > \frac{\log \pi}{\pi}$. That is, $e^{\pi} > \pi^{e}$.

(ii) Show that $e^x \geq 1 + x$ for all $x \in R$.

Proof: By **Taylor Theorem with Remainder Term**, we know that

$$e^x = 1 + x + \frac{e^c}{2}x^2, \text{ for some } c.$$

So, we finally have $e^x \geq 1 + x$ for all $x \in R$.

Note: (a) The method in (ii) tells us one thing, we can give a theorem as follows. Let $f \in C^{2n-1}([a, b])$, and $f^{(2n)}(x)$ exists and $f^{(2n)}(x) \geq 0$ on (a, b). Then we have

$$f(x) \geq \sum_{k=0}^{2n-1} \frac{f^{(k)}(a)}{k!}.$$

Proof: By **Generalized Mean Value Theorem**, we complete it.

(b) There are many proofs about that $e^x \geq 1+x$ for all $x \in R$. We list them as a reference.

(b-1) Let $f(x) = e^x - 1 - x$, and thus consider the extremum.
(b-2) Use **Mean Value Theorem.**
(b-3) Since $e^x - 1 \geq 0$ for $x \geq 0$ and $e^x - 1 \leq 0$ for $x \leq 0$, we then have

$$\int_0^x (e^t - 1)dt \geq 0 \text{ and } \int_x^0 (e^t - 1)dt \leq 0.$$

So, $e^x \geq 1+x$ for all $x \in R$.

(iii) Let f be continuous function on $[a,b]$, and differentiabel on (a,b). Prove that there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(c) - f(a)}{b-c}.$$

Proof: (STUDY) Since $f'(c) = \frac{f(c)-f(a)}{b-c}$, we consider $f'(c)(b-c) - (f(c) - f(a))$. Hence, we choose $g(x) = (f(x) - f(a))(b-x)$, then by **Rolle's Theorem,**

$$g(a) - g(b) = g'(c)(a-b) \text{ where } c \in (a,b)$$

which implies that $f'(c) = \frac{f(c)-f(a)}{b-c}$.

(iv) Let f be a polynomial of degree n, if $f \geq 0$ on R, then we have

$$f + f' + .. + f^{(n)} \geq 0 \text{ on } R.$$

Proof: Let $g(x) = f + f' + .. + f^{(n)}$, then we have

$$g - g' = f \geq 0 \text{ on } R \text{ since } f \text{ is a polynomial of degree } n. \quad *$$

Consider $h(x) = g(x)e^{-x}$, then $h'(x) = (g'(x) - g(x))e^{-x} \leq 0$ on R by (*). It means that h is a decreasing function on R. Since $\lim_{x\to+\infty} h(x) = 0$ by the fact g is still a polynomial, then $h(x) \geq 0$ on R. That is, $g(x) \geq 0$ on R.

(v) Suppose that f is continuous on $[a,b]$, $f(a) = 0 = f(b)$, and $x^2 f''(x) + 4xf'(x) + 2f(x) \geq 0$ for all $x \in (a,b)$. Prove that $f(x) \leq 0$ on $[a,b]$.

Proof: (STUDY) Since $x^2 f''(x) + 4xf'(x) + 2f(x) = [x^2 f(x)]''$ by **Leibnitz Rule,** let $g(x) = x^2 f(x)$, then claim that $g(x) \leq 0$ on $[a,b]$.

Suppose **NOT**, there is a point $p \in (a,b)$ such that $g(p) > 0$. Note that since $f(a) = 0$, and $f(b) = 0$, So, $g(x)$ has an absolute maximum at $c \in (a,b)$. Hence, we have $g'(c) = 0$. By **Taylor Theorem with Remainder term**, we have

$$\begin{aligned} g(x) &= g(c) + g'(c)(x-c) + \frac{g''(\xi)}{2!}(x-c)^2, \text{ where } \xi \in (x,c) \text{ or } (c,x) \\ &\geq g(c) \text{ since } g'(c) = 0, \text{ and } g''(x) \geq 0 \text{ for all } x \in (a,b) \\ &> 0 \text{ since } g(c) \text{ is absolute maximum.} \end{aligned}$$

So,

$$x^2 f(x) \geq c^2 f(c) > 0 \quad **$$

which is absurb since let $x = a$ in (**).

(vi) Suppose that f is continuous and differentiable on $[0,\infty)$, and $\lim_{x\to\infty} f'(x) + f(x) = 0$, show that $\lim_{x\to\infty} f(x) = 0$.

Proof: Since $\lim_{x\to\infty} f'(x) + f(x) = 0$, then given $\varepsilon > 0$, there is $M > 0$ such that as $x \geq M$, we have

$$-\varepsilon < f'(x) + f(x) < \varepsilon.$$

So, as $x \geq M,$ we have

$$\begin{aligned}[-\varepsilon e^x + \varepsilon e^M + \varepsilon e^M f(M)]' &= -\varepsilon e^x \\ &< [e^x f(x)]' \\ &< \varepsilon e^x = [\varepsilon e^x - \varepsilon e^M + \varepsilon e^M f(M)]'.\end{aligned}$$

If we let $-\varepsilon e^x + \varepsilon e^M + \varepsilon e^M f(M) = g(x),$ and $\varepsilon e^x - \varepsilon e^M + \varepsilon e^M f(M) = h(x),$ then we have

$$g'(x) \leq [e^x f(x)]' \leq h'(x)$$

and

$$g(M) = e^M f(M) = h(M).$$

Hence, for $x \geq M,$

$$\begin{aligned}-\varepsilon e^x + \varepsilon e^M + \varepsilon e^M f(M) &= g(x) \\ &\leq e^x f(x) \\ &\leq h(x) = \varepsilon e^x - \varepsilon e^M + \varepsilon e^M f(M)\end{aligned}$$

It implies that, for $x \geq M,$

$$-\varepsilon + e^{-x}[\varepsilon e^M + \varepsilon e^M f(M)] \leq f(x) \leq \varepsilon - e^{-x}[\varepsilon e^M - \varepsilon e^M f(M)]$$

which implies that

$$\lim_{x\to+\infty} f(x) = 0 \text{ since } \varepsilon \text{ is arbitrary.}$$

Note: In the process of proof, we use the result on **Mean Value Theorem**. Let $f, g,$ and h be continuous on $[a,b]$ and differentiable on $(a,b).$ Suppose $f(a) = g(a) = h(a)$ and $f'(x) \leq g'(x) \leq h'(x)$ on $(a,b).$ Show that $f(x) \leq g(x) \leq h(x)$ on $[a,b].$

Proof: By **Mean Value theorem**, we have

$$\begin{aligned}[g(x) - f(x)] - [g(a) - f(a)] &= g(x) - f(x) \\ &= g'(c) - f'(c), \text{ where } c \in (a,x). \\ &\leq 0 \text{ by hypothesis.}\end{aligned}$$

So, $f(x) \leq g(x)$ on $[a,b].$ Similarly for $g(x) \leq h(x)$ on $[a,b].$ Hence, $f(x) \leq g(x) \leq h(x)$ on $[a,b].$

(vii) Let $f(x) = a_1 \sin x + \ldots + a_n \sin nx,$ where a_i are real for $i = 1,2,..n.$ Suppose that $|f(x)| \leq |x|$ for all real $x.$ Prove that $|a_1 + .. + na_n| \leq 1.$

Proof: Let $x > 0,$ and by **Mean Value Theorem,** we have

$$\begin{aligned}|f(x) - f(0)| = |f(x)| &= |a_1 \sin x + \ldots + a_n \sin nx| \\ &= |f'(c)x|, \text{ where } c \in (0,x) \\ &= |(a_1 \cos c + \ldots + na_n \cos nc)x| \\ &\leq |x| \text{ by hypothesis.}\end{aligned}$$

So,

$$|a_1 \cos c + \ldots + na_n \cos nc| \leq 1$$

Note that as $x \to 0^+,$ we have $c \to 0^+;$ hence, $|a_1 + .. + na_n| \leq 1.$

Note: Here are another type:
(a) $|\sin^2 x - \sin^2 y| \leq |x - y|$ for all $x, y.$
(b) $|\tan x - \tan y| \geq |x - y|$ for all $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2}).$

(viii) Let $f : R \to R$ be differentiable with $f'(x) \geq c$ for all $x,$ where $c > 0.$ Show that there is a point p such that $f(p) = 0$.

Proof: By **Mean Vaule Theorem,** we have

$$f(x) = f(0) + f'(x_1)x \geq f(0) + cx \text{ if } x \geq 0$$
$$= f(0) + f'(x_2)x \leq f(0) + cx \text{ if } x \leq 0.$$

So, as x large enough, we have $f(x) > 0$ and as x is smalle enough, we have $f(x) < 0$. Since f is differentiable on R, it is continuous on R. Hence, by **Intermediate Value Theorem**, we know that there is a point p such that $f(p) = 0$.

(3) Here is another type about integral, but it is worth learning. Compare with (2)-(vii). If

$$c_0 + \frac{c_1}{2} + \ldots + \frac{c_n}{n+1} = 0, \text{ where } c_i \text{ are real constants for } i = 1, 2, .. n.$$

Prove that $c_0 + \ldots + c_n x^n$ has at least one real root between 0 and 1.

Proof: Suppose **NOT,** i.e., (i) $f(x) := c_0 + \ldots + c_n x^n > 0$ for all $x \in [0,1]$ or (ii) $f(x) < 0$ for all $x \in [0,1]$.

In case (i), consider

$$0 < \int_0^1 f(x)dx = c_0 + \frac{c_1}{2} + \ldots + \frac{c_n}{n+1} = 0$$

which is absurb. Similarly for case (ii).

So, we know that $c_0 + \ldots + c_n x^n$ has at least one real root between 0 and 1.

5.19. Assume f is continuous on $[a,b]$ and has a finite second derivative f'' in the open interval (a,b). Assume that the line segment joining the points $A = (a, f(a))$ and $B = (b, f(b))$ intersects the graph of f in a third point P different from A and B. Prove that $f''(c) = 0$ for some c in (a,b).

Proof: Consider a straight line equation, called $g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. Then $h(x) := f(x) - g(x)$, we knwo that there are three point $x = a$, p and b such that

$$h(a) = h(p) = h(b) = 0.$$

So, by Mean Value Theorem twice, we know that there is a point $c \in (a,b)$ such that

$$h''(c) = 0$$

which implies that $f''(c) = 0$ since g is a polynomial of degree at least 1.

5.20 If f has a finite third derivative f''' in $[a,b]$ and if

$$f(a) = f'(a) = f(b) = f'(b) = 0,$$

prove that $f'''(c) = 0$ for some c in (a,b).

Proof: Since $f(a) = f(b) = 0$, we have $f'(p) = 0$ where $p \in (a,b)$ by **Rolle's Theorem.** Since $f'(a) = f'(p) = 0$, we have $f''(q_1) = 0$ where $q_1 \in (a,p)$ and since $f'(p) = f'(b) = 0$, we have $f''(q_2) = 0$ where $q_2 \in (p,b)$ by **Rolle's Theorem.** Since $f''(q_1) = f''(q_2) = 0$, we have $f'''(c) = 0$ where $c \in (q_1, q_2)$ by **Rolle's Theorem.**

5.21 Assume f is nonnegative and has a finite third derivative f''' in the open interval $(0,1)$. If $f(x) = 0$ for at least two values of x in $(0,1)$, prove that $f'''(c) = 0$ for some c in $(0,1)$.

Proof: Since $f(x) = 0$ for at least two values of x in $(0,1)$, say $f(a) = f(b) = 0$, where $a, b \in (0,1)$. By Rolle's Theorem, we have $f'(p) = 0$ where $p \in (a,b)$. Note that f is nonnegative and differentiable on $(0,1)$, so both $f(a)$ and $f(b)$ are local minima, where a and b are interior to (a,b). Hence, $f'(a) = f'(b) = 0$.

Since $f'(a) = f'(p) = 0$, we have $f''(q_1) = 0$ where $q_1 \in (a,p)$ and since $f'(p) = f'(b) = 0$, we have $f''(q_2) = 0$ where $q_2 \in (p,b)$ by **Rolle's Theorem.** Since $f''(q_1) = f''(q_2) = 0$, we have $f'''(c) = 0$ where $c \in (q_1,q_2)$ by **Rolle's Theorem.**

5.22 Assume f has a finite derivative in some interval $(a,+\infty)$.

(a) If $f(x) \to 1$ and $f'(x) \to c$ as $x \to +\infty$, prove that $c = 0$.

Proof: Consider $f(x+1) - f(x) = f'(y)$ where $y \in (x,x+1)$ by **Mean Value Theorem,** since

$$\lim_{x\to+\infty} f(x) = 1$$

which implies that

$$\lim_{x\to+\infty}[f(x+1) - f(x)] = 0$$

which implies that $(x \to +\infty \Leftrightarrow y \to +\infty)$

$$\lim_{x\to+\infty} f'(y) = 0 = \lim_{y\to+\infty} f(y)$$

Since $f'(x) \to c$ as $x \to +\infty$, we know that $c = 0$.

Remark: (i) There is a similar exercise; we write it as follows. If $f(x) \to L$ and $f'(x) \to c$ as $x \to +\infty$, prove that $c = 0$.

Proof: By the same method metioned in (a), we complete it.

(ii) The exercise tells that the function is smooth; its first derivative is smooth too.

(b) If $f'(x) \to 1$ as $x \to +\infty$, prove that $f(x)/x \to 1$ as $x \to +\infty$.

Proof: Given $\varepsilon > 0$, we want to find $M > 0$ such that as $x \geq M$

$$\left|\frac{f(x)}{x} - 1\right| < \varepsilon.$$

Since $f'(x) \to 1$ as $x \to +\infty$, then given $\varepsilon' = \frac{\varepsilon}{3}$, there is $M' > 0$ such that as $x \geq M'$, we have

$$|f'(x) - 1| < \frac{\varepsilon}{3} \Rightarrow |f'(x)| < 1 + \frac{\varepsilon}{3} \qquad *$$

By **Taylor Theorem with Remainder Term**,

$$\begin{aligned} f(x) &= f(M') + f'(\xi)(x - M') \\ &\Rightarrow f(x) - x = f(M') + (f'(\xi) - 1)x - f'(\xi)M', \end{aligned}$$

then for $x \geq M'$,

$$\begin{aligned} \left|\frac{f(x)}{x} - 1\right| &\leq \left|\frac{f(M')}{x}\right| + |f'(\xi) - 1| + \left|\frac{f'(\xi)M'}{x}\right| \qquad ** \\ &\leq \left|\frac{f(M')}{x}\right| + \frac{\varepsilon}{3} + \left(1 + \frac{\varepsilon}{3}\right)\left|\frac{M'}{x}\right| \text{ by (*)} \end{aligned}$$

Choose $M > 0$ such that as $x \geq M \geq M'$, we have

$$\left|\frac{f(M)}{x}\right| < \frac{\varepsilon}{3} \text{ and } \left|\frac{M'}{x}\right| < \frac{\varepsilon/3}{(1 + \frac{\varepsilon}{3})}. \qquad ***$$

Combine (**) with (***), we have proved that given $\varepsilon > 0$, there is a $M > 0$ such that as $x \geq M$, we have

$$\left|\frac{f(x)}{x} - 1\right| < \varepsilon.$$

That is, $\lim_{x\to+\infty} \frac{f(x)}{x} = 1$.

Remark: If we can make sure that $f(x) \to \infty$ as $x \to +\infty$, we can use **L-Hopital Rule**. We give another proof as follows. It suffices to show that $f(x) \to \infty$ as $x \to +\infty$.

Proof: Since $f'(x) \to 1$ as $x \to +\infty$, then given $\varepsilon = 1$, there is $M > 0$ such that as $x \geq M$, we have

$$|f'(x)| < 1 + 1 = 2.$$

Consider

$$f(x) = f(M) + f'(\xi)(x - M)$$

by **Taylor Theorem with Remainder Term**, then

$$\lim_{x \to +\infty} f(x) = +\infty \text{ since } f'(x) \text{ is bounded for } x \geq M.$$

(c) If $f'(x) \to 0$ as $x \to +\infty$, prove that $f(x)/x \to 0$ as $x \to +\infty$.

Proof: The method metioned in (b). We omit the proof.

Remark: (i) There is a similar exercise; we write it as follows. If $f'(x) \to L$ as $x \to +\infty$, prove that $f(x)/x \to L$ as $x \to +\infty$. The proof is mentioned in (b), so we omit it.

(ii) It should be careful that we **CANNOT** use **L-Hospital Rule** since we may not have the fact $f(x) \to \infty$ as $x \to +\infty$. Hence, **L-Hospital Rule** cannot be used here. For example, f is a constant function.

5.23 Let h be a fixed positive number. Show that there is no function f satisfying the following three conditions: $f'(x)$ exists for $x \geq 0$, $f'(0) = 0$, $f'(x) \geq h$ for $x > 0$.

Proof: It is called **Intermediate Value Theorem for Derivatives. (**Sometimes, we also call this theorem **Darboux.)** See the text book in **Theorem 5.16.**

(**Supplement**) 1. Suppose that $a \in R$, and f is a twice-differentiable real function on (a, ∞). Let M_0, M_1, and M_2 are the least upper bound of $|f(x)|$, $|f'(x)|$, and $|f''(x)|$, respectively, on (a, ∞). Prove that $M_1^2 \leq 4M_0M_2$.

Proof: Consider **Taylor's Theorem with Remainder Term**,

$$f(a + 2h) = f(a) + f'(a)(2h) + \frac{f''(\xi)}{2!}(2h)^2, \text{ where } h > 0.$$

then we have

$$f'(a) = \frac{1}{2h}[f(a + 2h) - f(a)] - f''(\xi)h$$

which implies that

$$|f'(a)| \leq \frac{M_0}{h} + hM_2 \Rightarrow M_1 \leq \frac{M_0}{h} + hM_2. \qquad *$$

Since $g(h) := \frac{M_0}{h} + hM_2$ has an absolute maximum at $\sqrt{\frac{M_0}{M_2}}$, hence by (*), we know that

$$M_1^2 \leq 4M_0M_2.$$

Remark:

2. Suppose that f is a twice-differentiable real function on $(0, \infty)$, and f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Proof: Since $M_1^2 \leq 4M_0M_2$ in **Supplement 1**, we have prove it.

3. Suppose that f is real, three times differentiable on $[-1, 1]$, such that $f(-1) = 0$, $f(0) = 0$, $f(1) = 1$, and $f'(0) = 0$. Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

Proof: Consider **Taylor's Theorem with Remainder Term**,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(c)}{3!}x^3, \text{ where } c \in (x,0) \text{ or } (0,x),$$

Then let $x = \pm 1$, and subtract one from another, we get

$$f^{(3)}(c_1) + f^{(3)}(c_2) \geq 6, \text{ where } c_1 \text{ and } c_2 \text{ in } (-1,1).$$

So, we have prove $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$.

5.24 If $h > 0$ and if $f'(x)$ exists (and is finite) for every x in $(a-h, a+h)$, and if f is continuous on $[a-h, a+h]$, show that we have:

(a)

$$\frac{f(a+h) - f(a-h)}{h} = f'(a+\theta h) + f'(a-\theta h), 0 < \theta < 1;$$

Proof: Let $g(h) = f(a+h) - f(a-h)$, then by **Mean Vaule Theorem**, we have

$$\begin{aligned} g(h) - g(0) &= g(h) \\ &= g'(\theta h)h, \text{ where } 0 < \theta < 1 \\ &= [f'(a+\theta h) + f'(a-\theta h)]h \end{aligned}$$

which implies that

$$\frac{f(a+h) - f(a-h)}{h} = f'(a+\theta h) + f'(a-\theta h), 0 < \theta < 1.$$

(b)

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h} = f'(a+\lambda h) - f'(a-\lambda h), 0 < \lambda < 1.$$

Proof: Let $g(h) = f(a+h) - 2f(a) + f(a-h)$, then by **Mean Vaule Theorem**, we have

$$\begin{aligned} g(h) - g(0) &= g(h) \\ &= g'(\lambda h)h, \text{ where } 0 < \lambda < 1 \\ &= [f'(a+\lambda h) - f'(a-\lambda h)]h \end{aligned}$$

which implies that

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h} = f'(a+\lambda h) - f'(a-\lambda h), 0 < \lambda < 1.$$

(c) If $f''(a)$ exists, show that.

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Proof: Since

$$\begin{aligned} &\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \\ &= \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h} \text{ by **L-Hospital Rule**} \\ &= \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{2h} + \frac{f'(a) - f'(a-h)}{2h} \\ &= \frac{1}{2}(2f''(a)) \text{ since } f''(a) \text{ exists.} \\ &= f''(a). \end{aligned}$$

Remark: There is another proof by using **Generalized Mean Value theorem.**

Proof: Let $g_1(h) = f(a+h) - 2f(a) + f(a-h)$ and $g_2(h) = h^2$, then by **Generalized Mean Value theorem,** we have

$$[g_1(h) - g_1(0)]g_2'(\theta h) = g_1'(\theta h)[g_2(h) - g_2(0)]$$

which implies that

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \frac{f'(a+\theta h) - f'(a-\theta h)}{2\theta h}$$

Hence,

$$\begin{aligned}&\lim_{h\to 0}\frac{f(a+h) - 2f(a) + f(a-h)}{h^2}\\ &= \lim_{h\to 0}\frac{f'(a+\theta h) - f'(a-\theta h)}{2\theta h}\\ &= f''(a) \text{ since } f''(a) \text{ exists.}\end{aligned}$$

(d) Give an example where the limit of the quotient in (c) exists but where $f''(a)$ does not exist.

Solution: (**STUDY**) Note that in the proof of (c) by using **L-Hospital Rule**. We know that $|x|$ is not differentiable at $x = 0$, and $|x|$ satisfies that

$$\lim_{h\to 0}\frac{|0+h| - |0-h|}{2h} = 0 = \lim_{h\to 0}\frac{f'(0+h) - f'(0-h)}{2h}$$

So, let us try to find a function f so that $f'(x) = |x|$. So, consider its integral, we know that

$$f(x) = \begin{cases} \frac{x^2}{2} \text{ if } x \geq 0 \\ -\frac{x^2}{2} \text{ if } x < 0 \end{cases}$$

Hence, we complete it.

Remark: (i) There is a related statement; we write it as follows. Suppose that f defined on (a,b) and has a derivative at $c \in (a,b)$. If $\{x_n\} \subseteq (a,c)$ and $\{y_n\} \subseteq (c,b)$ with such that $(x_n - y_n) \to 0$ as $n \to \infty$. Then we have

$$f'(c) = \lim_{n\to\infty}\frac{f(y_n) - f(x_n)}{y_n - x_n}.$$

Proof: Since $f'(c)$ exists, we have

$$f(y_n) = f(c) + f'(c)(y_n - c) + o(y_n - c) \qquad *$$

and

$$f(x_n) = f(c) + f'(c)(x_n - c) + o(x_n - c). \qquad *'$$

If we combine (*) and (*'), we have

$$\frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c) + \frac{o(y_n - c)}{y_n - x_n} + \frac{o(x_n - c)}{y_n - x_n}. \qquad **$$

Note that

$$\left|\frac{y_n - c}{y_n - x_n}\right| < 1 \text{ and } \left|\frac{x_n - c}{y_n - x_n}\right| < 1 \text{ for all } n,$$

we have

$$\begin{aligned}&\lim_{n\to\infty}\left[\frac{o(y_n - c)}{y_n - x_n} + \frac{o(x_n - c)}{y_n - x_n}\right]\\ &= \lim_{n\to\infty}\left[\frac{o(y_n - c)}{y_n - c}\frac{y_n - c}{y_n - x_n} + \frac{o(x_n - c)}{x_n - c}\frac{x_n - c}{y_n - x_n}\right]\\ &= 0.\end{aligned}$$

which implies that, by (**)

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n}.$$

(ii) There is a good exercise; we write it as follows. Let $f \in C^2(a,b)$, and $c \in (a,b)$. For small $|h|$ such that $c+h \in (a,b)$, write

$$f(c+h) = f(c) + f'(c+\theta(h)h)h$$

where $0 < \theta < 1$. Show that if $f''(c) \neq 0$, then $\lim_{h\to 0}\theta(h) = 1/2$.

Proof: Since $f \in C^2(a,b)$, by **Taylor Theorem with Remainder Term,** we have

$$\begin{aligned} f(c+h) - f(c) &= f'(c)h + \frac{f''(\xi)}{2!}h^2, \text{ where } \xi \in (c,h) \text{ or } (h,c) \\ &= f'(c+\theta(h)h)h \text{ by hypothesis.} \end{aligned}$$

So,

$$\frac{f'(c+\theta(h)h) - f'(c)}{\theta(h)}\theta(h) = \frac{f''(\xi)}{2!},$$

and let $h \to 0$, we have $\xi \to c$ by continuity of f'' at c. Hence,

$$\lim_{h\to 0}\theta(h) = 1/2 \text{ since } f''(c) \neq 0.$$

Note: We can modify our statement as follows. Let f be defined on (a,b), and $c \in (a,b)$. For small $|h|$ such that $c+h \in (a,b)$, write

$$f(c+h) = f(c) + f'(c+\theta(h)h)h$$

where $0 < \theta < 1$. Show that if $f''(c) \neq 0$, and $\theta(-x) = \theta(x)$ for $x \in (a-h, a+h)$, then $\lim_{h\to 0}\theta(h) = 1/2$.

Proof: Use the exercise (c), we have

$$\begin{aligned} f''(c) &= \lim_{h\to 0}\frac{f(c+h) - 2f(c) + f(c-h)}{h^2} \\ &= \lim_{h\to 0}\frac{f'(c+\theta(h)h) - f'(c-\theta(-h)h)}{h} \text{ by hypothesis} \\ &= \lim_{h\to 0}\frac{f'(c+\theta(h)h) - f'(c-\theta(h)h)}{2\theta(h)h}2\theta(h) \text{ since } \theta(-x) = \theta(x) \text{ for } x \in (a-h,a+h). \end{aligned}$$

Since $f''(c) \neq 0$, we finally have $\lim_{h\to 0}\theta(h) = 1/2$.

5.25 Let f have a finite derivatiive in (a,b) and assume that $c \in (a,b)$. Consider the following condition: For every $\varepsilon > 0$, there exists a 1-ball $B(c;\delta)$, whose radius δ depends only on ε and not on c, such that if $x \in B(c;\delta)$, and $x \neq c$, then

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \varepsilon.$$

Show that f' is continuous on (a,b) if this condition holds throughout (a,b).

Proof: Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $d(x,y) < \delta$, $x,y \in (a,b)$, we have

$$|f'(x) - f'(y)| < \varepsilon.$$

Choose any point $y \in (a,b)$, and thus by hypothesis, given $\varepsilon' = \varepsilon/2$, there is a 1-ball $B(y;\delta)$, whose radius δ depends only on ε' and not on y, such that if $x \in B(y;\delta)$, and $x \neq y$, then ,

$$\left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \varepsilon/2 = \varepsilon'. \quad *$$

Note that $y \in B(x,\delta)$, so, we also have

$$\left|\frac{f(x)-f(y)}{x-y}-f'(x)\right| < \varepsilon/2 = \varepsilon' \qquad *'$$

Combine (*) with (*'), we have

$$|f'(x)-f'(y)| < \varepsilon.$$

Hence, we have proved f' is continuous on (a,b).

Remark: (i) The open interval can be changed into a closed interval; it just need to consider its endpoints. That is, f' is continuous on $[a,b]$ if this condition holds throughout $[a,b]$. The proof is similar, so we omit it.

(ii) The converse of statement in the exercise is alos true. We write it as follows. Let f' be continuous on $[a,b]$, and $\varepsilon > 0$. Prove that there exists a $\delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right| < \varepsilon$$

whenever $0 < |x-c| < \delta,\ a \le x,c,\le b.$

Proof: Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right| < \varepsilon$$

whenever $0 < |x-c| < \delta,\ a \le x,c \le b.$ Since f' is continuous on $[a,b]$, we know that f' is uniformly continuous on $[a,b]$. That is, given $\varepsilon' = \varepsilon > 0$, there is a $\delta > 0$ such that as $d(x,y) < \delta$, we have

$$|f'(x)-f'(y)| < \varepsilon. \qquad *$$

Consider $d(x,c) < \delta,\ x \in [a,b]$, then by (*), we have

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right| = |f'(x')-f'(c)| < \varepsilon \text{ by } \textbf{Mean Value Theorem}$$

where $d(x',x) < \delta$. So, we complete it.

Note: This could be expressed by saying that f is uniformly differentiable on $[a,b]$ if f' is continuous on $[a,b]$.

5.26 Assume f has a finite derivative in (a,b) and is continuous on $[a,b]$, with $a \le f(x) \le b$ for all x in $[a,b]$ and $|f'(x)| \le \alpha < 1$ for all x in (a,b). Prove that f has a unique fixed point in $[a,b]$.

Proof: Given any $x,y \in [a,b]$, thus, by **Mean Value Theorem**, we have

$$|f(x)-f(y)| = |f'(z)||x-y| \le \alpha|x-y| \text{ by hypothesis.}$$

So, we know that f is a contraction on a complete metric space $[a,b]$. So, f has a unique fixed point in $[a,b]$.

5.27 Give an example of a pair of functions f and g having a finite derivatives in $(0,1)$, such that

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = 0,$$

but such that $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ does not exist, choosing g so that $g'(x)$ is never zero.

Proof: Let $f(x) = \sin(1/x)$ and $g(x) = 1/x$. Then it is trivial for that $g'(x)$ is never zero. In addition, we have

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = 0, \text{ and } \lim_{x\to 0}\frac{f'(x)}{g'(x)} \text{ does not exist.}$$

Remark: In this exercise, it tells us that **the converse of L-Hospital Rule** is **NOT** necessary true. Here is a good exercise very like **L-Hospital Rule**, but it does not! We write it as follows.

Suppose that $f'(a)$ and $g'(a)$ exist with $g'(a) \neq 0,$ and $f(a) = g(a) = 0.$ Prove that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof: Consider

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{[f(x) - f(a)]/(x-a)}{[g(x) - g(a)]/(x-a)}$$

$$= \frac{f'(a)}{g'(a)} \text{ since } f'(a) \text{ and } g'(a) \text{ exist with } g'(a) \neq 0.$$

Note: (i) It should be noticed that we **CANNOT** use **L-Hospital Rule** since the statement tells that f and g have a derivative at $a,$ we do not make sure of the situation of other points.

(ii) This holds also for complex functions. Let us recall the proof of **L-Hospital Rule,** we need use the order field $R;$ however, C is not an order field. Hence, **L-Hospital Rule** does not hold for $C.$ In fact, no order can be defined in the complex field since $i^2 = -1.$

Supplement on L-Hospital Rule

We do not give a proof about the following fact. The reader may see the book named **A First Course in Real Analysis written by Protter and Morrey**, Charpter 4, pp 88-91.

Theorem $(\frac{0}{0})$ Let f and g be continuous and differentiable on (a,b) with $g' \neq 0$ on $(a,b).$ If

$$\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x) = 0 \text{ and} \qquad *$$

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

Remark: 1. The size of the interval (a,b) is of no importance; it suffices to have $g' \neq 0$ on $(a, a+\delta),$ for some $\delta > 0.$

2. (*) is a sufficient condition, not a necessary condition. For example, $f(x) = x^2,$ and $g(x) = \sin 1/x$ both defined on $(0,1).$

3. We have some similar results: $x \to a^-;$ $x \to a;$ $x \to +\infty (\Leftrightarrow 1/x \to 0^+);$ $x \to -\infty (\Leftrightarrow 1/x \to 0^-).$

Theorem $(\frac{\infty}{\infty})$ Let f and g be continuous and differentiable on (a,b) with $g' \neq 0$ on $(a,b).$ If

$$\lim_{x \to a^+} f(x) = \infty = \lim_{x \to a^+} g(x) = \infty \text{ and} \qquad *$$

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = L.$$

Remark: 1. The proof is skilled, and it needs an algebraic identity.

2. We have some similar results: $x \to a^-$; $x \to a$; $x \to +\infty(\Leftrightarrow 1/x \to 0^+)$; $x \to -\infty(\Leftrightarrow 1/x \to 0^-)$.

3. (*) is a sufficient condition, not a necessary condition. For example, $f(x) = x + \sin x$, and $g(x) = x$.

Theorem (O. Stolz) Suppose that $y_n \to \infty$, and $\{y_n\}$ is increasing. If

$$\lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} = L, \text{ (or } +\infty)$$

then

$$\lim_{n\to\infty}\frac{x_n}{y_n} = L. \text{ (or } +\infty)$$

Remark: 1. The proof is skilled, and it needs an algebraic identity.

2. The difference between Theorem 2 and Theorem 3 is that x is a **continuous varibale** but x_n is not.

Theorem (Taylor Theorem with Remainder) Suppose that f is a real function defined on $[a,b]$. If $f^n(x)$ is continuous on $[a,b]$, and differentiable on (a,b), then (let $x, c \in [a,b]$, with $x \neq c$) there is a $\tilde{x}$, interior to the interval joining x and c such that

$$f(x) = P_f(x) + \frac{f^{(n+1)}(\tilde{x})}{(n+1)!}(x-c)^{n+1},$$

where

$$P_f(x) := \sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Remark: 1. As $n = 1$, it is exactly **Mean Value Theorem.**

2. The part

$$\frac{f^{(n+1)}(\tilde{x})}{(n+1)!}(x-c)^{n+1} := R_n(x)$$

is called the **remainder term.**

3. There are some types about remainder term. (Lagrange, Cauchy, Berstein, etc.)

Lagrange

$$R_n(x) = \frac{f^{(n+1)}(\tilde{x})}{(n+1)!}(x-c)^{n+1}.$$

Cauchy

$$R_n(x) = \frac{f^{(n+1)}(c+\theta(x-c))}{n!}[(1-\theta)^n](x-c)^{n+1}, \text{ where } 0 < \theta < 1.$$

Berstein

$$R_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt$$

5.28 Prove the following theorem:

Let f and g be two functions having finite nth derivatives in (a,b). For some interior point c in (a,b), assume that $f(c) = f'(c) = \ldots = f^{(n-1)}(c) = 0$, and that $g(c) = g'(c) = \ldots = g^{(n-1)}(c) = 0$, but that $g^{(n)}(x)$ is never zero in (a,b). Show that

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

NOTE. $f^{(n)}$ and $g^{(n)}$ are not assumed to be continuous at c.

Hint. Let

$$F(x) = f(x) - \frac{(x-c)^{n-2}f^{(n)}(c)}{(n-2)!},$$

define G similarly, and apply Theorem 5.20 to the functions F and G.

Proof: Let

$$F(x) = f(x) - \frac{f^{(n)}(c)}{(n-2)!}(x-c)^{n-2}$$

and

$$G(x) = g(x) - \frac{g^{(n)}(c)}{(n-2)!}(x-c)^{n-2}$$

then inductively,

$$F^{(k)}(x) = f^{(k)}(x) - \frac{f^{(n)}(c)}{(n-2-k)!}(x-c)^{n-2-k}$$

and note that

$$F^{(k)}(c) = 0 \text{ for all } k = 0,1,..,n-3, \text{ and } F^{(n-2)}(c) = -f^{(n)}(c).$$

Similarly for G. Hence, by **Theorem 5.20**, we have

$$\left[F(x) - \sum_{k=0}^{n-2}\frac{F^{(k)}}{k!}(x-c)^k\right][G^{(n-1)}(x_1)] = [F^{(n-1)}(x_1)]\left[G(x) - \sum_{k=0}^{n-2}\frac{G^{(k)}}{k!}(x-c)^k\right]$$

where x_1 between x and c, which implies that

$$[f(x)][g^{(n-1)}(x_1)] = [f^{(n-1)}(x_1)][g(x)]. \qquad *$$

Note that since $g^{(n)}$ is never zero on (a,b); it implies that there exists a $\delta > 0$ such that every $g^{(k)}$ is never zero in $(c-\delta, c+\delta) - \{c\}$, where $k = 0,1,2\ldots,n$. Hence, we have, by (*),

$$\begin{aligned}\lim_{x\to c}\frac{f(x)}{g(x)} &= \lim_{x\to c}\frac{f^{(n-1)}(x_1)}{g^{(n-1)}(x_1)} \\ &= \lim_{x_1\to c}\frac{f^{(n-1)}(x_1) - f^{(n-1)}(c)}{g^{(n-1)}(x_1) - g^{(n-1)}(c)} \text{ since } x \to c (\Rightarrow x_1 \to c) \\ &= \lim_{x_1\to c}\frac{(f^{(n-1)}(x_1) - f^{(n-1)}(c))/(x_1-c)}{(g^{(n-1)}(x_1) - g^{(n-1)}(c))/(x_1-c)} \\ &= \frac{f^{(n)}(c)}{g^{(n)}(c)} \text{ since } f^{(n)} \text{ exists and } g^{(n)} \text{ exists}(\neq 0) \text{ on } (a,b).\end{aligned}$$

Remark: (1) The hint is not correct from text book. The reader should find the difference between them.

(2) Here ia another proof by **L-Hospital Rule and Remark in Exercise 5.27.**

Proof: Since $g^{(n)}$ is never zero on (a,b), it implies that there exists a $\delta > 0$ such that every $g^{(k)}$ is never zero in $(c-\delta, c+\delta) - \{c\}$, where $k = 0,1,2\ldots,n$. So, we can apply $(n-1)$ **−times L-Hospital Rule methoned in Supplement,** and thus get

$$\begin{aligned}\lim_{x\to c}\frac{f(x)}{g(x)} &= \lim_{x\to c}\frac{f^{(n-1)}(x)}{g^{(n-1)}(x)}\\ &= \lim_{x\to c}\frac{f^{(n-1)}(x)-f^{(n-1)}(c)}{g^{(n-1)}(x)-g^{(n-1)}(c)}\\ &= \lim_{x\to c}\frac{(f^{(n-1)}(x)-f^{(n-1)}(c))/(x-c)}{(g^{(n-1)}(x)-g^{(n-1)}(c))/(x-c)}\\ &= \frac{f^{(n)}(c)}{g^{(n)}(c)} \text{ since } f^{(n)} \text{ exists and } g^{(n)} \text{ exists}(\neq 0) \text{ on } (a,b).\end{aligned}$$

5.29 Show that the formula in Taylor's theorem can also be written as follows:

$$f(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{(x-c)(x-x_1)^{n-1}}{(n-1)!}f^{(n)}(x_1),$$

where x_1 is interior to the interval joining x and c. Let $1-\theta = (x-x_1)/(x-c)$. Show that $0 < \theta < 1$ and deduce the following form of the remainder term (due to Cauchy):

$$\frac{(1-\theta)^{n-1}(x-c)^n}{(n-1)!}f^{(n)}[\theta x + (1-\theta)c].$$

Hint. Take $G(t) = t$ in the proof of Theorem 5.20.

Proof: Let

$$F(t) = \sum_{k=0}^{n-1}\frac{f^{(k)}(t)}{k!}(x-t)^k, \text{ and } G(t) = t,$$

and note that

$$F'(t) = \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}$$

then by **Generalized Mean Value Theorem**, we have

$$[F(x)-F(c)][G'(x_1)] = [G(x)-G(c)][F'(x_1)]$$

which implies that

$$\begin{aligned}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k &= \frac{f^{(n)}(x_1)}{(n-1)!}(x-x_1)^n\\ &= \frac{f^{(n)}(\theta x+(1-\theta)c)}{(n-1)!}(x-c)^n(1-\theta)^n, \text{ where } x_1 = \theta x + (1-\theta)c.\end{aligned}$$

So, we have prove that

$$f(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k + R_{n-1}(x),$$

where

$$R_{n-1}(x) = \frac{f^{(n)}(\theta x+(1-\theta)c)}{(n-1)!}(x-c)^n(1-\theta)^n, \text{ where } x_1 = \theta x + (1-\theta)c$$

is called a **Cauchy Remainder.**

Supplement on some questions.

1. Let f be continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose that $f(0) = 0$ and

$|f'(x)| \leq |f(x)|$ for $x \in (0,1)$. Prove that f is constant.

Proof: Given any $x_1 \in (0,1]$, by Mean Value Theorem and hypothesis, we know that

$$|f(x_1) - f(0)| = x_1|f'(x_2)| \leq x_1|f(x_2)|, \text{ where } x_2 \in (0,x_1).$$

So, we have

$$|f(x_1)| \leq x_1 \cdots x_n|f(x_{n+1})| \leq M(x_1 \cdots x_n), \text{ where } x_{n+1} \in (0,x_n), \text{ and } M = \sup_{x\in[a,b]}|f(x)|$$

Since $M(x_1 \cdots x_n) \to 0$, as $n \to \infty$, we finally have $f(x_1) = 0$. Since x_1 is arbirary, we find that $f(x) = 0$ on $[0,1]$.

2. Suppose that g is real function defined on R, with bounded derivative, say $|g'| \leq M$. Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Show that f is 1-1 if ε is small enough. (It implies that f is strictly monotonic.)

Proof: Suppose that $f(x) = f(y)$, i.e., $x + \varepsilon g(x) = y + \varepsilon g(y)$ which implies that

$|y - x| = \varepsilon|g(y) - g(x)| \leq \varepsilon M|y - x|$ by **Mean Value Theorem**, and hypothesis.

So, as ε is small enough, we have $x = y$. That is, f is 1-1.

Supplement on Convex Function.

Definition(Convex Function) Let f be defined on an interval I, and given $0 < \lambda < 1$, we say that f is a convex function if for any two points $x,y \in I$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

For example, x^2 is a convex function on R. Sometimes, the reader may see another weak definition of convex function in case $\lambda = 1/2$. We will show that under continuity, two definitions are equivalent. In addition, it should be noted that a convex function is not necessarily continuous since we may give a jump on a continuous convex function on its boundary points, for example, $f(x) = x$ is a continuous convex function on $[0,1]$, and define a function g as follows:

$$g(x) = x, \text{ if } x \in (0,1) \text{ and } g(1) = g(0) = 2.$$

The function g is not continuous but convex. Note that if $-f$ is convex, we call f is concave, vice versa. Note that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f.) It is clear only by definition.

Theorem(Equivalence) Under continuity, two definitions are equivalent.

Proof: It suffices to consider if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \qquad *$$

then

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \text{ for all } 0 < \lambda < 1.$$

Since (*) holds, then by **Mathematical Induction**, it is easy to show that

$$f\left(\frac{x_1 + \ldots + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + \ldots + f(x_{2^n})}{2^n}.$$

Claim that

$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) \leq \frac{f(x_1) + \ldots + f(x_n)}{n} \text{ for all } n \in N. \qquad **$$

Using **Reverse Induction**, let $x_n = \frac{x_1+\ldots+x_{n-1}}{n-1}$, then

$$
\begin{aligned}
f\left(\frac{x_1+\ldots+x_n}{n}\right) &= f\left(\frac{x_1+\ldots+x_{n-1}}{n}+\frac{x_n}{n}\right)\\
&= f(x_n)\\
&\le \frac{f(x_1)+\ldots+f(x_n)}{n} \text{ by induction hypothesis.}
\end{aligned}
$$

So, we have

$$f\left(\frac{x_1+\ldots+x_{n-1}}{n-1}\right) \le \frac{f(x_1)+\ldots+f(x_{n-1})}{n-1}.$$

Hence, we have proved (**). Given a rational number $m/n \in (0,1)$, where $g.c.d.(m,n) = 1$; we choose $x := x_1 = \ldots = x_m$, and $y := x_{m+1} = \ldots = x_n$, then by (**), we finally have

$$f\left(\frac{mx}{n}+\frac{(n-m)y}{n}\right) \le \frac{mf(x)}{n}+\frac{(n-m)f(y)}{n} = \frac{m}{n}f(x)+(1-\frac{m}{n})f(y). \qquad ***$$

Given $\lambda \in (0,1)$, then there is a sequence $\{q_n\}(\subseteq Q)$ such that $q_n \to \lambda$ as $n \to \infty$. Then by continuity and (***), we get

$$f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y).$$

Remark: **The Reverse Induction** is that let $S \subseteq N$ and S has two properties:(1) For every $k \ge 0$, $2^k \in S$ and (2) $k \in S$ and $k-1 \in N$, then $k-1 \in S$. Then $S = N$.

(**Lemma**) Let f be a convex function on $[a,b]$, then f is bounded.

Proof: Let $M = \max(f(a),f(b))$, then every point $z \in I$, write $z = a\lambda+(1-\lambda)b$, we have

$$f(z) = f(a\lambda+(1-\lambda)b) \le \lambda f(a)+(1-\lambda)f(b) \le M.$$

In addition, we may write $z = \frac{a+b}{2}-t$, where t is chosen so that z runs through $[a,b]$. So, we have

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}f\left(\frac{a+b}{2}-t\right)+\frac{1}{2}f\left(\frac{a+b}{2}+t\right)$$

which implies that

$$2f\left(\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}+t\right) \le f\left(\frac{a+b}{2}-t\right) = f(z)$$

which implies that

$$2f\left(\frac{a+b}{2}\right)-M := m \le f(z).$$

Hence, we have proved that f is bounded above by M and bounded below by m.

(**Theorem**) If $f: I \to R$ is convex, then f satisfies a **Lipschitz condition** on any closed interval $[a,b] \subseteq int(I)$. In addition, f is **absolutely continuous** on $[a,b]$ and continuous on $int(I)$.

Proof: We choose $\varepsilon(>0)$ so that $[a-\varepsilon,b+\varepsilon](\subseteq int(I))$. By preceding lemma, we know that f is bounded, say $m \le f(x) \le M$ on $[a-\varepsilon,b+\varepsilon]$. Given any two points x, and y, with $a \le x < y \le b$ We consider an auxiliary point $z = y+\varepsilon$, and a suitable $\lambda = \frac{y-x}{\varepsilon+y-x}$, then $y = \lambda z+(1-\lambda)x$. So,

$$f(y) = f(\lambda z+(1-\lambda)x) \le \lambda f(z)+(1-\lambda)f(x) = \lambda[f(z)-f(x)]+f(x)$$

which implies that

$$f(y)-f(x) \le \lambda(M-m) \le \frac{y-x}{\varepsilon}(M-m).$$

Change roles of x and y, we finally have

$$|f(y) - f(x)| \leq K|y - x|, \text{ where } K = \frac{M - m}{\varepsilon}.$$

That is, f satisfies a **Lipschitz condition** on any closed interval $[a,b]$.

We call that f is absolutely continuous on $[a,b]$ if given any $\varepsilon > 0$, there is a $\delta > 0$ such that for any collection of $\{(a_i,b_i)\}_{i=1}^n$ of disjoint open intervals of $[a,b]$ with $\sum_{k=1}^n b_i - a_i < \delta$, we have

$$\sum_{k=1}^n |f(b_i) - f(a_i)| < \varepsilon.$$

Clearly, the choice $\delta = \varepsilon/K$ meets this requirement. Finally, the continuity of f on $int(I)$ is obvious.

(**Theorem**) Let f be a differentiable real function defined on (a,b). Prove that f is convex if and only if f' is monotonically increasing.

Proof: ($\Rightarrow$) Suppose f is convex, and given $x < y$, we want to show that $f'(x) \leq f'(y)$. Choose s and t such that $x < u < s < y$, then it is clear that we have

$$\frac{f(u) - f(x)}{u - x} \leq \frac{f(s) - f(u)}{s - u} \leq \frac{f(y) - f(s)}{y - s}. \quad *$$

Let $s \to y^-$, we have by (*)

$$\frac{f(u) - f(x)}{u - x} \leq f'(y)$$

which implies that, let $u \to x^+$

$$f'(x) \leq f'(y).$$

($\Leftarrow$) Suppose that f' is monotonically increasing, it suffices to consider $\lambda = 1/2$, if $x < y$, then

$$\begin{aligned}\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) &= \frac{[f(x) - f(\frac{x+y}{2})] + [f(y) - f(\frac{x+y}{2})]}{2} \\ &= \frac{x-y}{4}[f'(\xi_1) - f'(\xi_2)], \text{ where } \xi_1 \leq \xi_2 \\ &\leq 0.\end{aligned}$$

Similarly for $x > y$, and there is nothing to prove $x = y$. Hence, we know that f is convex.

(**Corollary 1**) Assume next that $f''(x)$ exists for every $x \in (a,b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a,b)$.

Proof: ($\Rightarrow$) Suppose that f is convex, we have shown that f' is monotonically increasing. So, we know that $f''(x) \geq 0$ for all $x \in (a,b)$.

($\Leftarrow$) Suppose that $f''(x) \geq 0$ for all $x \in (a,b)$, it implies that f' is monotonically increasing. So, we know that f is convex.

(**Corollary 2**) Let $0 < \beta \leq \alpha$, then we have

$$\left(\frac{|y_1|^\beta + \ldots + |y_n|^\beta}{n}\right)^{1/\beta} \leq \left(\frac{|y_1|^\alpha + \ldots + |y_n|^\alpha}{n}\right)^{1/\alpha}$$

Proof: Let $p \geq 1$, and since $(x^p)'' = p(p-1)x^{p-2} \geq 0$ for all $x > 0$, we know that $f(x) = x^p$ is convex. So, we have (let $p = \frac{\alpha}{\beta}$)

$$\left(\frac{x_1 + \ldots + x_n}{n}\right)^{\alpha/\beta} \leq \frac{x_1^{\alpha/\beta} + \ldots + x_n^{\alpha/\beta}}{n} \quad *$$

by

$$f\left(\frac{x_1+\ldots+x_n}{n}\right) \le \frac{f(x_1)+\ldots+f(x_n)}{n}.$$

Choose $x_i = |y_i|^\beta$, where $i = 1,2,..,n$. Then by (*), we have

$$\left(\frac{|y_1|^\beta+\ldots+|y_n|^\beta}{n}\right)^{1/\beta} \le \left(\frac{|y_1|^\alpha+\ldots+|y_n|^\alpha}{n}\right)^{1/\alpha}.$$

(**Corollary 3**) Define

$$M_r(y) = \left(\frac{|y_1|^r+\ldots+|y_n|^r}{n}\right)^{1/r}, \text{ where } r > 0.$$

Then $M_r(y)$ is a monotonic function of r on $(0,\infty)$. In particular, we have

$$M_1(y) \le M_2(y),$$

that is,

$$\frac{|y_1|+\ldots+|y_n|}{n} \le \left(\frac{|y_1|^2+\ldots+|y_n|^2}{n}\right)^{1/2}.$$

Proof: It is clear by **Corollary 2.**

(**Corollary 4**) By definition of $M_r(y)$ in **Corollary 3**, we have

$$\lim_{r\to 0^+} M_r(y) = (|y_1|\cdots|y_n|)^{1/n} := M_0(y)$$

and

$$\lim_{r\to\infty} M_r(y) = \max(|y_1|,\ldots,|y_n|) := M_\infty(y)$$

Proof: 1. Since $M_r(y) = \left(\frac{|y_1|^r+\ldots+|y_n|^r}{n}\right)^{1/r}$, taking log and thus by **Mean Value Theorem**, we have

$$\frac{\log\left(\frac{|y_1|^r+\ldots+|y_n|^r}{n}\right)-0}{r-0} = \frac{(\frac{1}{n})\sum_{i=1}^n |y_i|^{r'}\log|y_i|}{(\frac{1}{n})\sum_{i=1}^n |y_i|^{r'}}, \text{ where } 0 < r' < r.$$

So,

$$\begin{aligned}\lim_{r\to 0^+} M_r(y) &= \lim_{r\to 0^+} e^{\frac{\log\left(\frac{|y_1|^r+\ldots+|y_n|^r}{n}\right)}{r}} \\ &= \lim_{r\to 0^+} e^{\frac{(\frac{1}{n})\sum_{i=1}^n |y_i|^{r'}\log|y_i|}{(\frac{1}{n})\sum_{i=1}^n |y_i|^{r'}}} \\ &= e^{\frac{\sum_{i=1}^n \log|y_i|}{n}} \\ &= (|y_1|\cdots|y_n|)^{1/n}.\end{aligned}$$

2. As $r > 0$, we have

$$\left\{\frac{[\max(|y_1|,\ldots,|y_n|)]^r}{n}\right\}^{1/r} \le M_r(y) \le \{[\max(|y_1|,\ldots,|y_n|)]^r\}^{1/r}$$

which implies that, by **Sandwich Theorem**,

$$\lim_{r\to\infty} M_r(y) = \max(|y_1|,\ldots,|y_n|)$$

since $\lim_{r\to\infty}(\frac{1}{n})^{1/r} = 1$.

(**Inequality 1**) Let f be convex on $[a,b]$, and let $c \in (a,b)$. Define

$$l(x) = f(a) + \frac{f(c) - f(a)}{c - a}(x - a),$$
then $f(x) \geq l(x)$ for all $x \in [c, b]$.

Proof: Consider $x \in [c, d]$, then $c = \frac{x-c}{x-a}a + \frac{c-a}{x-a}x$, we have
$$f(c) \leq \frac{x - c}{x - a}f(a) + \frac{c - a}{x - a}f(x)$$
which implies that
$$f(x) \geq f(a) + \frac{f(c) - f(a)}{c - a}(x - a) = l(x).$$

(**Inequality 2**) Let f be a convex function defined on (a, b). Let $a < s < t < u < b$, then we have
$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Proof: By definition of convex, we know that
$$f(x) \leq f(s) + \frac{f(u) - f(s)}{u - s}(x - s),\ x \in [s, u] \qquad *$$
and by **inequality 1**, we know that
$$f(s) + \frac{f(t) - f(s)}{t - s}(x - s) \leq f(x),\ x \in [t, u]. \qquad **$$
So, as $x \in [t, u]$, by (*) and (**), we finally have
$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s}.$$
Similarly, we have
$$\frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$
Hence, we have
$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Remark: Using abvoe method, it is easy to verify that if f is a convex function on (a, b), then $f'_-(x)$ and $f'_+(x)$ exist for all $x \in (a, b)$. In addition, if $x < y$, where $x, y \in (a, b)$, then we have
$$f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y).$$
That is, $f'_+(x)$ and $f'_-(x)$ are increasing on (a, b). We omit the proof.

(**Exercise 1**) Let $f(x)$ be convex on (a, b), and assume that f is differentiable at $c \in (a, b)$, we have
$$l(x) = f(c) + f'(c)(x - c) \leq f(x).$$
That is, the equation of tangent line is below $f(x)$ if the equation of tangent line exists.

Proof: Since f is differentiable at $c \in (a, b)$, we write the equation of tangent line at c,
$$l(x) = f(c) + f'(c)(x - c).$$
Define
$$m(s) = \frac{f(s) - f(c)}{s - c} \text{ where } a < s < c \text{ and } m(t) = \frac{f(t) - f(c)}{t - c} \text{ where } b > t > c,$$
then it is clear that
$$m(s) \leq f'(c) \leq m(t)$$
which implies that

$$l(x) = f(c) + f'(c)(x - c) \le f(x).$$

(**Exercise 2**) Let $f : R \to R$ be convex. If f is bounded above, then f is a constant function.

Proof: Suppose that f is not constant, say $f(a) \ne f(b)$, where $a < b$. If $f(b) > f(a)$, we consider

$$\frac{f(x) - f(b)}{x - b} \ge \frac{f(b) - f(a)}{b - a}, \text{ where } x > b$$

which implies that as $x > b$,

$$f(x) \ge \frac{f(b) - f(a)}{b - a}(x - b) + f(b) \to +\infty \text{ as } x \to +\infty$$

And if $f(b) < f(a)$, we consider

$$\frac{f(x) - f(a)}{x - a} \le \frac{f(b) - f(a)}{b - a}, \text{ where } x < a$$

which implies that as $x < a$,

$$f(x) \ge \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \to +\infty \text{ as } x \to -\infty.$$

So, we obtain that f is not bouded above. So, f must be a constant function.

(**Exercise 3**) Note that e^x is convex on R. Use this to show that $A.P. \ge G.P.$

Proof: Since $(e^x)'' = e^x \ge 0$ on R, we know that e^x is convex. So,

$$e^{\frac{x_1 + \ldots + x_n}{n}} \le \frac{e^{x_1} + \ldots + e^{x_n}}{n}, \text{ where } x_i \in R,\ i = 1, 2, \ldots, n.$$

So, let $e^{x_i} = y_i > 0$, for $i = 1, 2, \ldots, n$. Then

$$(y_1 \bullet \bullet \bullet y_n)^{1/n} \le \frac{y_1 + \ldots + y_n}{n}.$$

Vector-Valued functions

5.30 If a vector valued function f is differentiable at c, prove that

$$f'(c) = \lim_{h \to 0} \frac{1}{h}[f(c + h) - f(c)].$$

Conversely, if this limit exists, prove that f is differentiable at c.

Proof: Write $f = (f_1, \ldots, f_n) : S(\subseteq R) \to R^n$, and let c be an interior point of S. Then if f is differentiable at c, each f_k is differentiable at c. Hence,

$$\begin{aligned}
&\lim_{h \to 0} \frac{1}{h}[f(c + h) - f(c)] \\
&= \lim_{h \to 0}\left(\frac{f_1(c + h) - f_1(c)}{h}, \ldots, \frac{f_n(c + h) - f_n(c)}{h}\right) \\
&= \left(\lim_{h \to 0}\frac{f_1(c + h) - f_1(c)}{h}, \ldots, \lim_{h \to 0}\frac{f_n(c + h) - f_n(c)}{h}\right) \\
&= (f_1'(c), \ldots, f_n'(c)) \\
&= f'(c).
\end{aligned}$$

Conversly, it is obvious by above.

Remark: We give a summary about this. Let f be a vector valued function defined on S. Write $f : S(\subseteq R^n) \to R^m$, c is a interior point.

$$f = (f_1, \ldots, f_n) \text{ is differentiable at } c \Leftrightarrow \text{ each } f_k \text{ is differentiable at } c, \qquad *$$

and

$$f = (f_1,\ldots,f_n) \text{ is continuous at } c \Leftrightarrow \text{ each } f_k \text{ is continuous at } c.$$

Note: The set S can be a subset in R^n, the definition of differentiation in higher dimensional space makes (*) holds. The reader can see textbook, Charpter 12.

5.31 A vector-valued function f is differentiable at each point of (a,b) and has constant norm $\|f\|$. Prove that $f(t) \cdot f'(t) = 0$ on (a,b).

Proof: Since $\langle f,f\rangle = \|f\|^2$ is constant on (a,b), we have $\langle f,f\rangle' = 0$ on (a,b). It implies that $2\langle f,f'\rangle = 0$ on (a,b). That is, $f(t) \cdot f'(t) = 0$ on (a,b).

Remark: The proof of $\langle f,g\rangle' = \langle f',g\rangle + \langle f,g'\rangle$ is easy from definition of differentiation. So, we omit it.

5.32 A vector-valued function f is never zero and has a derivative f' which exists and is continuous on R. If there is a real function λ such that $f'(t) = \lambda(t)f(t)$ for all t, prove that there is a positive real function u and a constant vector c such that $f(t) = u(t)c$ for all t.

Proof: Since $f'(t) = \lambda(t)f(t)$ for all t, we have

$$(f_1'(t),\ldots,f_n'(t)) = f'(t) = \lambda(t)f(t) = (\lambda(t)f_1(t),\ldots,\lambda(t)f_n(t))$$

which implies that

$$\frac{f_i'(t)}{fi(t)} = \lambda(t) \text{ since } f \text{ is never zero.} \qquad *$$

Note that $\frac{f_i'(t)}{f_i(t)}$ is a continuous function from R to R for each $i = 1,2,\ldots,n$, since f' is continuous on R, we have, by (*)

$$\int_a^x \frac{f_1'(t)}{f_1(t)}dt = \int_a^x \lambda(t)dt \Rightarrow f_i(t) = \frac{f_i(a)}{e^{\lambda(a)}}e^{\lambda(t)} \text{ for } i = 1,2,\ldots,n.$$

So, we finally have

$$\begin{aligned} f(t) &= (f_1(t),\ldots,f_n(t)) \\ &= e^{\lambda(t)}\left(\frac{f_1(a)}{e^{\lambda(a)}},\ldots,\frac{f_n(a)}{e^{\lambda(a)}}\right) \\ &= u(t)c \end{aligned}$$

where $u(t) = e^{\lambda(t)}$ and $c = \left(\frac{f_1(a)}{e^{\lambda(a)}},\ldots,\frac{f_n(a)}{e^{\lambda(a)}}\right)$.

Supplement on Mean Value Theorem in higher dimensional space.

In the future, we will learn so called **Mean Value Theorem in higher dimensional space** from the text book in Charpter 12. We give a similar result as supplement.

Suppose that f is continuous mapping of $[a,b]$ into R^n and f is differentiable in (a,b). Then there exists $x \in (a,b)$ such that

$$\|f(b) - f(a)\| \leq (b-a)\|f'(x)\|.$$

Proof: Let $z = f(b) - f(a)$, and define $\phi(x) = f(x) \cdot z$ which is a real valued function defined on (a,b). It is clear that $\phi(x)$ is continuous on $[a,b]$ and differentiable on (a,b). So, by **Mean Value Theorem**, we know that

$$\phi(b) - \phi(a) = \phi'(x)(b-a), \text{ where } x \in (a,b)$$

which implies that

$$\begin{aligned} |\phi(b) - \phi(a)| &= |\phi'(x)(b-a)| \\ &\leq \|f(b) - f(a)\|\,\|\phi'(x)\|(b-a) \text{ by Cauchy-Schwarz inequality.} \end{aligned}$$

So, we have

$$\|f(b) - f(a)\| \leq (b-a)\|f'(x)\|.$$

Partial derivatives

5.33 Consider the function f defined on R^2 by the following formulas:

$$f(x,y) = \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0)\ f(0,0) = 0.$$

Prove that the partial derivatives $D_1f(x,y)$ and $D_2f(x,y)$ exist for every (x,y) in R^2 and evaluate these derivatives explicitly in terms of x and y. Also, show that f is not continuous at $(0,0)$.

Proof: It is clear that for all $(x,y) \neq (0,0)$, we have

$$D_1f(x,y) = y\frac{y^2-x^2}{(x^2+y^2)^2} \text{ and } D_2f(x,y) = x\frac{x^2-y^2}{(x^2+y^2)^2}.$$

For $(x,y) = (0,0)$, we have

$$D_1f(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = 0.$$

Similarly, we have

$$D_2f(0,0) = 0.$$

In addition, let $y = x$ and $y = 2x$, we have

$$\lim_{x\to 0} f(x,x) = 1/2 \neq \lim_{x\to 0} f(x,2x) = 2/5.$$

Hence, f is not continuous at $(0,0)$.

Remark: The existence of all partial derivatives does not make sure the continuity of f. The trouble with partial derivatives is that they treat a function of several variables as a function of one variable at a time.

5.34 Let f be defined on R^2 as follows:

$$f(x,y) = y\frac{x^2-y^2}{x^2+y^2} \text{ if } (x,,y) \neq (0,0),\ f(0,0) = 0.$$

Compute the first- and second-order partial derivatives of f at the origin, when they exist.

Proof: For $(x,y) \neq (0,0)$, it is clear that we have

$$D_1f(x,y) = \frac{4xy^3}{(x^2+y^2)^2} \text{ and } D_2f(x,y) = \frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}$$

and for $(x,y) = (0,0)$, we have

$$D_1f(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = 0,\ D_2f(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = -1.$$

Hence,

$$D_{1,1}f(0,0) = \lim_{x\to 0}\frac{D_1f(x,0)-D_1f(0,0)}{x-0} = 0,$$

$$D_{1,2}f(0,0) = \lim_{x\to 0}\frac{D_2f(x,0)-D_2f(0,0)}{x-0} = \lim_{x\to 0}\frac{2}{x} \text{ does not exist,}$$

$$D_{2,1}f(0,0) = \lim_{y\to 0}\frac{D_1f(0,y)-D_1f(0,0)}{y-0} = 0,$$

and

$$D_{2,2}f(0,0) = \lim_{y\to 0}\frac{D_2f(0,y) - D_2f(0,0)}{y-0} = \lim_{y\to 0}\frac{0}{y} = 0.$$

Remark: We do not give a detail computation, but here are answers. Leave to the reader as a practice. For $(x,y) \neq (0,0)$, we have

$$D_{1,1}f(x,y) = \frac{4y^3(y^2-3x^2)}{(x^2+y^2)^3},$$

$$D_{1,2}f(x,y) = \frac{4xy^2(3x^2-y^2)}{(x^2+y^2)^3},$$

$$D_{2,1}f(x,y) = \frac{4xy^2(3x^2-y^2)}{(x^2+y^2)^3},$$

and

$$D_{2,2}f(x,y) = \frac{4x^2y(y^2-3x^2)}{(x^2+y^2)^3}.$$

complex-valued functions

5.35 Let S be an open set in C and let S^* be the set of complex conjugates $\bar{z}$, where $z \in S$. If f is defined on S, define g on S^* as follows: $g(\bar{z}) = \bar{f}(z)$, the complex conjugate of $f(z)$. If f is differentiable at c, prove that g is differentiable at $\bar{c}$ and that $g'(\bar{c}) = \bar{f}'(c)$.

Proof: Since $c \in S$, we know that c is an interior point. Thus, it is clear that $\bar{c}$ is also an interior point of S^*. Note that we have

$$\text{the conjugate of } \left(\frac{f(z)-f(c)}{z-c}\right) = \frac{\bar{f}(z)-\bar{f}(c)}{\bar{z}-\bar{c}}$$
$$= \frac{g(\bar{z})-g(\bar{c})}{\bar{z}-\bar{c}} \text{ by } g(\bar{z}) = \bar{f}(z).$$

Note that $z \to c (\Leftrightarrow \bar{z} \to \bar{c})$, so we know that if f is differentiable at c, prove that g is differentiable at $\bar{c}$ and that $g'(\bar{c}) = \bar{f}'(c)$.

5.36 (i) In each of the following examples write $f = u + iv$ and find explicit formulas for $u(x,y)$ and $v(x,y)$: (These functions are to be defined as indicated in Charpter 1.)

(a) $f(z) = \sin z$,

Solution: Since $e^{iz} = \cos z + i\sin z$, we know that

$$\sin z = \frac{1}{2}[(e^y + e^{-y})\sin x + i(e^y - e^{-y})\cos x]$$

from $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$. So, we have

$$u(x,y) = \frac{(e^{-y}+e^y)\sin x}{2}$$

and

$$v(x,y) = \frac{(e^y - e^{-y})\cos x}{2}.$$

(b) $f(z) = \cos z$,

Solution: Since $e^{iz} = \cos z + i\sin z$, we know that

$$\cos z = \frac{1}{2}[(e^{-y}+e^y)\cos x + (e^{-y}-e^y)\sin x]$$

from $\cos z = \frac{e^{iz}+e^{-iz}}{2}$. So, we have

$$u(x,y) = \frac{(e^{-y} + e^{y})\cos x}{2}$$

and

$$v(x,y) = \frac{(e^{-y} - e^{y})\sin x}{2}.$$

(c) $f(z) = |z|$,

Solution: Since $|z| = (x^2 + y^2)^{1/2}$, we know that

$$u(x,y) = (x^2 + y^2)^{1/2}$$

and

$$v(x,y) = 0.$$

(d) $f(z) = \bar{z}$,

Solution: Since $\bar{z} = x - iy$, we know that

$$u(x,y) = x$$

and

$$v(x,y) = -y.$$

(e) $f(z) = \arg z$, $(z \neq 0)$,

Solution: Since $\arg z \in R$, we know that

$$u(x,y) = \arg(x + iy)$$

and

$$v(x,y) = 0.$$

(f) $f(z) = Log\, z$, $(z \neq 0)$,

Solution: Since $Log\, z = \log|z| + i\arg(z)$, we know that

$$u(x,y) = \log(x^2 + y^2)^{1/2}$$

and

$$v(x,y) = \arg(x + iy).$$

(g) $f(z) = e^{z^2}$,

Solution: Since $e^{z^2} = e^{(x^2-y^2)+i(2xy)}$, we know that

$$u(x,y) = e^{x^2-y^2}\cos(2xy)$$

and

$$v(x,y) = e^{x^2-y^2}\sin(2xy).$$

(h) $f(z) = z^\alpha$, (α complex, $z \neq 0$).

Solution: Since $z^\alpha = e^{\alpha Log\, z}$, then we have (let $\alpha = \alpha_1 + i\alpha_2$)

$$\begin{aligned} z^\alpha &= e^{(\alpha_1 + i\alpha_2)(\log|z| + i\arg z)} \\ &= e^{(\alpha_1 \log|z| - \alpha_2 \arg z) + i(\alpha_2 \log|z| + \alpha_1 \arg z)}. \end{aligned}$$

So, we know that

$$\begin{aligned} u(x,y) &= e^{\alpha_1 \log|z| - \alpha_2 \arg z}\cos(\alpha_2 \log|z| + \alpha_1 \arg z) \\ &= e^{\alpha_1 \log(x^2+y^2)^{1/2} - \alpha_2 \arg(x+iy)}\cos\left(\alpha_2 \log(x^2 + y^2)^{1/2} + \alpha_1 \arg(x + iy)\right) \end{aligned}$$

and

$$\begin{aligned} v(x,y) &= e^{\alpha_1 \log|z| - \alpha_2 \arg z} \sin(\alpha_2 \log|z| + \alpha_1 \arg z) \\ &= e^{\alpha_1 \log(x^2+y^2)^{1/2} - \alpha_2 \arg(x+iy)} \sin\left(\alpha_2 \log(x^2+y^2)^{1/2} + \alpha_1 \arg(x+iy)\right). \end{aligned}$$

(ii) Show that u and v satisfy the Cauchy -Riemanns equation for the following values of z : All z in (a), (b), (g); no z in (c), (d), (e); all z except real $z \leq 0$ in (f), (h).

Proof: (a) $\sin z = u + iv,$ where

$$u(x,y) = \frac{(e^{-y} + e^{y})\sin x}{2} \text{ and } v(x,y) = \frac{(e^{y} - e^{-y})\cos x}{2}.$$

So,

$$u_x = v_y = \frac{(e^{-y} + e^{y})\cos x}{2} \text{ for all } z = x + iy$$

and

$$u_y = -v_x = \frac{(e^{y} - e^{-y})\sin x}{2} \text{ for all } z = x + iy.$$

(b) $\cos z = u + iv,$ where

$$u(x,y) = \frac{(e^{-y} + e^{y})\cos x}{2} \text{ and } v(x,y) = \frac{(e^{-y} - e^{y})\sin x}{2}.$$

So,

$$u_x = v_y = -\frac{(e^{-y} + e^{y})\sin x}{2} \text{ for all } z = x + iy.$$

and

$$u_y = -v_x = \frac{(e^{y} - e^{-y})\cos x}{2} \text{ for all } z = x + iy.$$

(c) $|z| = u + iv,$ where

$$u(x,y) = (x^2 + y^2)^{1/2} \text{ and } v(x,y) = 0.$$

So,

$$u_x = x(x^2 + y^2)^{-1/2} = v_y = 0 \text{ if } x = 0,\ y \neq 0.$$

and

$$u_y = y(x^2 + y^2)^{-1/2} = -v_x = 0 \text{ if } x \neq 0,\ y = 0.$$

So, we know that no z makes Cauchy-Riemann equations hold.

(d) $\bar{z} = u + iv,$ where

$$u(x,y) = x \text{ and } v(x,y) = -y.$$

So,

$$u_x = 1 \neq -1 = v_y.$$

So, we know that no z makes Cauchy-Riemann equations hold.

(e) $\arg z = u + iv,$ where

$$-\pi < u(x,y) = \arg(x^2 + y^2)^{1/2} \leq \pi \text{ and } v(x,y) = 0.$$

Note that

$$u(x,y) = \begin{cases} (1)\ \arctan(y/x),\ \text{if } x > 0, y \in R \\ (2)\ \pi/2,\ \text{if } x = 0, y > 0 \\ (3)\ \arctan(y/x) + \pi,\ \text{if } x < 0,\ y \geq 0 \\ (4)\ \arctan(y/x) - \pi,\ \text{if } x < 0,\ y < 0 \\ (5)\ -\pi/2,\ \text{if } x = 0, y < 0. \end{cases}$$

and

$$v_x = v_y = 0.$$

So, we know that by (1)-(5), for $(x,y) \neq (0,0)$

$$u_x = \frac{-y}{x^2 + y^2}$$

and for $(x,y) \notin \{(x,y) : x \leq 0, y = 0\}$, we have

$$u_y = \frac{x}{x^2 + y^2}.$$

Hence, we know that no z makes Cauchy-Riemann equations hold.

Remark: We can give the conclusion as follows:

$$(\arg z)_x = \frac{-y}{x^2 + y^2} \text{ for } (x,y) \neq (0,0)$$

and

$$(\arg z)_y = \frac{x}{x^2 + y^2} \text{ for } (x,y) \notin \{(x,y) : x \leq 0, y = 0\}.$$

(f) $Log\, z = u + iv$, where

$$u(x,y) = \log(x^2 + y^2)^{1/2} \text{ and } v(x,y) = \arg(x^2 + y^2)^{1/2}.$$

Since

$$u_x = \frac{x}{x^2 + y^2} \text{ and } u_y = \frac{y}{x^2 + y^2}$$

and

$$v_x = \frac{-y}{x^2 + y^2} \text{ for } (x,y) \neq (0,0) \text{ and } v_y = \frac{x}{x^2 + y^2} \text{ for } (x,y) \notin \{(x,y) : x \leq 0, y = 0\},$$

we know that all z except real $z \leq 0$ make Cauchy-Riemann equations hold.

Remark: $Log\, z$ is differentiable on $C - \{(x,y) : x \leq 0, y = 0\}$ since Cauchy-Riemann equations along with continuity of $u_x + iv_x$, and $u_y + iv_y$.

(g) $e^{z^2} = u + iv$, where

$$u(x,y) = e^{x^2 - y^2}\cos(2xy) \text{ and } v(x,y) = e^{x^2 - y^2}\sin(2xy).$$

So,

$$u_x = v_y = 2e^{x^2 - y^2}[x(\cos 2xy) - y(\sin 2xy)] \text{ for all } z = x + iy.$$

and

$$u_y = -v_x = -2e^{x^2 - y^2}[y(\cos 2xy) + x(\sin 2xy)] \text{ for all } z = x + iy.$$

Hence, we know that all z make Cauchy-Riemann equations hold.

(h) Since $z^\alpha = e^{\alpha Log\, z}$, and e^z is differentiable on C, we know that, by the remark of (f), we know that z^α is differentiable for all z except real $z \leq 0$. So, we know that all z except real $z \leq 0$ make Cauchy-Riemann equations hold.

(In part (h), the Cauchy-Riemann equations hold for all z if α is a nonnegative integer, and they hold for all $z \neq 0$ if α is a negative integer.)

Solution: It is clear from definition of differentiability.

(iii) Compute the derivative $f'(z)$ in (a), (b), (f), (g), (h), assuming it exists.

Solution: Since $f'(z) = u_x + iv_x,$ if it exists. So, we know all results by (ii).

5.37 Write $f = u + iv$ and assume that f has a derivative at each point of an open disk D centered at $(0,0).$ If $au^2 + bv^2$ is constant on D for some real a and $b,$ not both 0. Prove that f is constant on $D.$

Proof: Let $au^2 + bv^2$ be constant on $D.$ We consider three cases as follows.

1. As $a = 0, b \neq 0,$ then we have

$$v^2 \text{ is constant on } D$$

which implies that

$$vv_x = 0.$$

If $v = 0$ on $D,$ it is clear that f is constant.
If $v \neq 0$ on $D,$ that is $v_x = 0$ on $D.$ So, we still have f is contant.

2. As $a \neq 0, b = 0,$ then it is similar. We omit it.
3. As $a \neq 0, b \neq 0,$ Taking partial derivatives we find

$$auu_x + bvv_x = 0 \text{ on } D. \quad 1$$

and

$$auu_y + bvv_y = 0 \text{ on } D.$$

By **Cauchy-Riemann equations** the second equation can be written as we have

$$-auv_x + bvu_x = 0 \text{ on } D. \quad 2$$

We consider $(1)(v_x) + (2)(u_x)$ and $(1)(u_x) + (2)(v_x),$ then we have

$$bv(v_x^2 + u_x^2) = 0 \quad 3$$

and

$$au(v_x^2 + u_x^2) = 0 \quad 4$$

which imply that

$$(au^2 + bv^2)(v_x^2 + u_x^2) = 0. \quad 5$$

If $au^2 + bv^2 = c,$ constant on $D,$ where $c \neq 0,$ then $v_x^2 + u_x^2 = 0.$ So, f is constant.
If $au^2 + bv^2 = c,$ constant on $D,$ where $c = 0,$ then if there exists (x,y) such that $v_x^2 + u_x^2 \neq 0,$ then by (3) and (4), $u(x,y) = v(x,y) = 0.$ By continuity of $v_x^2 + u_x^2,$ we know that there exists an open region $S(\subseteq D)$ such that $u = v = 0$ on $S.$ Hence, by **Uniqueness Theorem**, we know that f is constant.

Remark: In complex theory, the Uniqueness theorem is fundamental and important. The reader can see this from the book named **Complex Analysis by Joseph Bak and Donald J. Newman.**