Limits And Continuity

Limits of sequence

4.1 Prove each of the following statements about sequences in C.

(a) $z^n \to 0$ if $|z| < 1$; $\{z^n\}$ diverges if $|z| > 1$.

Proof: For the part: $z^n \to 0$ if $|z| < 1$. Given $\varepsilon > 0$, we want to find that there exists a positive integer N such that as $n \geq N$, we have

$$|z^n - 0| < \varepsilon.$$

Note that $\log|z| < 0$ since $|z| < 1$, hence if we choose a positive integer $N \geq \left[\log_{|z|}\varepsilon\right] + 1$, then as $n \geq N$, we have

$$|z^n - 0| < \varepsilon.$$

For the part: $\{z^n\}$ diverges if $|z| > 1$. Assume that $\{z^n\}$ converges to L, then given $\varepsilon = 1$, there exists a positive integer N_1 such that as $n \geq N_1$, we have

$$|z^n - L| < 1 (= \varepsilon)$$

$$\Rightarrow |z|^n < 1 + |L|. \quad *$$

However, note that $\log|z| > 0$ since $|z| > 1$, if we choose a positive integer $N \geq \max\left(\left[\log_{|z|} 1 + |L|\right] + 1, N_1\right)$, then we have

$$|z|^N > 1 + |L|$$

which contradicts (*). Hence, $\{z^n\}$ diverges if $|z| > 1$.

Remark: 1. Given any complex number $z \in C - \{0\}$, $\lim_{n\to\infty}|z|^{1/n} = 1$.

2. Keep $\lim_{n\to\infty}(n!)^{1/n} = \infty$ in mind.

3. In fact, $\{z^n\}$ is unbounded if $|z| > 1$. ($\Rightarrow$ $\{z^n\}$ diverges if $|z| > 1$.) Since given $M > 1$, and choose a positive integer $N = \left[\log_{|z|}M\right] + 1$, then $|z|^N \geq M$.

(b) If $z_n \to 0$ and if $\{c_n\}$ is bounded, then $\{c_n z_n\} \to 0$.

Proof: Since $\{c_n\}$ is bounded, say its bound M, i.e., $|c_n| \leq M$ for all $n \in N$. In addition, since $z_n \to 0$, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|z_n - 0| < \varepsilon/M$$

which implies that as $n \geq N$, we have

$$|c_n z_n| \leq M|z_n| < \varepsilon.$$

That is, $\lim_{n\to\infty} c_n z_n = 0$.

(c) $z^n/n! \to 0$ for every complex z.

Proof: Given a complex z, and thus find a positive integer N such that $|z| \leq N/2$. Consider (let $n > N$).

$$\left|\frac{z^n}{n!}\right| = \left|\left(\frac{z^N}{N!}\right)\left(\frac{z^{n-N}}{(N+1)(N+2)\cdots n}\right)\right| \leq \left|\frac{z^N}{N!}\right|\left(\frac{1}{2}\right)^{n-N} \to 0 \text{ as } n \to \infty.$$

Hence, $z^n/n! \to 0$ for every complex z.

Remark: There is another proof by using the fact $\sum_{n=1}^{\infty} a_n$ converges which implies $a_n \to 0$. Since $\sum_{n=1}^{\infty} \frac{z^n}{n!}$ converges by **ratio test** for every complex z, then we have

$z^n/n! \to 0$ for every complex z.

(d) If $a_n = \sqrt{n^2+2} - n$, then $a_n \to 0$.

Proof: Since

$$0 < a_n = \sqrt{n^2+2} - n = \frac{2}{\sqrt{n^2+2}+n} \le \frac{1}{n} \text{ for all } n \in N,$$

and $\lim_{n\to\infty} 1/n = 0$, we have $a_n \to 0$ as $n \to \infty$ by **Sandwich Theorem.**

4.2 If $a_{n+2} = (a_{n+1} + a_n)/2$ for all $n \ge 1$, show that $a_n \to (a_1 + 2a_2)/3$. Hint: $a_{n+2} - a_{n+1} = \frac{1}{2}(a_n - a_{n+1})$.

Proof: Since $a_{n+2} = (a_{n+1} + a_n)/2$ for all $n \ge 1$, we have $b_{n+1} = -b_n/2$, where $b_n = a_{n+1} - a_n$. So, we have

$$b_{n+1} = \left(\frac{-1}{2}\right)^n b_1 \to 0 \text{ as } n \to \infty. \qquad *$$

Consider

$$a_{n+2} - a_2 = \sum_{k=2}^{n+1} b_k = \frac{-1}{2}\sum_{k=1}^{n} b_k = \left(\frac{-1}{2}\right)(a_{n+1} - a_1)$$

which implies that

$$b_n + \left(\frac{3a_{n+1}}{2}\right) = \frac{a_1 + 2a_2}{2}.$$

So we have

$$a_n \to (a_1 + 2a_2)/3 \text{ by } (*).$$

4.3 If $0 < x_1 < 1$ and if $x_{n+1} = 1 - \sqrt{1 - x_n}$ for all $n \ge 1$, prove that $\{x_n\}$ is a decreasing sequence with limit 0. Prove also that $x_{n+1}/x_n \to \frac{1}{2}$.

Proof: Claim that $0 < x_n < 1$ for all $n \in N$. We prove the claim by **Mathematical Induction.** As $n = 1$, there is nothing to prove. Suppose that $n = k$ holds, i.e., $0 < x_k < 1$, then as $n = k+1$, we have

$$0 < x_{k+1} = 1 - \sqrt{1 - x_k} < 1 \text{ by induction hypothesis.}$$

So, by **Mathematical Induction,** we have proved the claim. Use the claim, and then we have

$$x_{n+1} - x_n = (1 - x_n) - \sqrt{1 - x_n} = \frac{x_n(x_n - 1)}{(1 - x_n)^2 + (1 - x_n)} < 0 \text{ since } 0 < x_n < 1.$$

So, we know that the sequence $\{x_n\}$ is a decreasing sequence. Since $0 < x_n < 1$ for all $n \in N$, by Completeness of R, (That is, **a monotonic sequence in R which is bounded is a convergent sequence.**) Hence, we have proved that $\{x_n\}$ is a convergent sequence, denoted its limit by x. Note that since

$$x_{n+1} = 1 - \sqrt{1 - x_n} \text{ for all } n \in N,$$

we have $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} 1 - \sqrt{1 - x_n} = 1 - \sqrt{1 - x}$ which implies $x(x-1) = 0$. Since $\{x_n\}$ is a decreasing sequence with $0 < x_n < 1$ for all $n \in N$, we finally have $x = 0$.

For proof of $x_{n+1}/x_n \to \frac{1}{2}$. Since

$$\lim_{x\to 0} \frac{1 - \sqrt{1 - x}}{x} = \frac{1}{2} \qquad *$$

then we have

$$\frac{x_{n+1}}{x_n} = \frac{1-\sqrt{1-x_n}}{x_n} \to \frac{1}{2}.$$

Remark: In (*), it is the derivative of $1-\sqrt{1-x}$ at the point $x = 0$. Of course, we can prove (*) by **L-Hospital Rule.**

4.4 Two sequences of positive integers $\{a_n\}$ and $\{b_n\}$ are defined recursively by taking $a_1 = b_1 = 1$ and equating rational and irrational parts in the equation

$$a_n + b_n\sqrt{2} = \left(a_{n-1} + b_{n-1}\sqrt{2}\right)^2 \text{ for } n \geq 2.$$

Prove that $a_n^2 - 2b_n^2 = 1$ for all $n \geq 2$. Deduce that $a_n/b_n \to \sqrt{2}$ through values $> \sqrt{2}$, and that $2b_n/a_n \to \sqrt{2}$ through values $< \sqrt{2}$.

Proof: Note $a_n + b_n\sqrt{2} = \left(a_{n-1} + b_{n-1}\sqrt{2}\right)^2$ for $n \geq 2$, we have

$$a_n = a_{n-1}^2 + 2b_{n-1}^2 \text{ for } n \geq 2, \text{and} \qquad *$$

$$b_n = 2a_{n-1}b_{n-1} \text{ for } n \geq 2$$

since if $A, B, C,$ and $D \in N$ with $A + B\sqrt{2} = C + D\sqrt{2}$, then $A = C$, and $B = D$.

Claim that $a_n^2 - 2b_n^2 = 1$ for all $n \geq 2$. We prove the claim by **Mathematical Induction.** As $n = 2$, we have by (*)

$a_2^2 - 2b_2^2 = (a_1^2 + 2b_1^2)^2 - 2(2a_1b_1)^2 = (1+2)^2 - 2(2)^2 = 1$. Suppose that as $n = k (\geq 2)$ holds, i.e., $a_k^2 - 2b_k^2 = 1$, then as $n = k+1$, we have by (*)

$$\begin{aligned} a_{k+1}^2 - 2b_{k+1}^2 &= (a_k^2 + 2b_k^2)^2 - 2(2a_kb_k)^2 \\ &= a_k^4 + 4b_k^4 - 4a_k^2b_k^2 \\ &= (a_k^2 - 2b_k^2)^2 \\ &= 1 \text{ by induction hypothesis.} \end{aligned}$$

Hence, by **Mathematical Induction,** we have proved the claim. Note that $a_n^2 - 2b_n^2 = 1$ for all $n \geq 2$, we have

$$\left(\frac{a_n}{b_n}\right)^2 = \left(\frac{1}{b_n}\right)^2 + 2 > 2$$

and

$$\left(\frac{2b_n}{a_n}\right)^2 = 2 - \frac{2}{a_n^2} < 2.$$

Hence, $\lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{2}$ by $\lim_{n\to\infty} \frac{1}{b_n} = 0$ from (*) through values $> \sqrt{2}$, and $\lim_{n\to\infty} \frac{2b_n}{a_n} = \sqrt{2}$ by $\lim_{n\to\infty} \frac{1}{a_n} = 0$ from (*) through values $< \sqrt{2}$.

Remark: From (*), we know that $\{a_n\}$ and $\{b_n\}$ is increasing since $\{a_n\} \subseteq N$ and $\{b_n\} \subseteq N$. That is, we have $\lim_{n\to\infty} a_n = \infty$, and $\lim_{n\to\infty} b_n = \infty$.

4.5 A real sequence $\{x_n\}$ satisfies $7x_{n+1} = x_n^3 + 6$ for $n \geq 1$. If $x_1 = \frac{1}{2}$, prove that the sequence increases and find its limit. What happens if $x_1 = \frac{3}{2}$ or if $x_1 = \frac{5}{2}$?

Proof: Claim that if $x_1 = \frac{1}{2}$, then $0 < x_n < 1$ for all $n \in N$. We prove the claim by **Mathematical Induction**. As $n = 1$, $0 < x_1 = \frac{1}{2} < 1$. Suppose that $n = k$ holds, i.e., $0 < x_k < 1$, then as $n = k+1$, we have

$$0 < x_{k+1} = \frac{x_k^3 + 6}{7} < \frac{1+6}{7} = 1 \text{ by induction hypothesis.}$$

Hence, we have prove the claim by **Mathematical Induction**. Since $x^3 - 7x + 6 = (x+3)(x-1)(x-2)$, then

$$\begin{aligned} x_{n+1} - x_n &= \frac{x_n^3 + 6}{7} - x_n \\ &= \frac{x_n^3 - 7x_n + 6}{7} \\ &> 0 \text{ since } 0 < x_n < 1 \text{ for all } n \in N. \end{aligned}$$

It means that the sequence $\{x_n\}$ (strictly) increasing. Since $\{x_n\}$ is bounded, by completeness of R, we know that he sequence $\{x_n\}$ is convergent, denote its limit by x. Since

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n^3 + 6}{7} = \frac{x^3 + 6}{7},$$

we find that $x = -3, 1,$ or 2. Since $0 < x_n < 1$ for all $n \in N$, we finally have $x = 1$.

Claim that if $x_1 = \frac{3}{2}$, then $1 < x_n < 2$ for all $n \in N$. We prove the claim by **Mathematical Induction**. As $n = 1$, there is nothing to prove. Suppose $n = k$ holds, i.e., $1 < x_k < 2$, then as $n = k + 1$, we have

$$1 = \frac{1 + 6}{7} < x_{k+1} = \frac{x_k^3 + 6}{7} < \frac{2^3 + 6}{7} = 2.$$

Hence, we have prove the claim by **Mathematical Induction**. Since $x^3 - 7x + 6 = (x + 3)(x - 1)(x - 2)$, then

$$\begin{aligned} x_{n+1} - x_n &= \frac{x_n^3 + 6}{7} - x_n \\ &= \frac{x_n^3 - 7x_n + 6}{7} \\ &< 0 \text{ since } 1 < x_n < 2 \text{ for all } n \in N. \end{aligned}$$

It means that the sequence $\{x_n\}$ (strictly) decreasing. Since $\{x_n\}$ is bounded, by completeness of R, we know that he sequence $\{x_n\}$ is convergent, denote its limit by x. Since

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n^3 + 6}{7} = \frac{x^3 + 6}{7},$$

we find that $x = -3, 1,$ or 2. Since $1 < x_n < 2$ for all $n \in N$, we finally have $x = 1$.

Claim that if $x_1 = \frac{5}{2}$, then $x_n > \frac{5}{2}$ for all $n \in N$. We prove the claim by **Mathematical Induction**. As $n = 1$, there is nothing to prove. Suppose $n = k$ holds, i.e., $x_k > \frac{5}{2}$, then as $n = k + 1$,

$$x_{k+1} = \frac{x_k^3 + 6}{7} > \frac{\left(\frac{5}{2}\right)^3 + 6}{7} = \frac{173}{56} > 3 > \frac{5}{2}.$$

Hence, we have proved the claim by **Mathematical Induction**. If $\{x_n\}$ was convergent, say its limit x. Then the possibilities for $x = -3, 1,$ or 2. However, $x_n > \frac{5}{2}$ for all $n \in N$. So, $\{x_n\}$ diverges if $x_1 = \frac{5}{2}$.

Remark: Note that in the case $x_1 = 5/2$, we can show that $\{x_n\}$ is increasing by the same method. So, it implies that $\{x_n\}$ is unbounded.

4.6 If $|a_n| < 2$ and $|a_{n+2} - a_{n+1}| \le \frac{1}{8}|a_{n+1}^2 - a_n^2|$ for all $n \ge 1$, prove that $\{a_n\}$ converges.

Proof: Let $a_{n+1} - a_n = b_n$, then we have $|b_{n+1}| \le \frac{1}{8}|b_n||a_{n+1} + a_n| \le \frac{1}{2}|b_n|$, since $|a_n| < 2$ for all $n \ge 1$. So, we have $|b_{n+1}| \le \left(\frac{1}{2}\right)^n |b_1|$. Consider (Let $m > n$)

$$\begin{aligned}|a_m - a_n| &= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \ldots + (a_{n+1} - a_n)| \\ &\leq |b_{m-1}| + \ldots + |b_n| \\ &\leq |b_1| \left[\left(\frac{1}{2}\right)^{m-2} + \ldots + \left(\frac{1}{2}\right)^{n-1} \right],\end{aligned} \quad *$$

then $\{a_n\}$ is a Cauchy sequence since $\sum(\frac{1}{2})^k$ converges. Hence, we know that $\{a_n\}$ is a convergent sequence.

Remark: In this exercise, we use the very important theorem, every Cauchy sequence in the Euclidean space R^k is convergent.

4.7 In a metric space (S,d), assume that $x_n \to x$ and $y_n \to y$. Prove that $d(x_n,y_n) \to d(x,y)$.

Proof: Since $x_n \to x$ and $y_n \to y$, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$d(x_n,x) < \varepsilon/2 \text{ and } d(y_n,y) < \varepsilon/2.$$

Hence, as $n \geq N$, we have

$$\begin{aligned}|d(x_n,y_n) - d(x,y)| &\leq |d(x_n,x) + d(y_n,y)| \\ &= d(x_n,x) + d(y_n,y) < \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon.\end{aligned}$$

So, it means that $d(x_n,y_n) \to d(x,y)$.

4.8 Prove that in a compact meric space (S,d), every sequence in S has a subsequence which converges in S. This property also implies that S is compact but you are not required to prove this. (For a proof see either Reference 4.2 or 4.3.)

Proof: Given a sequence $\{x_n\} \subseteq S$, and let $T = \{x_1,x_2,\ldots\}$. If the range of T is finite, there is nothing to prove. So, we assume that the range of T is infinite. Since S is compact, and $T \subseteq S$, we have T has a accumulation point x in S. So, there exists a point y_n in T such that $B(y_n,x) < \frac{1}{n}$. It implies that $y_n \to x$. Hence, we have proved that every sequence in S has a subsequence which converges in S.

Remark: If every sequence in S has a subsequence which converges in S, then S is compact. We give a proof as follows.

Proof: In order to show S is compact, it suffices to show that every infinite subset of S has an accumulation point in S. Given any infinite subset T of S, and thus we choose $\{x_n\} \subseteq T$ (of course in S). By hypothesis, $\{x_n\}$ has a subsequence $\{x_{k(n)}\}$ which converges in S, say its limit x. From definition of limit of a sequence, we know that x is an accumulation of T. So, S is compact.

4.9 Let A be a subset of a metric space S. If A is complete, prove that A is closed. Prove that the converse also holds if S is complete.

Proof: Let x be an accumulation point of A, then there exists a sequence $\{x_n\}$ such that $x_n \to x$. Since $\{x_n\}$ is convergent, we know that $\{x_n\}$ is a Cauchy sequence. And A is complete, we have $\{x_n\}$ converges to a point $y \in A$. By uniqueness, we know $x = y \in A$. So, A contains its all accumulation points. That is, A is closed.

Suppose that S is complete and A is closed in S. Given any Cauchy sequence $\{x_n\} \subseteq A$, we want to show $\{x_n\}$ is converges to a point in A. Trivially, $\{x_n\}$ is also a Cauchy sequence in S. Since S is complete, we know that $\{x_n\}$ is convergent to a point x in S. By definition of limit of a sequence, it is easy to know that x is an adherent point of A. So, $x \in A$ since A is closed. That is, every Cauchy sequence in A is convergent. So, A is complete.

Supplement

1. Show that the sequence

$$\lim_{n\to+\infty}\frac{(2n)!!}{(2n+1)!!} = 0.$$

Proof: Let a and b be positive integers satisfying $a \geq b > 1.$ Then we have

$$a!b \leq a!b! \leq (a+b)! \leq (ab)!. \qquad *$$

So, if we let $f(n) = (2n)!,$ then we have, by (*)

$$\frac{(2n)!!}{(2n+1)!!} = \frac{f(n)!}{(f(n)(2n+1))!} \leq \frac{1}{2n+1} \to 0.$$

Hence, we know that $\lim_{n\to+\infty}\frac{(2n)!!}{(2n+1)!!} = 0.$

2. Show that

$$a_n = \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{[\sqrt{n}\,]}{n}\right) \to e^{1/2} \text{ as } n\to\infty,$$

where $[x]$ means **Gauss Symbol.**

Proof: Since

$$x - \frac{1}{2}x^2 \leq \log(1+x) \leq x, \text{ for all } x \in (-1,1)$$

we have

$$\sum_{k=1}^{k=[\sqrt{n}\,]}\frac{k}{n} - \frac{1}{2}\left(\frac{k}{n}\right)^2 \leq \log a_n = \sum_{k=1}^{k=[\sqrt{n}\,]}\log\left(1+\frac{k}{n}\right) \leq \sum_{k=1}^{k=[\sqrt{n}\,]}\frac{k}{n}$$

Consider $i^2 < n \leq (i+1)^2,$ then by Sandwish Theorem, we know that

$$\lim_{n\to\infty}\log a_n = 1/2$$

which implies that $a_n \to e^{1/2}$ as $n\to\infty.$

3. Show that $(n!)^{1/n} \geq \sqrt{n}$ for all $n \in N.$ ($\Rightarrow (n!)^{1/n} \to \infty$ as $n \to \infty.$)

Proof: We prove it by a special method following Gauss' method. Consider

$$n! = 1\cdots\cdots\cdots k\cdots\cdots\cdots n$$
$$= n\cdots(n-k+1)\cdots\cdots 1$$

and thus let $f(k) := k(n-k+1),$ it is easy to show that $f(k) \geq f(1) = n$ for all $k = 1,2,\ldots,n.$ So, we have prove that

$$(n!)^2 \geq n^n$$

which implies that

$$(n!)^{1/n} \geq \sqrt{n}.$$

Remark: There are many and many method to show $(n!)^{1/n} \to \infty$ as $n\to\infty.$ We do not give a detail proofs about it. But We method it as follows as references.

(a) By $A.P. \geq G.P.,$ we have

$$\frac{\sum_{k=1}^{n}\frac{1}{k}}{n} \geq \left(\frac{1}{n!}\right)^{1/n}$$

and use the fact if $\{a_n\}$ converges to a, then so is $\left\{\frac{\sum_{k=1}^{n} a_k}{n}\right\}$.

(b) Use the fact, by **Mathematical Induction**, $(n!)^{1/n} \geq n/3$ for all n.

(c) Use the fact, $A^n/n! \to 0$ as $n \to \infty$ for any real A.

(d) Consider $p(n) = \left(\frac{n!}{n^n}\right)^{1/n}$, and thus taking $\log p(n)$.

(e) Use the famuos formula, a_n are positive for all n.

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf (a_n)^{1/n} \leq \limsup (a_n)^{1/n} \leq \limsup \frac{a_{n+1}}{a_n}$$

and let $a_n = \left(\frac{n!}{n^n}\right)$.

(f) The radius of the power series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is ∞.

(g) Ue the fact, $(1+1/n)^n \leq e \leq (1+1/n)^{n+1}$, then $e(n^n e^{-n}) \leq n! \leq e(n^{n+1} e^{-n})$.

(h) More.

Limits of functions

Note. In Exercise 4.10 through 4.28, all functions are real valued.

4.10 Let f be defined on an opne interval (a,b) and assume $x \in (a,b)$. Consider the two statements

(a) $\lim_{h\to 0} |f(x+h) - f(x)| = 0$;

(b) $\lim_{h\to 0} |f(x+h) - f(x-h)| = 0$.

Prove that (a) always implies (b), and give an example in which (b) holds but (a) does not.

Proof: (a) Since

$$\lim_{h\to 0} |f(x+h) - f(x)| = 0 \Leftrightarrow \lim_{h\to 0} |f(x-h) - f(x)| = 0,$$

we consider

$$\begin{aligned} & |f(x+h) - f(x-h)| \\ &= |(f(x+h) - f(x)) + (f(x) - f(x-h))| \\ &\leq |f(x+h) - f(x)| + |f(x) - f(x-h)| \to 0 \text{ as } h \to 0. \end{aligned}$$

So, we have

$$\lim_{h\to 0} |f(x+h) - f(x-h)| = 0.$$

(b) Let

$$f(x) = \begin{cases} |x| \text{ if } x \neq 0, \\ 1 \text{ if } x = 0. \end{cases}$$

Then

$$\lim_{h\to 0} |f(0+h) - f(0-h)| = 0,$$

but

$$\lim_{h\to 0} |f(0+h) - f(0)| = \lim_{h\to 0} ||h| - 1| = 1.$$

So, (b) holds but (a) does not.

Remark: In case (b), there is another example,

$$g(x) = \begin{cases} 1/|x| \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

The difference of two examples is that the limit of $|g(0+h) - g(0)|$ does not exist as h tends to 0.

4.11 Let f be defined on R^2. If

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

and if the one-dimensional $\lim_{x\to a} f(x,y)$ and $\lim_{y\to b} f(x,y)$ both exist, prove that

$$\lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right] = \lim_{y\to b}\left[\lim_{x\to a} f(x,y)\right] = L.$$

Proof: Since $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $0 < |(x,y) - (a,b)| < \delta$, we have

$$|f(x,y) - L| < \varepsilon/2,$$

which implies

$$\lim_{y\to b}|f(x,y) - L| = \left|\lim_{y\to b} f(x,y) - L\right| \leq \varepsilon/2 \text{ if } 0 < |(x,y) - (a,b)| < \delta$$

which implies

$$\lim_{x\to a}\left|\lim_{y\to b} f(x,y) - L\right| \leq \varepsilon/2 \text{ if } 0 < |(x,y) - (a,b)| < \delta.$$

Hence, we have proved $\lim_{x\to a}|\lim_{y\to b}(x,y) - L| \leq \varepsilon/2 < \varepsilon$. Since ε is arbitrary, we have

$$\lim_{x\to a}\left|\lim_{y\to b} f(x,y) - L\right| = 0$$

which implies that

$$\left|\lim_{x\to a}\lim_{y\to b} f(x,y) - L\right| = 0.$$

So, $\lim_{x\to a}[\lim_{y\to b} f(x,y)] = L$. The proof of $\lim_{y\to b}[\lim_{x\to a} f(x,y)] = L$ is similar.

Remark: 1. The exercise is much important since in mathematics, we would encounter many and many similar questions about the interchange of the order of limits. So, we should keep the exercise in mind.

2. In the proof, we use the concept: $|\lim_{x\to a} f(x)| = 0$ if, and only if $\lim_{x\to a} f(x) = 0$.

3. The hypothesis $f(x,y) \to L$ as $(x,y) \to (a,b)$ tells us that every approach form these points (x,y) to the point (a,b), $f(x,y)$ approaches to L. Use this concept, and consider the special approach from points (x,y) to (x,b) and thus from (x,b) to (a,b). Note that since $\lim_{y\to b} f(x,y)$ exists, it means that we can regrad this special approach as one of approaches from these points (x,y) to the point (a,b). So, it is natural to have the statement.

4. The converse of statement is not necessarily true. For example,

$$f(x,y) = \begin{cases} x + y \text{ if } x = 0 \text{ or } y = 0 \\ 1 \quad \text{otherwise.} \end{cases}$$

Trivially, we have the limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$. However,

$$\lim_{y\to 0} f(x,y) = \begin{cases} 0 \text{ if } x = 0, \\ 1 \text{ if } x \neq 0. \end{cases} \quad \text{and } \lim_{x\to 0} f(x,y) = \begin{cases} 0 \text{ if } y = 0, \\ 1 \text{ if } y \neq 0. \end{cases}$$

$$\lim_{x\to 0}\Big[\lim_{y\to 0} f(x,y)\Big] = \lim_{y\to 0}\Big[\lim_{x\to 0} f(x,y)\Big] = 1.$$

In each of the preceding examples, determine whether the following limits exist and evaluate those limits that do exist:

$$\lim_{x\to 0}\Big[\lim_{y\to 0} f(x,y)\Big]; \ \lim_{y\to 0}\Big[\lim_{x\to 0} f(x,y)\Big]; \ \lim_{(x,y)\to(0,0)} f(x,y).$$

Now consider the functions f defined on R^2 as follows:

(a) $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$, $f(0,0) = 0$.

Proof: 1. Since$(x \neq 0)$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2} = \begin{cases} \frac{-y^2}{y^2} = -1 \text{ if } y \neq 0, \\ 1 \text{ if } y = 0, \end{cases}$$

we have

$$\lim_{y\to 0}\Big[\lim_{x\to 0} f(x,y)\Big] = -1.$$

2. Since $(y \neq 0)$

$$\lim_{y\to 0} f(x,y) = \lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2} = \begin{cases} \frac{x^2}{x^2} = 1 \text{ if } x \neq 0, \\ 1 \text{ if } x = 0, \end{cases} \qquad **$$

we have

$$\lim_{x\to 0}\Big[\lim_{y\to 0} f(x,y)\Big] = 1.$$

3. $((x,y) \neq (0,0))$ Let $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \leq \theta < 2\pi$, and note that $(x,y) \to (0,0) \Leftrightarrow r \to 0$. Then

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \\ &= \lim_{r\to 0} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2} \\ &= \cos^2\theta - \sin^2\theta. \end{aligned}$$

So, if we choose $\theta = \pi$ and $\theta = \pi/2$, we find the limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$.

Remark: 1. This case shows that

$$1 = \lim_{x\to 0}\Big[\lim_{y\to 0} f(x,y)\Big] \neq \lim_{y\to 0}\Big[\lim_{x\to 0} f(x,y)\Big] = -1$$

2. Obviously, the limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$. Since if it was, then by (*), (**), and the preceding theorem, we know that

$$\lim_{x\to 0}\Big[\lim_{y\to 0} f(x,y)\Big] = \lim_{y\to 0}\Big[\lim_{x\to 0} f(x,y)\Big]$$

which is absurb.

(b) $f(x,y) = \frac{(xy)^2}{(xy)^2+(x-y)^2}$ if $(x,y) \neq (0,0)$, $f(0,0) = 0$.

Proof: 1. Since $(x \neq 0)$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{(xy)^2}{(xy)^2+(x-y)^2} = 0 \text{ for all } y,$$

we have

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 0.$$

2. Since $(y \neq 0)$

$$\lim_{y\to 0} f(x,y) = \lim_{y\to 0} \frac{(xy)^2}{(xy)^2+(x-y)^2} = 0 \text{ for all } x,$$

we have

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 0.$$

3. $((x,y) \neq (0,0))$ Let $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \leq \theta < 2\pi$, and note that $(x,y) \to (0,0) \Leftrightarrow r \to 0$. Then

$$\begin{aligned} f(x,y) &= \frac{(xy)^2}{(xy)^2+(x-y)^2} \\ &= \frac{r^4\cos^2\theta\sin^2\theta}{r^4\cos^2\theta\sin^2\theta + r^2 - 2r^2\cos\theta\sin\theta} \\ &= \frac{\cos^2\theta\sin^2\theta}{\cos^2\theta\sin^2\theta + \frac{1-2\cos\theta\sin\theta}{r^2}}. \end{aligned}$$

So,

$$f(x,y)\begin{cases} \to 0 \text{ if } r \to 0 \\ 1 \quad \text{ if } \theta = \pi/4 \text{ or } \theta = 5\pi/4. \end{cases}$$

Hence, we know that the limit of $f(x,y)$ does not exists as $(x,y) \to (0,0)$.

(c) $f(x,y) = \frac{1}{x}\sin(xy)$ if $x \neq 0$, $f(0,y) = y$.

Proof: 1. Since $(x \neq 0)$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{1}{x}\sin(xy) = y \qquad *$$

we have

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 0.$$

2. Since $(y \neq 0)$

$$\lim_{y\to 0} f(x,y) = \begin{cases} \lim_{y\to 0}\frac{1}{x}\sin(xy) = 0 \text{ if } x \neq 0, \\ \lim_{y\to 0} y = 0 \text{ if } x = 0, \end{cases}$$

we have

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 0.$$

3. $((x,y) \neq (0,0))$ Let $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \leq \theta < 2\pi$, and note that $(x,y) \to (0,0) \Leftrightarrow r \to 0$. Then

$$f(x,y) = \begin{cases} \frac{1}{r\cos\theta}\sin(r^2\cos\theta\sin\theta) & \text{if } x = r\cos\theta \neq 0, \\ r\sin\theta & \text{if } x = r\cos\theta = 0. \end{cases}$$

$$\to \begin{cases} 0 \text{ if } r \to 0, \\ 0 \text{ if } r \to 0. \end{cases} \qquad **$$

So, we know that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Remark: In (*) and (**), we use the famuos limit, that is,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

There are some similar limits, we write them without proofs.

(a) $\lim_{t\to\infty} t\sin(1/t) = 1$.
(b) $\lim_{x\to 0} x\sin(1/x) = 0$.
(c) $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$, if $b \neq 0$.

(d) $f(x,y) = \begin{cases} (x+y)\sin(1/x)\sin(1/y) & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$

Proof: 1. Since $(x \neq 0)$

$$f(x,y) = \begin{cases} (x+y)\sin(1/x)\sin(1/y) = x\sin(1/x)\sin(1/y) + y\sin(1/x)\sin(1/y) \text{ if } y \neq 0 \\ 0 \text{ if } y = 0 \end{cases}$$

we have if $y \neq 0$, the limit $f(x,y)$ does not exist as $x \to 0$, and if $y = 0$, $\lim_{x\to 0} f(x,y) = 0$. Hence, we have $(x \neq 0,\ y \neq 0)$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] \text{ does not exist.}$$

2. Since $(y \neq 0)$

$$f(x,y) = \begin{cases} (x+y)\sin(1/x)\sin(1/y) = x\sin(1/x)\sin(1/y) + y\sin(1/x)\sin(1/y) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

we have if $x \neq 0$, the limit $f(x,y)$ does not exist as $y \to 0$, and if $x = 0$, $\lim_{y\to 0} f(x,y) = 0$. Hence, we have $(x \neq 0,\ y \neq 0)$

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] \text{ does not exist.}$$

3. $((x,y) \neq (0,0))$ Consider

$$|f(x,y)| \leq \begin{cases} |x+y| \text{ if } x \neq 0 \text{ and } y \neq 0, \\ 0 \text{ if } x = 0 \text{ or } y = 0. \end{cases}$$

we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

(e) $f(x,y) = \begin{cases} \frac{\sin x - \sin y}{\tan x - \tan y}, & \text{if } \tan x \neq \tan y, \\ \cos^3 x & \text{if } \tan x = \tan y. \end{cases}$

Proof: Since we consider the three approaches whose tend to $(0,0)$, we may assume that $x,y \in (-\pi/2, \pi/2)$. and note that in this assumption, $x = y \Leftrightarrow \tan x = \tan y$. Consider

1. $(x \neq 0)$

$$\lim_{x\to 0} f(x,y) = \begin{cases} \lim_{x\to 0} \frac{\sin x - \sin y}{\tan x - \tan y} = \cos y \text{ if } x \neq y. \\ 1 \qquad\qquad\qquad\qquad\quad \text{if } x = y. \end{cases}$$

So,

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 1.$$

2. ($y \neq 0$)

$$\lim_{y\to 0} f(x,y) = \begin{cases} \lim_{y\to 0} \frac{\sin x - \sin y}{\tan x - \tan y} = \cos x \text{ if } x \neq y. \\ \cos^3 x \qquad\qquad\qquad\quad \text{if } x = y. \end{cases}$$

So,

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 1.$$

3. Let $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \leq \theta < 2\pi$, and note that $(x,y) \to (0,0) \Leftrightarrow r \to 0$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \begin{cases} \lim_{r\to 0} \frac{\sin(r\cos\theta) - \sin(r\sin\theta)}{\tan(r\cos\theta) - \tan(r\sin\theta)} \text{ if } \cos\theta \neq \sin\theta. \\ \lim_{r\to 0} \cos^3(r\cos\theta) \text{ if } \cos\theta = \sin\theta. \end{cases}$$

$$= \begin{cases} 1 \text{ if } \cos\theta \neq \sin\theta, \text{ by } \textbf{L-Hospital Rule.} \\ 1 \text{ if } \cos\theta = \sin\theta. \end{cases}$$

So, we know that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

Remark: 1. There is another proof about (e)-(3). Consider

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

and

$$\tan x - \tan y = \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y},$$

then

$$\frac{\sin x - \sin y}{\tan x - \tan y} = \frac{\cos\left(\frac{x+y}{2}\right)\cos x \cos y}{\cos\left(\frac{x-y}{2}\right)}.$$

So,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \begin{cases} \lim_{(x,y)\to(0,0)} \frac{\cos\left(\frac{x+y}{2}\right)\cos x \cos y}{\cos\left(\frac{x-y}{2}\right)} = 1 \text{ if } x \neq y, \\ \lim_{(x,y)\to(0,0)} \cos^3 x = 1 \text{ if } x = y. \end{cases}$$

That is, $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

2. In the process of proof, we use the concept that we write it as follows. Since its proof is easy, we omit it. If

$$\lim_{(x,y)\to(a,b)} f(x,y) = \begin{cases} L \text{ if } x = y \\ L \text{ if } x \neq y \end{cases}$$

or

$$\lim_{(x,y)\to(a,b)} f(x,y) = \begin{cases} L \text{ if } x \neq 0 \text{ and } y \neq 0, \\ L \text{ if } x = 0 \text{ or } y = 0. \end{cases}$$

then we have

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

4.12 If $x \in [0,1]$ prove that the following limit exists,

$$\lim_{m\to\infty}\Big[\lim_{n\to\infty} \cos^{2n}(m!\pi x)\Big],$$

and that its value is 0 or 1, according to whether x is irrational or rational.

Proof: If x is rational, say $x = q/p$, where $g.c.d.(q,p) = 1$, then $p!x \in N$. So,

$$\lim_{n\to\infty} \cos^{2n}(m!\pi x) = \begin{cases} 1 \text{ if } m \geq p, \\ 0 \text{ if } m < p. \end{cases}$$

Hence,

$$\lim_{m\to\infty}\Big[\lim_{n\to\infty} \cos^{2n}(m!\pi x)\Big] = 1.$$

If x is irrational, then $m!x \notin N$ for all $m \in N$. So, $\cos^{2n}(m!\pi x) < 1$ for all irrational x. Hence,

$$\lim_{n\to\infty} \cos^{2n}(m!\pi x) = 0 \Rightarrow \lim_{m\to\infty}\Big[\lim_{n\to\infty} \cos^{2n}(m!\pi x)\Big] = 0.$$

Continuity of real-valued functions

4.13 Let f be continuous on $[a,b]$ and let $f(x) = 0$ when x is rational. Prove that $f(x) = 0$ for every $x \in [a,b]$.

Proof: Given any irrational number x in $[a,b]$, and thus choose a sequence $\{x_n\} \subseteq Q$ such that $x_n \to x$ as $n \to \infty$. Note that $f(x_n) = 0$ for all n. Hence,

$$\begin{aligned} 0 &= \lim_{n\to\infty} 0 \\ &= \lim_{n\to\infty} f(x_n) \\ &= f\Big(\lim_{n\to\infty} x_n\Big) \text{ by continuity of } f \text{ at } x \\ &= f(x). \end{aligned}$$

Since x is arbitrary, we have shown $f(x) = 0$ for all $x \in [a,b]$. That is, f is constant 0.

Remark: Here is another good exercise, we write it as a reference. Let f be continuous on R, and if $f(x) = f(x^2)$, then f is constant.

Proof: Since $f(-x) = f\big((-x)^2\big) = f(x^2) = f(x)$, we know that f is an even function. So, in order to show f is constant on R, it suffices to show that f is constant on $[0,\infty)$. Given any $x \in (0,\infty)$, since $f(x^2) = f(x)$ for all $x \in R$, we have $f(x^{1/2n}) = f(x)$ for all n. Hence,

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f(x) \\ &= \lim_{n\to\infty} f(x^{1/2n}) \\ &= f\Big(\lim_{n\to\infty} x^{1/2n}\Big) \text{ by continuity of } f \text{ at } 1 \\ &= f(1) \text{ since } x \neq 0. \end{aligned}$$

So, we have $f(x) = f(1) := c$ for all $x \in (0,\infty)$. In addition, given a sequence $\{x_n\} \subseteq (0,\infty)$ such that $x_n \to 0$, then we have

$$\begin{aligned} c &= \lim_{n\to\infty} c \\ &= \lim_{n\to\infty} f(x_n) \\ &= f\left(\lim_{n\to\infty} x_n\right) \text{ by continuity of } f \text{ at } 0 \\ &= f(0) \end{aligned}$$

From the preceding, we have proved that f is constant.

4.14 Let f be continuous at the point $a = (a_1, a_2, \ldots, a_n) \in R^n$. Keep $a_2, a_3, \ldots, a_n$ fixed and define a new function g of one real variable by the equation

$$g(x) = f(x, a_2, \ldots, a_n).$$

Prove that g is continuous at the point $x = a_1$. (This is sometimes stated as follows: **A continuous function of *n* variables is continuous in each variable separately**.)

Proof: Given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $y \in B(a;\delta) \cap D$, where D is a domain of f, we have

$$|f(y) - f(a)| < \varepsilon. \qquad *$$

So, as $|x - a_1| < \delta$, which implies $|(x, a_2, \ldots, a_n) - (a_1, a_2, \ldots, a_n)| < \delta$, we have

$$|g(x) - g(a_1)| = |f(x, a_2, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)| < \varepsilon.$$

Hence, we have proved g is continuous at $x = a_1$

Remark: Here is an important example like the exercise, we write it as follows. Let $\pi_j : R^n \to R^n$, and $\pi_j : (x_1, x_2, \ldots, x_n) = (0, .., x_j, .., 0)$. Then π_j is continuous on R^n for $1 \leq j \leq n$. Note that π_j is called a **projection**. Note that a projection P is sometimes defined as $P^2 = P$.

Proof: Given any point $a \in R^n$,and given $\varepsilon > 0$, and choose $\delta = \varepsilon$, then as $x \in B(a;\delta)$, we have

$$|\pi_j(x) - \pi_j(a)| = |x_j - a_j| \leq \|x - a\| < \delta = \varepsilon \text{ for each } 1 \leq j \leq n$$

Hence, we prove that $\pi_j(x)$ is continuous on R^n for $1 \leq j \leq n$.

4.15 Show by an example that the converse of statement in Exercise 4.14 is not true in general.

Proof: Let

$$f(x,y) = \begin{cases} x + y \text{ if } x = 0 \text{ or } y = 0 \\ 1 \text{ otherwise.} \end{cases}$$

Define $g_1(x) = f(x,0)$ and $g_2(y) = f(0,y)$, then we have

$$\lim_{x\to 0} g_1(x) = 0 = g_1(0)$$

and

$$\lim_{y\to 0} g_2(y) = 0 = g_2(0).$$

So, $g_1(x)$ and $g_2(y)$ are continuous at 0. However, f is not continuous at $(0,0)$ since

$$\lim_{x\to 0} f(x,x) = 1 \neq 0 = f(0,0).$$

Remark: 1. For continuity, if f is continuous at $x = a$, then it is **NOT** necessary for us to have

$$\lim_{x\to a} f(x) = f(a)$$

this is because a can be an isolated point. However, if a is an accumulation point, we then have

$$f \text{ is continuous at } a \text{ if, and only if, } \lim_{x \to a} f(x) = f(a).$$

4.16 Let f, g, and h be defined on $[0,1]$ as follows:

$f(x) = g(x) = h(x) = 0$, whenever x is irrational;

$f(x) = 1$ and $g(x) = x$, whenever x is rational;

$h(x) = 1/n$, if x is the rational number m/n (in lowest terms);

$h(0) = 1$.

Prove that f is not continuous anywhere in $[0,1]$, that g is continuous only at $x = 0$, and that h is continuous only at the irrational points in $[0,1]$.

Proof: 1. Write

$$f(x) = \begin{cases} 0 \text{ if } x \in (R - Q) \cap [0,1], \\ 1 \text{ if } x \in Q \cap [0,1]. \end{cases}$$

Given any $x \in (R - Q) \cap [0,1]$, and $y \in Q \cap [0,1]$, and thus choose $\{x_n\} \subseteq Q \cap [0,1]$ such that $x_n \to x$, and $\{y_n\} \subseteq (R - Q) \cap [0,1]$ such that $y_n \to y$. If f is continuous at x, then

$$\begin{aligned} 1 &= \lim_{n \to \infty} f(x_n) \\ &= f\left(\lim_{n \to \infty} x_n\right) \text{ by continuity of } f \text{ at } x \\ &= f(x) \\ &= 0 \end{aligned}$$

which is absurb. And if f is continuous at y, then

$$\begin{aligned} 0 &= \lim_{n \to \infty} f(y_n) \\ &= f\left(\lim_{n \to \infty} y_n\right) \text{ by continuity of } f \text{ at } y \\ &= f(y) \\ &= 1 \end{aligned}$$

which is absurb. Hence, f is not continuous on $[0,1]$.

2. Write

$$g(x) = \begin{cases} 0 \text{ if } x \in (R - Q) \cap [0,1], \\ x \text{ if } x \in Q \cap [0,1]. \end{cases}$$

Given any $x \in (R - Q) \cap [0,1]$, and choose $\{x_n\} \subseteq Q \cap [0,1]$ such that $x_n \to x$. Then

$$\begin{aligned} &x \\ &= \lim_{n \to \infty} x_n \\ &= \lim_{n \to \infty} g(x_n) \\ &= \lim g\left(\lim_{n \to \infty} x_n\right) \text{ by continuity of } g \text{ at } x \\ &= g(x) \\ &= 0 \end{aligned}$$

which is absurb since x is irrational. So, f is not continous on $(R - Q) \cap [0,1]$.

Given any $x \in Q \cap [0,1]$, and choose $\{x_n\} \subseteq (R - Q) \cap [0,1]$ such that $x_n \to x$. If g is continuous at x, then

$$\begin{aligned} & 0 \\ = & \lim_{n\to\infty} g(x_n) \\ = & g\left(\lim_{n\to\infty} x_n\right) \text{ by continuity of } f \text{ at } x \\ = & g(x) \\ = & x. \end{aligned}$$

So, the function g may be continuous at 0. In fact, g is continuous at 0 which prove as follows. Given $\varepsilon > 0$, choose $\delta = \varepsilon$, as $|x| < \delta$, we have $|g(x) - g(0)| = |g(x)| \leq |x| < \varepsilon (= \delta)$. So, g is continuous at 0. Hence, from the preceding, we know that g is continuous only at $x = 0$.

3. Write

$$h(x) = \begin{cases} 1 \text{ if } x = 0, \\ 0 \text{ if } x \in (R - Q) \cap [0,1], \\ 1/n \text{ if } x = m/n,\ g.c.d.(m,n) = 1. \end{cases}$$

Consider $a \in (0,1)$ and given $\varepsilon > 0$, there exists the largest positive integer N such that $N \leq 1/\varepsilon$. Let $T = \{x : h(x) \geq \varepsilon\}$, then

$$T = \begin{cases} \{0,1\} \cup \{x : h(x) = 1\} \cup \{x : h(x) = 1/2\} \ldots \cup \{x : h(x) = 1/N\} \text{ if } \varepsilon \leq 1, \\ \phi \text{ if } \varepsilon > 1. \end{cases}$$

Note that T is at most a finite set, and then we can choose a $\delta > 0$ such that $(a - \delta, a + \delta) - \{a\}$contains no points of T and $(a - \delta, a + \delta) \subseteq (0,1)$. So, if $x \in (a - \delta, a + \delta) - \{a\}$, we have $h(x) < \varepsilon$. It menas that

$$\lim_{x\to a} h(x) = 0.$$

Hence, we know that h is continuous at $x \in (0,1) \cap (R - Q)$. For two points $x = 1$, and $y = 0$, it is clear that h is not continuous at $x = 1$, and not continuous at $y = 1$ by the method mentioned in the exercise of part 1 and part 2. Hence, we have proved that h is continuous only at the irrational points in $[0,1]$.

Remark: 1. Sometimes we call f **Dirichlet function.**

2. Here is another proof about g, we write it down to make the reader get more.

Proof: Write

$$g(x) = \begin{cases} 0 \text{ if } x \in (R - Q) \cap [0,1], \\ x \text{ if } x \in Q \cap [0,1]. \end{cases}$$

Given $a \in (0,1]$, and if g is continuous at a, then given $0 < \varepsilon < a$, there exists a $\delta > 0$ such that as $x \in (a - \delta, a + \delta) \subseteq [0,1]$, we have

$$|g(x) - g(a)| < \varepsilon.$$

If $a \in R - Q$, choose $0 < \delta' < \delta$ so that $a + \delta' \in Q$. Then $a + \delta' \in (a - \delta, a + \delta)$ which implies $|g(a + \delta') - g(a)| = |g(a + \delta')| = a + \delta' < \varepsilon < a$. But it is impossible.

If $a \in Q$, choose $0 < \delta' < \delta$ so that $a + \delta' \in R - Q$. $a + \delta' \in (a - \delta, a + \delta)$ which implies $|g(a + \delta') - g(a)| = |-a| = a < \varepsilon < a$. But it is impossible.

If $a = 0$, given $\varepsilon > 0$ and choose $\delta = \varepsilon$, then as $0 \leq x < \delta$, we have $|g(x) - g(0)| = |g(x)| \leq |x| = x < \varepsilon (= \delta)$. It means that g is continuous at 0.

4.17 For each $x \in [0,1]$, let $f(x) = x$ if x is rational, and let $f(x) = 1 - x$ if x is irrational. Prove that:

(a) $f(f(x)) = x$ for all x in $[0,1]$.

Proof: If x is rational, then $f(f(x)) = f(x) = x$. And if x is irrarional, so is $1-x(\in [0,1])$. Then $f(f(x)) = f(1-x) = 1-(1-x) = x$. Hence, $f(f(x)) = x$ for all x in $[0,1]$.

(b) $f(x)+f(1-x) = 1$ for all x in $[0,1]$.

Proof: If x is rational, so is $1-x \in [0,1]$. Then $f(x)+f(1-x) = x+(1-x) = 1$. And if x is irrarional, so is $1-x(\in [0,1])$. Then $f(x)+f(1-x) = (1-x)+1-(1-x) = 1$. Hence, $f(x)+f(1-x) = 1$ for all x in $[0,1]$.

(c) f is continuous only at the point $x = \frac{1}{2}$.

Proof: If f is continuous at x, then choose $\{x_n\} \subseteq Q$ and $\{y_n\} \subseteq Q^c$ such that $x_n \to x$, and $y_n \to x$. Then we have, by continuity of f at x,

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x$$

and

$$f(x) = f\left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 1-y_n = 1-x.$$

So, $x = 1/2$ is the only possibility for f. Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $x \in (1/2-\delta, 1/2+\delta) \subseteq [0,1]$, we have

$$|f(x)-f(1/2)| = |f(x)-1/2| < \varepsilon.$$

Choose $(0 <)\delta < \varepsilon$ so that $(1/2-\delta, 1/2+\delta) \subseteq [0,1]$, then as $x \in (1/2-\delta, 1/2+\delta) \subseteq [0,1]$, we have

$$|f(x)-1/2| = |x-1/2| < \delta < \varepsilon \text{ if } x \in Q,$$

$$|f(x)-1/2| = |(1-x)-1/2| = |1/2-x| < \delta < \varepsilon \text{ if } x \in Q^c.$$

Hence, we have proved that f is continuous at $x = 1/2$.

(d) f assumes every value between 0 and 1.

Proof: Given $a \in [0,1]$, we want to find $x \in [0,1]$ such that $f(x) = a$. If $a \in Q$, then choose $x = a$, we have $f(x = a) = a$. If $a \in R-Q$, then choose $x = 1-a(\in R-Q)$, we have $f(x = 1-a) = 1-(1-a) = a$.

Remark: The range of f on $[0,1]$ is $[0,1]$. In addition, f is an **one-to-one** mapping since if $f(x) = f(y)$, then $x = y$. (The proof is easy, just by definition of 1-1, so we omit it.)

(e) $f(x+y)-f(x)-f(y)$ is rational for all x and y in $[0,1]$.

Proof: We prove it by four steps.

1. If $x \in Q$ and $y \in Q$, then $x+y \in Q$. So,

$$f(x+y)-f(x)-f(y) = x+y-x-y = 0 \in Q.$$

2. If $x \in Q$ and $y \in Q^c$, then $x+y \in Q^c$. So,

$$f(x+y)-f(x)-f(y) = [1-(x+y)]-x-(1-y) = -2x \in Q.$$

3. If $x \in Q^c$ and $y \in Q$, then $x+y \in Q^c$. So,

$$f(x+y)-f(x)-f(y) = [1-(x+y)]-(1-x)-y = -2y \in Q.$$

4. If $x \in Q^c$ and $y \in Q^c$, then $x+y \in Q^c$ or $x+y \in Q$. So,

$$f(x+y) - f(x) - f(y) = \begin{cases} [1-(x+y)] - (1-x) - (1-y) = -1 \in Q \text{ if } x+y \in Q^c, \\ (x+y) - (1-x) - (1-y) = -2 \in Q \text{ if } x+y \in Q. \end{cases}$$

Remark: Here is an interesting question about functions. Let $f : R - \{0,1\} \to R$. If f satisfies that

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x,$$

then $f(x) = \frac{x^3 - x^2 - 1}{2x(x-1)}$.

Proof: Let $\phi(x) = \frac{x-1}{x}$, then we have $\phi^2(x) = \frac{-1}{x-1}$, and $\phi^3(x) = x$. So,

$$f(x) + f\left(\frac{x-1}{x}\right) = f(x) + f(\phi(x)) = 1 + x \qquad *$$

which implies that

$$f(\phi(x)) + f(\phi^2(x)) = 1 + \phi(x) \qquad **$$

and

$$f(\phi^2(x)) + f(\phi^3(x)) = f(\phi^2(x)) + f(x) = 1 + \phi^2(x). \qquad ***$$

So, by (*), (**), and (***), we finally have

$$\begin{aligned} f(x) &= \frac{1}{2}[1 + x - \phi(x) + \phi^2(x)] \\ &= \frac{x^3 - x^2 - 1}{2x(x-1)}. \end{aligned}$$

4.18 Let f be defined on R and assume that there exists at least one x_0 in R at which f is continuous. Suppose also that, for every x and y in R, f satisfies the equation

$$f(x+y) = f(x) + f(y).$$

Prove that there exists a constant a such that $f(x) = ax$ for all x.

Proof: Let f be defined on R and assume that there exists at least one x_0 in R at which f is continuous. Suppose also that, for every x and y in R, f satisfies the equation

$$f(x+y) = f(x) + f(y).$$

Prove that there exists a constant a such that $f(x) = ax$ for all x.

Proof: Suppose that f is continuous at x_0, and given any $r \in R$. Since $f(x+y) = f(x) + f(y)$ for all x, then

$$f(x) = f(y - x_0) + f(r), \text{ where } y = x - r + x_0.$$

Note that $y \to x_0 \Leftrightarrow x \to r$, then

$$\begin{aligned} \lim_{x \to r} f(x) &= \lim_{x \to r} f(y - x_0) + f(r) \\ &= \lim_{y \to x_0} f(y - x_0) + f(r) \\ &= f(r) \text{ since } f \text{ is continuous at } x_0. \end{aligned}$$

So, f is continuous at r. Since r is arbitrary, we have f is continuous on R. Define $f(1) = a$, and then since $f(x+y) = f(x) + f(y)$, we have

$$\begin{aligned} f(1) &= f\left(\frac{1}{m} + .. + \frac{1}{m}\right)_{m\text{-times}} \\ &= mf\left(\frac{1}{m}\right) \\ &\Rightarrow f\left(\frac{1}{m}\right) = \frac{f(1)}{m} \qquad * \end{aligned}$$

In addition, since $f(-1) = -f(1)$ by $f(0) = 0$, we have

$$f(-1) = f\left(\frac{1}{-m}+..+\frac{1}{-m}\right)_{m-\text{times}}$$
$$= mf\left(\frac{-1}{m}\right)$$
$$\Rightarrow f\left(\frac{1}{-m}\right) = \frac{f(1)}{-m} \qquad *'$$

Thus we have

$$f\left(\frac{n}{m}\right) = f(1/m+\ldots+1/m)_{n-\text{times}} = nf(1/m) = \frac{n}{m}f(1) \text{ by (*) and (*')} \qquad **$$

So, given any $x \in R$, and thus choose a sequence $\{x_n\} \subseteq Q$ with $x_n \to x$. Then

$$f(x) = f\left(\lim_{n\to\infty} x_n\right)$$
$$= \lim_{n\to\infty} f(x_n) \text{ by continuity of } f \text{ on } R$$
$$= \lim_{n\to\infty} x_n f(1) \text{ by (**)}$$
$$= xf(1)$$
$$= ax.$$

Remark: There is a similar statement. Suppose that $f(x+y) = f(x)f(y)$ for all real x and y.

(1) If f is differentiable and non-zero, prove that $f(x) = e^{cx}$, where c is a constant.

Proof: Note that $f(0) = 1$ since $f(x+y) = f(x)f(y)$ and f is non-zero. Since f is differentiable, we define $f'(0) = c$. Consider

$$\frac{f(x+h)-f(x)}{h} = f(x)\frac{f(h)-f(0)}{h} \to f(x)f'(0) = cf(x) \text{ as } h \to 0,$$

we have for every $x \in R$, $f'(x) = cf(x)$. Hence,

$$f(x) = Ae^{cx}.$$

Since $f(0) = 1$, we have $A = 1$. Hence, $f(x) = e^{cx}$, where c is a constant.

Note: (i) If for every $x \in R$, $f'(x) = cf(x)$, then $f(x) = Ae^{cx}$.

Proof: Since $f'(x) = cf(x)$ for every x, we have for every x,

$$[f'(x) - cf(x)]e^{-cx} = 0 \Rightarrow [e^{-cx}f(x)]' = 0.$$

We note that by **Elementary Calculus**, $e^{-cx}f(x)$ is a constant function A. So, $f(x) = Ae^{cx}$ for all real x.

(ii) Suppose that $f(x+y) = f(x)f(y)$ for all real x and y. If $f(x_0) > 0$ for some x_0, then $f(x) > 0$ for all x.

Proof: Suppose **NOT**, then $f(a) = 0$ for some a. However,

$$0 < f(x_0) = f(x_0 - a + a) = f(x_0 - a)f(a) = 0.$$

Hence, $f(x) > 0$ for all x.

(iii) Suppose that $f(x+y) = f(x)f(y)$ for all real x and y. If f is differentiable at x_0 for some x_0, then f is differentiable for all x. And thus, $f(x) \in C^\infty(R)$.

Proof: Since

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x_0+h+x-x_0)-f(x_0+x-x_0)}{h}$$
$$= f(x-x_0)\frac{f(x_0+h)-f(x_0)}{h} \to f(x-x_0)f'(x_0) \text{ as } h \to 0,$$

we have $f'(x)$ is differentiable and $f'(x) = f(x-x_0)f'(x_0)$ for all x. And thus we have $f(x) \in C^\infty(R)$.

(iv) Here is another proof by (iii) and **Taylor Theorem with Remainder term** $R_n(x)$.

Proof: Since f is differentiable, by (iii), we have $f^{(n)}(x) = (f'(0))^n f(x)$ for all x. Consider $x \in [-r,r]$, then by **Taylor Theorem with Remainder term** $R_n(x)$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + R_n(x), \text{ where } R_n(x) := \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1},\ \xi \in (0,x) \text{ or } \in (x,0),$$

Then

$$\begin{aligned}|R_n(x)| &= \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}\right| \\ &= \left|\frac{(f'(0))^{n+1}f(\xi)}{(n+1)!}x^{n+1}\right| \\ &\le \left|\frac{(f'(0)r)^{n+1}}{(n+1)!}\right|M, \text{ where } M = \max_{x\in[-r,r]}|f(x)| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence, we have for every $x \in [-r,r]$

$$\begin{aligned}f(x) &= \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \\ &= f(0)\left(\sum_{k=0}^{\infty}\frac{[f'(0)x]^k}{k!}\right) \\ &= e^{cx}, \text{ where } c := f'(0).\end{aligned}$$

Since r is arbitrary, we have proved that $f(x) = e^{cx}$ for all x.

(2) If f is continuous and non-zero, prove that $f(x) = e^{cx}$, where c is a constant.

Proof: Since $f(x+y) = f(x)f(y)$, we have

$$0 < f(1) = f\left(\tfrac{1}{n}+\ldots+\tfrac{1}{n}\right)_{n-\text{times}} = f\left(\tfrac{1}{n}\right)^n \Rightarrow f\left(\tfrac{1}{n}\right) = f(1)^{1/n} \qquad *$$

and (note that $f(-1) = f(1)^{-1}$ by $f(0) = 1$,)

$$0 < f(-1) = f\left(\tfrac{-1}{n}+\ldots+\tfrac{-1}{n}\right)_{n-\text{times}} = f\left(\tfrac{-1}{n}\right)^n \Rightarrow f\left(\tfrac{1}{-n}\right) = f(1)^{-1/n} \qquad *'$$

$$f\left(\tfrac{m}{n}\right) = f\left(\tfrac{1}{n}+\ldots+\tfrac{1}{n}\right)_{m-\text{times}} = f\left(\tfrac{1}{n}\right)^m = f(1)^{\frac{m}{n}} \text{ by (*) and (*')} \qquad **$$

So, given any $x \in R$, and thus choose a sequence $\{x_n\} \subseteq Q$ with $x_n \to x$. Then

$$\begin{aligned} f(x) &= f\left(\lim_{n\to\infty} x_n\right) \\ &= \lim_{n\to\infty} f(x_n) \text{ by continuity of } f \\ &= \lim_{n\to\infty} f(1)^{x_n} \text{ by (**)} \\ &= f(1)^x \\ &= e^{cx}, \text{ where } \log f(1) = c. \end{aligned}$$

Note: (i) We can prove (2) by the exercise as follows. Note that $f(x) > 0$ for all x by the remark (1)-(ii) Consider the composite function $g(x) = \log f(x)$, then $g(x+y) = \log f(x+y) = \log f(x)f(y) = \log f(x) + \log f(y) = g(x) + g(y)$. Since log and f are continuous on R, its composite function g is continuous on R. Use the exercise, we have $g(x) = cx$ for some c. Therefore, $f(x) = e^{g(x)} = e^{cx}$.

(ii) We can prove (2) by the remark (1) as follows. It suffices to show that this f is differentiable at 0 by remark (1) and (1)-(iii). Since $f(\frac{m}{n}) = f(1)^{\frac{m}{n}}$ then for every real r, $f(r) = [f(1)]^r$ by continuity of f. Note that $\lim_{r\to 0} \frac{a^r - b^r}{r}$ exists. Given any sequence $\{r_n\}$ with $r_n \to 0$, and thus consider

$$\lim_{r_n\to 0} \frac{f(r_n) - f(0)}{r_n} = \frac{[f(1)]^{r_n} - 1}{r_n} = \frac{[f(1)]^{r_n} - 1^{r_n}}{r_n} \text{ exists,}$$

we have f is differentiable at $x = 0$. So, by remark (1), we have $f(x) = e^{cx}$.

(3) Give an example such that f is not continuous on R.

Solution: Consider $g(x+y) = g(x) + g(y)$ for all x,y. Then we have $g(q) = qg(1)$, where $q \in Q$. By **Zorn's Lemma**, we know that every vector space has a basis $\{v_\alpha : \alpha \in I\}$. Note that $\{v_\alpha : \alpha \in I\}$ is an uncountable set, so there exists a convergent sequence $\{s_n\} \subseteq \{v_\alpha : \alpha \in I\}$. Hence, $S := (\{v_\alpha : \alpha \in I\} - \{s_n\}_{n=1}^{\infty}) \cup \{\frac{s_n}{n}\}_{n=1}^{\infty}$ is a new basis of R over Q. Given $x,y \in R$, and we can find the same N such that

$$x = \sum_{k=1}^{N} q_k v_k \text{ and } y = \sum_{k=1}^{N} p_k v_k, \text{ where } v_k \in S$$

Define the sume

$$x + y := \sum_{k=1}^{N} (p_k + q_k) v_k$$

By uniqueness, we define $g(x)$ to be the sum of coefficients, i.e.,

$$g(x) := \sum_{k=1}^{N} q_k.$$

Note that

$$g\left(\frac{s_n}{n}\right) = 1 \text{ for all } n \Rightarrow \lim_{n\to\infty} g\left(\frac{s_n}{n}\right) = 1$$

and

$$\frac{s_n}{n} \to 0 \text{ as } n \to \infty$$

Hence, g is not continuous at $x = 0$ since if it was, then

$$\begin{aligned} 1 &= \lim_{n\to\infty} g\left(\frac{s_n}{n}\right) \\ &= g\left(\lim_{n\to\infty}\frac{s_n}{n}\right) \text{ by continuity of } g \text{ at } 0 \\ &= g(0) \\ &= 0 \end{aligned}$$

which is absurb. Hence, g is not continuous on R by the exercise. To find such f, it suffices to consider $f(x) = e^{g(x)}$.

Note: Such g (or f) is **not measurable** by **Lusin Theorem**.

4.19 Let f be continuous on $[a,b]$ and define g as follows: $g(a) = f(a)$ and, for $a < x \leq b$, let $g(x)$ be the maximum value of f in the subinterval $[a,x]$. Show that g is continuous on $[a,b]$.

Proof: Define $g(x) = \max\{f(t) : t \in [a,x]\}$, and choose any point $c \in [a,b]$, we want to show that g is continuous at c. Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $x \in (c-\delta, c+\delta) \cap [a,b]$, we have

$$|g(x) - g(c)| < \varepsilon.$$

Since f is continuous at $x = c$, then there exists a $\delta' > 0$ such that as $x \in (c-\delta', c+\delta') \cap [a,b]$, we have

$$f(c) - \varepsilon/2 < f(x) < f(c) + \varepsilon/2. \quad *$$

Consider two cases as follows.

(1) $\max\{f(t) : t \in [a, c+\delta'] \cap [a,b]\} = f(p_1)$, where $p_1 \leq c - \delta'$.
As $x \in (c-\delta', c+\delta') \cap [a,b]$, we have $g(x) = f(p_1)$ and $g(c) = f(p_1)$.
Hence, $|g(x) - g(c)| = 0$.
(2) $\max\{f(t) : t \in [a, c+\delta'] \cap [a,b]\} = f(p_1)$, where $p_1 > c - \delta'$.
As $x \in (c-\delta', c+\delta') \cap [a,b]$, we have by (*) $f(c) - \varepsilon/2 \leq g(x) \leq f(c) + \varepsilon/2$.
Hence, $|g(x) - g(c)| < \varepsilon$.
So, if we choose $\delta = \delta'$, then for $x \in (c-\delta, c+\delta) \cap [a,b]$,

$$|g(x) - g(c)| < \varepsilon \text{ by (1) and (2).}$$

Hence, $g(x)$ is continuous at c. And since c is arbitrary, we have $g(x)$ is continuous on $[a,b]$.

Remark: It is the same result for $\min\{f(t) : t \in [a,x]\}$ by the preceding method.

4.20 Let $f_1, \ldots, f_m$ be m real-valued functions defined on R^n. Assume that each f_k is continuous at the point a of S. Define a new function f as follows: For each x in S, $f(x)$ is the largest of the m numbers $f_1(x), \ldots, f_m(x)$. Discuss the continuity of f at a.

Proof: Assume that each f_k is continuous at the point a of S, then we have $(f_i + f_j)$ and $|f_i - f_j|$ are continuous at a, where $1 \leq i,j \leq m$. Since $\max(a,b) = \frac{(a+b)+|a-b|}{2}$, then $\max(f_1, f_2)$ is continuous at a since both $(f_1 + f_2)$ and $|f_1 - f_2|$ are continuous at a. Define $f(x) = \max(f_1, \ldots f_m)$, use **Mathematical Induction** to show that $f(x)$ is continuous at $x = a$ as follows. As $m = 2$, we have proved it. Suppose $m = k$ holds, i.e., $\max(f_1, \ldots f_k)$ is continuous at $x = a$. Then as $m = k+1$, we have

$$\max(f_1, \ldots f_{k+1}) = \max[\max(f_1, \ldots f_k), f_{k+1}]$$

is continuous at $x = a$ by induction hypothesis. Hence, by **Mathematical Induction**, we have prove that f is continuos at $x = a$.

It is possible that f and g is not continuous on R whihc implies that $\max(f,g)$ is continuous on R. For example, let $f(x) = 0$ if $x \in Q$, and $f(x) = 1$ if $x \in Q^c$ and $g(x) = 1$

if $x \in Q$, and $g(x) = 0$ if $x \in Q^c$.

Remark: It is the same rusult for $\min(f_1, \ldots f_m)$ since $\max(a,b) + \min(a,b) = a + b$.

4.21 Let $f : S \to R$ be continuous on an open set in R^n, assume that $p \in S$, and assume that $f(p) > 0$. Prove that there is an n –ball $B(p;r)$ such that $f(x) > 0$ for every x in the ball.

Proof: Since $(p \in)S$ is an open set in R^n, there exists a $\delta_1 > 0$ such that $B(p,\delta_1) \subseteq S$. Since $f(p) > 0$, given $\varepsilon = \frac{f(p)}{2} > 0$, then there exists an n –ball $B(p;\delta_2)$ such that as $x \in B(p;\delta_2) \cap S$, we have

$$\frac{f(p)}{2} = f(p) - \varepsilon < f(x) < f(p) + \varepsilon = \frac{3f(p)}{2}.$$

Let $\delta = \min(\delta_1, \delta_2)$, then as $x \in B(p;\delta)$, we have

$$f(x) > \frac{f(p)}{2} > 0.$$

Remark: The exercise tells us that under the assumption of continuity at p, we roughly have the same sign in a neighborhood of p, if $f(p) > 0$ $\big($or $f(p) < 0.\big)$

4.22 Let f be defined and continuous on a closed set S in R. Let

$$A = \left\{x : x \in S \text{ and } f(x) = 0\right\}.$$

Prove that A is a closed subset of R.

Proof: Since $A = f^{-1}(\{0\})$, and f is continous on S, we have A is closed in S. And since S is closed in R, we finally have A is closed in R.

Remark: 1. Roughly speaking, the property of being closed has **Transitivity**. That is, in (M,d) let $S \subseteq T \subseteq M$, if S is closed in T, and T is closed in M, then S is closed in M.

Proof: Let x be an adherent point of S in M, then $B_M(x,r) \cap S \neq \phi$ for every $r > 0$. Hence, $B_M(x,r) \cap T \neq \phi$ for every $r > 0$. It means that x is also an adherent point of T in M. Since T is closed in M, we find that $x \in T$. Note that since $B_M(x,r) \cap S \neq \phi$ for every $r > 0$, we have $(S \subseteq T)$

$$B_T(x,r) \cap S = (B_M(x,r) \cap T) \cap S = B_M(x,r) \cap (S \cap T) = B_M(x,r) \cap S \neq \phi.$$

So, we have x is an adherent point of S in T. And since S is closed in T, we have $x \in S$. Hence, we have proved that if x is an adherent point of S in M, then $x \in S$. That is, S is closed in M.

Note: (1) Another proof of remark 1, since S is closed in T, there exists a closed subset U in Msuch that $S = U \cap T$, and since T is closed in M, we have S is closed in M.

(2) There is a similar result, in (M,d) let $S \subseteq T \subseteq M$, if S is open in T, and T is open in M, then S is open in M. (Leave to the reader.)

2. Here is another statement like the exercise, but we should be cautioned. We write it as follows. Let f and g be continuous on (S,d_1) into (T,d_2). Let $A = \{x : f(x) = g(x)\}$, show that A is closed in S.

Proof: Let x be an accumulation point of A, then there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x$. So, we have $f(x_n) = g(x_n)$ for all n. Hence, by continuity of f and g, we have

$$f(x) = f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g\Big(\lim_{n\to\infty} x_n\Big) = g(x).$$

Hence, $x \in A$. That is, A contains its all adherent point. So, A is closed.

Note: In remark 2, we **CANNOT** use the relation

$$f(x) - g(x)$$

since the difference "$-$" are not necessarily defined on the metric space (T, d_2).

4.23 Given a function $f : R \to R$, define two sets A and B in R^2 as follows:

$$A = \{(x,y) : y < f(x)\},$$
$$B = \{(x,y) : y > f(x)\}.$$

and Prove that f is continuous on R if, and only if, both A and B are open subsets of R^2.

Proof: ($\Rightarrow$) Suppose that f is continuous on R. Let $(a,b) \in A$, then $f(a) > b$. Since f is continuous at a, then given $\varepsilon = \frac{f(a)-b}{2} > 0$, there exists a $(\varepsilon >)\delta > 0$ such that as $|x-a| < \delta$, we have

$$\frac{f(a)+b}{2} = f(a) - \varepsilon < f(x) < f(a) + \varepsilon. \qquad *$$

Consider $(x,y) \in B((a,b);\delta)$, then $|x-a|^2 + |y-b|^2 < \delta^2$ which implies that

1. $|x-a| < \delta \Rightarrow f(x) > \dfrac{f(a)+b}{2}$ by (*) and
2. $|y-b| < \delta \Rightarrow y < b + \delta < b + \varepsilon = \dfrac{f(a)+b}{2}$.

Hence, we have $f(x) > y$. That is, $B((a,b);\delta) \subseteq A$. So, A is open since every point of A is interior. Similarly for B.

($\Leftarrow$) Suppose that A and B are open in R^2. Trivially, $(a, f(a) - \varepsilon/2) := p_1 \in A$, and $(a, f(a) + \varepsilon/2) := p_2 \in B$. Since A and B are open in R^2, there exists a $(\varepsilon/2 >)\delta > 0$ such that

$$B(p_1,\delta) \subseteq A \text{ and } B(p_2,\delta) \subseteq B.$$

Hence, if $(x,y) \in B(p_1,\delta)$, then

$$(x-a)^2 + (y - (f(a) - \varepsilon/2))^2 < \delta^2 \text{ and } y < f(x).$$

So, it implies that

$$|x-a| < \delta,\ |y - f(a) + \varepsilon/2| < \delta, \text{ and } y < f(x).$$

Hence, as $|x-a| < \delta$, we have

$$\begin{aligned} -\delta &< y - f(a) + \varepsilon/2 \\ &\Rightarrow f(a) - \delta - \varepsilon/2 < y < f(x) \\ &\Rightarrow f(a) - \varepsilon < y < f(x) \\ &\Rightarrow f(a) - \varepsilon < f(x). \end{aligned} \qquad **$$

And if $(x,y) \in B(p_2,\delta)$, then

$$(x-a)^2 + (y - (f(a) + \varepsilon/2))^2 < \delta^2 \text{ and } y > f(x).$$

So, it implies that

$$|x-a| < \delta,\ |y - f(a) - \varepsilon/2| < \delta, \text{ and } y > f(x).$$

Hence, as $|x-a| < \delta$, we have

$$f(x) < y < f(a) + \varepsilon/2 + \delta < f(a) + \varepsilon. \qquad ***$$

So, given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $|x-a| < \delta$, we have by (**) and (***)

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon.$$

That is, f is continuous at a. Since a is arbitrary, we know that f is continuous on R.

4.24 Let f be defined and bounded on a compact interval S in R. If $T \subseteq S$, the number

$$\Omega_f(T) = \sup\{f(x) - f(y) : x, y \in T\}$$

is called the **oscillation (or span) of** f on T. If $x \in S$, the oscillation of f at x is defined to be the number

$$\omega_f(x) = \lim_{h \to 0^+} \Omega_f(B(x;h) \cap S).$$

Prove that this limit always exists and that $\omega_f(x) = 0$ if, and only if, f is continuous at x.

Proof: 1. Note that since f is bounded, say $|f(x)| \leq M$ for all x, we have $|f(x) - f(y)| \leq 2M$ for all $x, y \in S$. So, $\Omega_f(T)$, the oscillation of f on any subset T of S, exists. In addition, we define $g(h) = \Omega_f(B(x;h) \cap S)$. Note that if $T_1 \subseteq T_2 (\subseteq S)$, we have $\Omega_f(T_1) \leq \Omega_f(T_2)$. Hence, the oscillation of f at x, $\omega_f(x) = \lim_{h \to 0^+} g(h) = g(0+)$ since g is an increasing function. That is, the limit of $\Omega_f(B(x;h) \cap S)$ always exists as $h \to 0^+$.

2. ($\Rightarrow$) Suppose that $\omega_f(x) = 0$, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $h \in (0, \delta)$, we have

$$|g(h)| = g(h) = \Omega_f(B(x;h) \cap S) < \varepsilon/2.$$

That is, as $h \in (0, \delta)$, we have

$$\sup\{f(t) - f(s) : t, s \in B(x;h) \cap S\} < \varepsilon/2$$

which implies that

$$-\varepsilon/2 < f(t) - f(x) < \varepsilon/2 \text{ as } t \in (x - \delta, x + \delta) \cap S.$$

So, as $t \in (x - \delta, x + \delta) \cap S$. we have

$$|f(t) - f(x)| < \varepsilon.$$

That is, f is continuous at x.

($\Leftarrow$) Suppose that f is continous at x, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $t \in (x - \delta, x + \delta) \cap S$, we have

$$|f(t) - f(x)| < \varepsilon/3.$$

So, as $t, s \in (x - \delta, x + \delta) \cap S$, we have

$$|f(t) - f(s)| \leq |f(t) - f(x)| + |f(x) - f(s)| < \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3$$

which implies that

$$\sup\{(t) - f(s) : t, s \in (x - \delta, x + \delta) \cap S\} \leq 2\varepsilon/3 < \varepsilon.$$

So, as $h \in (0, \delta)$, we have

$$\Omega_f(B(x;h) \cap S) = \sup\{(t) - f(s) : t, s \in (x - \delta, x + \delta) \cap S\} < \varepsilon.$$

Hence, the oscillation of f at x, $\omega_f(x) = 0$.

Remark: 1. The compactness of S is not used here, we will see the advantage of the oscillation of f in text book, Theorem 7.48, in page 171. (On **Lebesgue's Criterion for Riemann-Integrability**.)

2. One of advantage of the oscillation of f is to show the statement: Let f be defined on $[a, b]$ Prove that a bounded f does **NOT** have the properties:

$$f \text{ is continuous on } Q \cap [a, b], \text{ and discontinuous on } (R - Q) \cap [a, b].$$

Proof: Since $\omega_f(x) = 0$ if, and only if, f is continuous at x, we know that $\omega_f(r) > 0$ for $r \in (R - Q) \cap [a, b]$. Define $J_{1/n} = \{r : \omega_f(r) \geq 1/n\}$, then by hypothesis, we know that $\cup_{n=1}^{\infty} J_{1/n} = (R - Q) \cap [a, b]$. It is easy to show that $J_{1/n}$ is closed in $[a, b]$. Hence, $int[cl(J_{1/n})] = int(J_{1/n}) = \phi$ for all $n \in N$. It means that $J_{1/n}$ is **nowhere dense** for all $n \in N$. Hence,

$$[a,b] = (\cup_{n=1}^{\infty} J_{1/n}) \cup (Q \cap [a,b])$$

is of the **firse category** which is absurb since every complete metric space is of the **second category**. So, this f cannot exist.

Note: 1 The Boundedness of f **can be removed** since we we can accept the concept $\infty > 0$.

2. ($J_{1/n}$ is closed in $[a,b]$) Given an accumulation point x of $J_{1/n}$, if $x \notin J_{1/n}$, we have $\omega_f(x) < 1/n$. So, there exists a 1 –ball $B(x)$ such that $\Omega_f(B(x) \cap [a,b]) < 1/n$. Thus, no points of $B(x)$ can belong to $J_{1/n}$, contradicting that x is an accumulation point of $J_{1/n}$. Hence, $x \in J_{1/n}$ and $J_{1/n}$ is closed.

3. (**Definition of a nowhere dense set)** In a metric space (M,d), let A be a subset of M, we say A is nowhere dense in M if, and only if $\bar{A}$ contains no balls of M, ($\Leftrightarrow$ $int(\bar{A}) = \phi$).

4. (**Definition of a set of the first category and of the second category)** A set A in a metric space M is of the first category if, and only if, A is the union of a countable number of nowhere dense sets. A set B is of the second category if, and only if, B is not of the first category.

5. (**Theorem**) A complete metric space is of the second category.
We write another important theorem about a set of the second category below.
(**Baire Category Theorem**) A nonempty open set in a complete metric space is of the second category.

6. In the notes 3,4 and 5, the reader can see the reference, A First Course in Real Analysis written by M. H. Protter and C. B. Morrey, in pages 375-377.

4.25 Let f be continuous on a compact interval $[a,b]$. Suppose that f has a local maximum at x_1 and a local maximum at x_2. Show that there must be a third point between x_1 and x_2 where f has a local minimum.
Note. To say that f has a local maximum at x_1 means that there is an 1 –ball $B(x_1)$ such that $f(x) \leq f(x_1)$ for all x in $B(x_1) \cap [a,b]$. Local minimum is similarly defined.

Proof: Let $x_2 > x_1$. Suppose **NOT**, i.e., no points on (x_1,x_2) can be a local minimum of f. Since f is continuous on $[x_1,x_2]$, then $\inf\{f(x) : x \in [x_1,x_2]\} = f(x_1)$ or $f(x_2)$ by hypothesis. We consider two cases as follows:
(1) If $\inf\{f(x) : x \in [x_1,x_2]\} = f(x_1)$, then

$$\begin{cases} \text{(i) } f(x) \text{ has a local maximum at } x_1 \text{ and} \\ \text{(ii) } f(x) \geq f(x_1) \text{ for all } x \in [x_1,x_2]. \end{cases}$$

By (i), there exists a $\delta > 0$ such that $x \in [x_1,x_1+\delta) \subseteq [x_1,x_2]$, we have

$$\text{(iii) } f(x) \leq f(x_1).$$

So, by (ii) and (iii), as $x \in [x_1,x_1+\delta)$, we have

$$f(x) = f(x_1)$$

which contradicts the hypothesis that no points on (x_1,x_2) can be a local minimum of f.
(2) If $\inf\{f(x) : x \in [x_1,x_2]\} = f(x_1)$, it is similar, we omit it.
Hence, from (1) and (2), we have there has a third point between x_1 and x_2 where f has a local minimum.

4.26 Let f be a real-valued function, continuous on $[0,1]$, with the following property: For every real y, either there is no x in $[0,1]$ for which $f(x) = y$ or there is exactly one such x. Prove that f is strictly monotonic on $[0,1]$.

Proof: Since the hypothesis says that f is one-to-one, then by Theorem*, we know that f is trictly monotonic on $[0,1]$.

Remark: (**Theorem***) Under assumption of continuity on a compact interval, 1-1 is equivalent to being strictly monotonic. We will prove it in Exercise 4.62.

4.27 Let f be a function defined on $[0,1]$ with the following property: For every real number y, either there is no x in $[0,1]$ for which $f(x) = y$ or there are exactly two values of x in $[0,1]$ for which $f(x) = y$.

(a) Prove that f cannot be continuous on $[0,1]$.

Proof: Assume that f is continuous on $[0,1]$, and thus consider $\max_{x\in[0,1]} f(x)$ and $\min_{x\in[0,1]} f(x)$. Then by hypothesis, there exist exactly two values $a_1 < a_2 \in [0,1]$ such that $f(a_1) = f(a_2) = \max_{x\in[0,1]} f(x)$, and there exist exactly two values $b_1 < b_2 \in [0,1]$ such that $f(b_1) = f(b_2) = \min_{x\in[0,1]} f(x)$.

Claim that $a_1 = 0$ and $a_2 = 1$. Suppose **NOT,** then there exists at least one belonging to $(0,1)$. Without loss of generality, say $a_1 \in (0,1)$. Since f has maximum at $a_1 \in (0,1)$ and $a_2 \in [0,1]$, we can find three points p_1, p_2, and p_3 such that

$$\begin{cases} 1.\ p_1 < a_1 < p_2 < p_3 < a_2, \\ 2.\ f(p_1) < f(a_1),\ f(p_2) < f(a_1), \text{ and } f(p_3) < f(a_2). \end{cases}$$

Since $f(a_1) = f(a_2)$, we choose a real number r so that

$$f(p_1) < r < f(a_1) \Rightarrow r = f(q_1), \text{ where } q_1 \in (p_1, a_1) \text{ by continuity of } f.$$

$$f(p_2) < r < f(a_1) \Rightarrow r = f(q_2), \text{ where } q_2 \in (a_1, p_2) \text{ by continuity of } f.$$

$$f(p_3) < r < f(a_2) \Rightarrow r = f(q_3), \text{ where } q_3 \in (p_3, a_2) \text{ by continuity of } f.$$

which contradicts the hypothesis that for every real number y, there are exactly two values of x in $[0,1]$ for which $f(x) = y$. Hence, we know that $a_1 = 0$ and $a_2 = 1$. Similarly, we also have $b_1 = 0$ and $b_2 = 1$.

So, $\max_{x\in[0,1]} f(x) = \min_{x\in[0,1]} f(x)$ which implies that f is constant. It is impossible. Hence, such f does not exist. That is, f is not continuous on $[0,1]$.

(b) Construct a function f which has the above property.

Proof: Consider $[0,1] = (Q^c \cap [0,1]) \cup (Q \cap [0,1])$, and write $Q \cap [0,1] = \{x_1, x_2, \ldots, x_n, \ldots\}$. Define

$$1.\ f(x_{2n-1}) = f(x_{2n}) = n,$$

$$2.\ f(x) = x \text{ if } x \in (0,1/2) \cap Q^c,$$

$$3.\ f(x) = 1 - x \text{ if } x \in (1/2,1) \cap Q^c.$$

Then if $x = y$, then it is clear that $f(x) = f(y)$. That is, f is well-defined. And from construction, we know that the function defined on $[0,1]$ with the following property: For every real number y, either there is no x in $[0,1]$ for which $f(x) = y$ or there are exactly two values of x in $[0,1]$ for which $f(x) = y$.

Remark: $\{x : f \text{ is discontinuous at } x\} = [0,1]$. Given $a \in [0,1]$. Note that since $f(x) \in N$ for all $x \in Q \cap [0,1]$ and Q is dense in R, for any 1 –ball $B(a;r) \cap (Q \cap [0,1])$, there is always a rational number $y \in B(a;r) \cap (Q \cap [0,1])$ such that $|f(y) - f(a)| \geq 1$.

(c) Prove that any function with this property has infinite many discontinuities on $[0,1]$.

Proof: In order to make the proof clear, **property** A **of** f means that

$$\text{for every real number } y, \text{ either there is no } x \text{ in } [0,1] \text{ for which } f(x) = y \text{ or}$$
$$\text{there are exactly two values of } x \text{ in } [0,1] \text{ for which } f(x) = y$$

Assume that there exist a finite many numbers of discontinuities of f, say these points $x_1, \ldots, x_n$. By property A, there exists a unique y_i such that $f(x_i) = f(y_i)$ for $1 \leq i \leq n$. Note that the number of the set
$S := \left(\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \cup \left\{x : f(x) = f(0), \text{ and } f(x) = f(1)\right\}\right)$ is even, say $2m$
We remove these points from S, and thus we have $2m + 1$ subintervals, say I_j, $1 \leq j \leq 2m + 1$. Consider the local extremum in every I_j, $1 \leq j \leq 2m + 1$ and note that every subinterval I_j, $1 \leq j \leq 2m + 1$, has at most finite many numbers of local extremum, say $\#\left(\left\{t \in I_j : f(x) \text{ is the local extremum}\right\} = \left\{t_1^{(j)}, .., t_{p_j}^{(j)}\right\}\right) = p_j$. And by property A, there exists a unique $s_k^{(j)}$ such that $f\left(t_k^{(j)}\right) = f\left(s_k^{(j)}\right)$ for $1 \leq k \leq p_j$. We again remove these points, and thus we have removed even number of points. And odd number of **open** intervals is left, call the odd number $2q - 1$. Note that since the function f is monotonic in every open interval left, R_l, $1 \leq l \leq 2q - 1$, the image of f on these open interval is also an **open** interval. If $R_a \cap R_b \neq \phi$, say $R_a = (a_1, a_1)$ and $R_b = (b_1, b_2)$ with (without loss of generality) $a_1 < b_1 < a_2 < b_2$, then

$$R_a = R_b \text{ by property } A.$$

(Otherwise, b_1 is only point such that $f(x) = f(b_1)$, which contradicts property A.) Note that given any R_a, there must has one and only one R_b such that $R_a = R_b$. However, we have $2q - 1$ **open** intervals is left, it is impossible. Hence, we know that f has infinite many discontinuities on $[0,1]$.

4.28 In each case, give an example of a real-valued function f, continuous on S and such that $f(S) = T$, or else explain why there can be no such f :

(a) $S = (0,1)$, $T = (0,1]$.
Solution: Let

$$f(x) = \begin{cases} 2x \text{ if } x \in (0,1/2], \\ 1 \text{ if } x \in (1/2,1). \end{cases}$$

(b) $S = (0,1)$, $T = (0,1) \cup (1,2)$.

Solution: NO! Since a continuous functions sends a connected set to a connected set. However, in this case, S is connected and T is not connected.

(c) $S = R^1$, $T =$ the set of rational numbers.

Solution: NO! Since a continuous functions sends a connected set to a connected set. However, in this case, S is connected and T is not connected.

(d) $S = [0,1] \cup [2,3]$, $T = \{0,1\}$.

Solution: Let

$$f(x) = \begin{cases} 0 \text{ if } x \in [0,1], \\ 1 \text{ if } x \in [2,3]. \end{cases}$$

(e) $S = [0,1] \times [0,1]$, $T = R^2$.

Solution: NO! Since a continuous functions sends a compact set to a compact set. However, in this case, S is compact and T is not compact.

(f) $S = [0,1] \times [0,1], T = (0,1) \times (0,1)$.

Solution: NO! Since a continuous functions sends a compact set to a compact set. However, in this case, S is compact and T is not compact.

(g) $S = (0,1) \times (0,1), T = R^2$.

Solution: Let

$$f(x,y) = (\cot \pi x, \cot \pi y).$$

Remark: 1. There is some important theorems. We write them as follows.

(**Theorem A**) Let $f : (S, d_s) \to (T, d_T)$ be continuous. If X is a compact subset of S, then $f(X)$ is a compact subset of T.

(**Theorem B**) Let $f : (S, d_s) \to (T, d_T)$ be continuous. If X is a connected subset of S, then $f(X)$ is a connected subset of T.

2. In (g), the key to the example is to find a continuous function $f : (0,1) \to R$ which is onto.

Supplement on Continuity of real valued functions

Exercise Suppose that $f(x) : (0,\infty) \to R$, is continuous with $a \leq f(x) \leq b$ for all $x \in (0,\infty)$, and for any real y, either there is no x in $(0,\infty)$ for which $f(x) = y$ or there are finitely many x in $(0,\infty)$ for which $f(x) = y$. Prove that $\lim_{x\to\infty} f(x)$ exists.

Proof: For convenience, we say property A, it means that for any real y, either there is no x in $(0,\infty)$ for which $f(x) = y$ or there are finitely many x in $(0,\infty)$ for which $f(x) = y$.

We partition $[a,b]$ into n subintervals. Then, by continuity and property A, as x is large enough, $f(x)$ is lying in one and only one subinterval. Given $\varepsilon > 0$, there exists N such that $2/N < \varepsilon$. For this N, we partition $[a,b]$ into N subintervals, then there is a $M > 0$ such that as $x,y \geq M$

$$|f(x) - f(y)| \leq 2/N < \varepsilon.$$

So, $\lim_{x\to\infty} f(x)$ exists.

Exercise Suppose that $f(x) : [0,1] \to R$ is continunous with $f(0) = f(1) = 0$. Prove that

(a) there exist two points x_1 and x_2 such that as $|x_1 - x_2| = 1/n$, we have $f(x_1) = f(x_2) \neq 0$ for all n. In this case, we call $1/n$ the length of horizontal strings.

Proof: Define a new function $g(x) = f(x + \frac{1}{n}) - f(x) : [0, 1 - \frac{1}{n}]$. Claim that there exists $p \in [0, 1 - \frac{1}{n}]$ such that $g(p) = 0$. Suppose **NOT**, by **Intermediate Value Theorem**, without loss of generality, let $g(x) > 0$, then

$$g(0) + g\left(\frac{1}{n}\right) + \ldots + g\left(1 - \frac{1}{n}\right) = f(1) > 0$$

which is absurb. Hence, we know that there exists $p \in [0, 1 - \frac{1}{n}]$ such that $g(p) = 0$. That is,

$$f\left(p + \frac{1}{n}\right) = f(p).$$

So, we have $1/n$ as the length of horizontal strings.

(b) Could you show that there exists 2/3 as the length of horizontal strings?

Proof: The horizontal strings does not exist, for example,

$$f(x) = \begin{cases} x, \text{ if } x \in [0, \frac{1}{4}] \\ -x + \frac{1}{2}, \text{ if } x \in [\frac{1}{4}, \frac{3}{4}] \\ x - 1, \text{ if } x \in [\frac{3}{4}, 1] \end{cases}.$$

Exercise Suppose that $f(x) : [a,b] \to R$ is a continuous and non-constant function. Prove that the function f cannot have any small periods.

Proof: Say f is continuous at $q \in [a,b]$, and by hypothesis that f is non-constant, there is a point $p \in [a,b]$ such that $|f(q) - f(p)| := M > 0$. Since f is continuous at q, then given $\varepsilon = M$, there is a $\delta > 0$ such that as $x \in (q-\delta, q+\delta) \cap [a,b]$, we have

$$|f(x) - f(q)| < M. \qquad *$$

If f has any small periods, then in the set $(q-\delta, q+\delta) \cap [a,b]$, there is a point $r \in (q-\delta, q+\delta) \cap [a,b]$ such that $f(r) = f(p)$. It contradicts to (*). Hence, the function f cannot have any small periods.

Remark 1. There is a function with any small periods.

Solution:The example is Dirichlet function,

$$f(x) = \begin{cases} 0, \text{ if } x \in Q^c \\ 1, \text{ if } x \in Q \end{cases}.$$

Since $f(x+q) = f(x)$, for any rational q, we know that f has any small periods.

2. Prove that there cannot have a non-constant continuous function which has two period p, and q such that q/p is irrational.

Proof: Since q/p is irrational, there is a sequence $\{\frac{q_n}{p_n}\}(\subseteq Q)$ such that

$$\left|\frac{q_n}{p_n} - \frac{q}{p}\right| < \frac{1}{p_n^2} \Rightarrow |pq_n - qp_n| < \left|\frac{p}{p_n}\right| \to 0 \text{ as } n \to \infty.$$

So, f has any small periods, by this exercise, we know that this f cannot a non-constant continuous function.

Note: The inequality is important; the reader should kepp it in mind. There are many ways to prove this inequality, we metion two methods without proofs. The reader can find the proofs in the following references.

(1) An Introduction To The Theory Of Numbers written by G.H. Hardy and E.M. Wright, charpter X, pp 137-138.

(2) In the text book, exercise 1.15 and 1.16, pp 26.

3. Suppose that $f(x)$ is differentiable on R prove that if f has any small periods, then f is constant.

Proof: Given $c \in R$, and consider

$$\frac{f(c+p_n) - f(c)}{p_n} = 0 \text{ for all } n.$$

where p_n is a sequence of periods of function such that $p_n \to 0$. Hence, by differentiability of f, we know that $f'(c) = 0$. Since c is arbitrary, we know that $f'(x) = 0$ on R. Hence, f is constant.

Continuity in metric spaces

In Exercises 4.29 through 4.33, we assume that $f : S \to T$ is a function from one metric space (S, d_S) to another (T, d_T).

4.29 Prove that f is continuous on S if, and only if,

$$f^{-1}(intB) \subseteq int(f^{-1}(B)) \text{ for every subset } B \text{ of } T.$$

Proof: ($\Rightarrow$) Suppose that f is continuous on S, and let B be a subset of T. Since $int(B) \subseteq B$, we have $f^{-1}(intB) \subseteq f^{-1}(B)$. Note that $f^{-1}(intB)$ is open since a pull back of an open set under a continuous function is open. Hence, we have

$$int[f^{-1}(intB)] = f^{-1}(intB) \subseteq int(f^{-1}(B)).$$

That is, $f^{-1}(intB) \subseteq int(f^{-1}(B))$ for every subset B of T.

($\Leftarrow$) Suppose that $f^{-1}(intB) \subseteq int(f^{-1}(B))$ for every subset B of T. Given an open subset $U(\subseteq T)$, i.e., $intU = U$, so we have

$$f^{-1}(U) = f^{-1}(intU) \subseteq int(f^{-1}(U)).$$

In addition, $int(f^{-1}(U)) \subseteq f^{-1}(U)$ by the fact, for any set A, $intA$ is a subset of A. So, f is continuous on S.

4.30 Prove that f is continuous on S if, and only if,

$$f(cl(A)) \subseteq cl(f(A)) \text{ for every subset } A \text{ of } S.$$

Proof: ($\Rightarrow$) Suppose that f is continuous on S, and let A be a subset of S. Since $f(A) \subseteq cl(f(A))$, then $(A \subseteq) f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A)))$. Note that $f^{-1}(cl(f(A)))$ is closed since a pull back of a closed set under a continuous function is closed. Hence, we have

$$cl(A) \subseteq cl[f^{-1}(cl(f(A)))] = f^{-1}(cl(f(A)))$$

which implies that

$$f(cl(A)) \subseteq f[f^{-1}(cl(f(A)))] \subseteq cl(f(A)).$$

($\Leftarrow$) Suppose that $f(cl(A)) \subseteq cl(f(A))$ for every subset A of S. Given a closed subset $C(\subseteq T)$, and consider $f^{-1}(C)$ as follows. Define $f^{-1}(C) = A$, then

$$\begin{aligned} f(cl(f^{-1}(C))) &= f(cl(A)) \\ &\subseteq cl(f(A)) = cl(f(f^{-1}(C))) \\ &\subseteq cl(C) = C \text{ since } C \text{ is closed.} \end{aligned}$$

So, we have by $(f(cl(A)) \subseteq C)$

$$cl(A) \subseteq f^{-1}(f(cl(A))) \subseteq f^{-1}(C) = A$$

which implies that $A = f^{-1}(C)$ is closed set. So, f is continuous on S.

4.31 Prove that f is continuous on S if, and only if, f is continuous on every compact subset of S. Hint. If $x_n \to p$ in S, the set $\{p, x_1, x_2, \ldots\}$ is compact.

Proof: ($\Rightarrow$) Suppose that f is continuous on S, then it is clear that f is continuous on every compact subset of S.

($\Leftarrow$) Suppose that f is continuous on every compact subset of S, Given $p \in S$, we consider two cases.

(1) p is an isolated point of S, then f is automatically continuous at p.

(2) p is not an isolated point of S, that is, p is an accumulation point p of S, then there exists a sequence $\{x_n\}(\subseteq)S$ with $x_n \to p$. Note that the set $\{p, x_1, x_2, \ldots\}$ is compact, so we know that f is continuous at p. Since p is arbitrary, we know that f is continuous on S.

Remark: If $x_n \to p$ in S, the set $\{p, x_1, x_2, \ldots\}$ is compact. The fact is immediately from the statement that every infinite subset $\{p, x_1, x_2, \ldots\}$ of has an accumulation point in $\{p, x_1, x_2, \ldots\}$.

4.32 A function $f : S \to T$ is called a closed mapping on S if the image $f(A)$ is closed in T for every closed subset A of S. Prove that f is continuous and closed on S if, and only if,

$$f(cl(A)) = cl(f(A)) \text{ for every subset } A \text{ of } S.$$

Proof: (⇒) Suppose that f is continuous and closed on S, and let A be a subset of S. Since $A \subseteq cl(A)$, we have $f(A) \subseteq f(cl(A))$. So, we have

$$cl(f(A)) \subseteq cl(f(cl(A))) = f(cl(A)) \text{ since } f \text{ is closed.} \quad *$$

In addition, since $f(A) \subseteq cl(f(A))$, we have $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A)))$. Note that $f^{-1}(cl(f(A)))$ is closed since f is continuous. So, we have

$$cl(A) \subseteq cl(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A)))$$

which implies that

$$f(cl(A)) \subseteq f(f^{-1}(cl(f(A)))) \subseteq cl(f(A)). \quad **$$

From (*) and (**), we know that $f(cl(A)) = cl(f(A))$ for every subset A of S.

(⇐) Suppose that $f(cl(A)) = cl(f(A))$ for every subset A of S. Gvien a closed subset C of S, i.e., $cl(C) \subseteq C$, then we have

$$f(C) \supseteq f(cl(C)) = cl(f(C)).$$

So, we have $f(C)$ is closed. That is, f is closed. Given any closed subset B of T, i.e., $cl(B) \subseteq B$, we want to show that $f^{-1}(B)$ is closed. Since $f^{-1}(B) := A \subseteq S$, we have

$$f(cl(f^{-1}(B))) = f(cl(A)) = cl(f(A)) = cl(f(f^{-1}(B))) \subseteq cl(B) \subseteq B$$

which implies that

$$f(cl(f^{-1}(B))) \subseteq B \Rightarrow cl(f^{-1}(B)) \subseteq f^{-1}(f(cl(f^{-1}(B)))) \subseteq f^{-1}(B).$$

That is, we have $cl(f^{-1}(B)) \subseteq f^{-1}(B)$. So, $f^{-1}(B)$ is closed. Hence, f is continuous on S.

4.33 Give an example of a continuous f and a Cauchy sequence $\{x_n\}$ in some metric space S for which $\{f(x_n)\}$ is not a Cauchy sequence in T.

Solution: Let $S = (0,1]$, $x_n = 1/n$ for all $n \in N$, and $f = 1/x : S \to R$. Then it is clear that f is continous on S, and $\{x_n\}$ is a Cauchy sequence on S. In addition, Trivially, $\{f(x_n) = n\}$ is not a Cauchy sequence.

Remark: The reader may compare the exercise with the Exercise 4.54.

4.34 Prove that the interval $(-1,1)$ in R^1 is homeomorphic to R^1. This shows that **neither boundedness nor completeness is a topological property.**

Proof: Since $f(x) = \tan(\frac{\pi x}{2}) : (-1,1) \to R$ is **bijection** and continuous, and its converse function $f^{-1}(x) = \arctan x : R \to (-1,1)$. Hence, we know that f is a Topologic mapping. (Or say f is a homeomorphism). Hence, $(-1,1)$ is homeomorphic to R^1.

Remark: A function f is called a bijection if, and only if, f is 1-1 and onto.

4.35 Section 9.7 contains an example of a function f, continuous on $[0,1]$, with $f([0,1]) = [0,1] \times [0,1]$. Prove that no such f can be one-to-one on $[0,1]$.

Proof: By section 9.7, let $f : [0,1] \to [0,1] \times [0,1]$ be an onto and continuous function. If f is 1-1, then so is its converse function f^{-1}. Note that since f is a 1-1 and continous function defined on a compact set $[0,1]$, then its converse function f^{-1} is also a continous function. Since $f([0,1]) = [0,1] \times [0,1]$, we have the domain of f^{-1} is $[0,1] \times [0,1]$ which is connected. Choose a special point $y \in [0,1] \times [0,1]$ so that $f^{-1}(y) := x \in (0,1)$. Consider a continous function $g = f^{-1}|_{[0,1]\times[0,1]-\{y\}}$, then $g : [0,1] \times [0,1] - \{y\} \to [0,x) \cup (x,1]$ which is continous. However, it is impossible since $[0,1] \times [0,1] - \{y\}$ is connected but $[0,x) \cup (x,1]$ is not connected. So, such f cannot exist.

Connectedness

4.36 Prove that a metric space S is disconnected if, and only if there is a nonempty subset A of S, $A \neq S$, which is both open and closed in S.

Proof: ($\Rightarrow$) Suppose that S is disconnected, then there exist two subset A, B in S such that

$$1.\ A,\ B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

Note that since A, B are open in S, we have $A = S - B$, $B = S - A$ are closed in S. So, if S is disconnected, then there is a nonempty subset A of S, $A \neq S$, which is both open and closed in S.

($\Leftarrow$) Suppose that there is a nonempty subset A of S, $A \neq S$, which is both open and closed in S. Then we have $S - A := B$ is nonempty and B is open in S. Hence, we have two sets A, B in S such that

$$1.\ A,\ B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

That is, S is disconnected.

4.37 Prove that a metric space S is connected if, and only if the only subsets of S which are both open and closed in S are empty set and S itself.

Proof: ($\Rightarrow$) Suppose that S is connected. If there exists a subset A of S such that

$$1.\ A \neq \phi,\ 2.\ A \text{ is a proper subset of } S,\ 3.\ A \text{ is open and closed in } S,$$

then let $B = S - A$, we have

$$1.\ A,\ B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

It is impossible since S is connected. So, this A cannot exist. That is, the only subsets of S which are both open and closed in S are empty set and S itself.

($\Leftarrow$) Suppose that the only subsets of S which are both open and closed in S are empty set and S itself. If S is disconnected, then we have two sets A, B in S such that

$$1.\ A,\ B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

It contradicts the hypothesis that the only subsets of S which are both open and closed in S are empty set and S itself.

Hence, we have proved that S is connected if, and only if the only subsets of S which are both open and closed in S are empty set and S itself.

4.38 Prove that the only connected subsets of R are
(a) the empty set,
(b) sets consisting of a single point, and
(c) intervals (open, closed, half-open, or infinite).

Proof: Let S be a connected subset of R. Denote the symbol $\#(A)$ to be the number of elements in a set A. We consider three cases as follows. (a) $\#(S) = 0$, (b) $\#(S) = 1$, (c) $\#(S) > 1$.

For case (a), it means that $S = \phi$, and for case (b), it means that S consists of a single point. It remains to consider the case (c). Note that since $\#(S) > 1$, we have $\inf S \neq \sup S$.

Since $S \subseteq R$, we have $S \subseteq [\inf S, \sup S]$. (Note that we accept that $\inf S = -\infty$ or $\sup S = \infty$.) If S is not an interval, then there exists $x \in (\inf S, \sup S)$ such that $x \notin S$. (Otherwise, $(\inf S, \sup S) \subseteq S$ which implies that S is an interval.) Then we have

$$1.\ (-\infty, x) \cap S\ :=A \text{ is open in } S$$

$$2.\ (x, +\infty) \cap S\ :=B \text{ is open in } S$$

$$3.\ A \cup B = S.$$

Claim that both A and B are not empty. Asume that A is empty, then every $s \in S$, we have $s > x > \inf S$. By the definition of infimum, it is impossible. So, A is not empty. Similarly for B. Hence, we have proved that S is disconnected, a contradiction. That is, S is an interval.

Remark: 1. We note that any interval in R is connected. It is immediate from Exercise 4.44. But we give another proof as follows. Suppose there exists an interval S is not connceted, then there exist two subsets A and B such that

1. A, B are open in S, 2. $A \neq \phi$ and $B \neq \phi$, 3. $A \cap B = \phi$, and 4. $A \cup B = S$.

Since $A \neq \phi$ and $B \neq \phi$, we choose $a \in A$ and $b \in B$, and let $a < b$. Consider

$$c := \sup\{A \cap [a,b]\}.$$

Note that $c \in cl(A) = A$ implies that $c \notin B$. Hence, we have $a \leq c < b$. In addition, $c \notin B = cl(B)$, then there exists a $B_S(c;\delta) \cap B = \phi$. Choose $d \in B_S(c;\delta) = (c-\delta, c+\delta) \cap S$ so that

1. $c < d < b$ and 2. $d \notin B$.

Then $d \notin A$. (Otherwise, it contradicts $c = \sup\{A \cap [a,b]\}$. Note that $d \in [a,b] \subseteq S = A \cup B$ which implies that $d \in A$ or $d \in B$. We reach a contradiction since $d \notin A$ and $d \notin B$. Hence, we have proved that any interval in R is connected.

2. Here is an application. Is there a continuous function $f : R \to R$ such that $f(Q) \subseteq Q^c$, and $f(Q^c) \subseteq Q$?

Ans: NO! If such f exists, then both $f(Q)$ and $f(Q^c)$ are countable. Hence, $f(R)$ is countable. In addition, $f(R)$ is connected. Since $f(R)$ contains rationals and irrationals, we know $f(R)$ is an interval which implies that $f(R)$ is uncountable, a cotradiction. Hence, such f does not exist.

4.39 Let X be a connected subset of a metric space S. Let Y be a subset of S such that $X \subseteq Y \subseteq cl(X)$, where $cl(X)$ is the closure of X. Prove that Y is also connected. In particular, this shows that $cl(X)$ is connected.

Proof: Given a two valued function f on Y, we know that f is also a two valued function on X. Hence, f is constant on X, (without loss of generality) say $f = 0$ on X. Consider $p \in Y - X$, it ,means that p is an accumulation point of X. Then there exists a dequence $\{x_n\} \subseteq X$ such that $x_n \to p$. Note that $f(x_n) = 0$ for all n. So, we have by continuity of f on Y,

$$f(p) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = 0.$$

Hence, we have f is constant 0 on Y. That is, Y is conneceted. In particular, $cl(X)$ is connected.

Remark: Of course, we can use definition of a connected set to show the exercise. But, it is too tedious to write. However, it is a good practice to use definition to show it. The reader may give it a try as a challenge.

4.40 If x is a point in a metric space S, let $U(x)$ be the component of S containing x. Prove that $U(x)$ is closed in S.

Proof: Let p be an accumulation point of $U(x)$. Let f be a two valued function defined on $U(x) \cup \{p\}$, then f is a two valud function defined on $U(x)$. Since $U(x)$ is a component of S containing x, then $U(x)$ is connected. That is, f is constant on $U(x)$, (without loss of generality) say $f = 0$ on $U(x)$. And since p is an accumulation point of $U(x)$, there exists a sequence $\{x_n\} \subseteq U(x)$ such that $x_n \to p$. Note that $f(x_n) = 0$ for all n. So, we have by continuity of f on $U(x) \cup \{p\}$,

$$f(p) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = 0.$$

So, $U(x) \cup \{p\}$ is a connected set containing x. Since $U(x)$ is a component of S containing x, we have $U(x) \cup \{p\} \subseteq U(x)$ which implies that $p \in U(x)$. Hence, $U(x)$ contains its all accumulation point. That is, $U(x)$ is closed in S.

4.41 Let S be an open subset of R. By Theorem 3.11, S is the union of a countable disjoint collection of open intervals in R. Prove that each of these open intervals is a component of the metric subspace S. Explain why this does not contradict Exercise 4.40.

Proof: By Theorem 3.11, $S = \cup_{n=1}^{\infty} I_n$, where I_i is open in R and $I_i \cap I_j = \phi$ if $i \neq j$. Assume that there exists a I_m such that I_m is not a component T of S. Then $T - I_m$ is not empty. So, there exists $x \in T - I_m$ and $x \in I_n$ for some n. Note that the component $U(x)$ is the union of all connected subsets containing x, then we have

$$T \cup I_n \subseteq U(x). \qquad *$$

In addition,

$$U(x) \subseteq T \qquad **$$

since T is a component containing x. Hence, by (*) and (**), we have $I_n \subseteq T$. So, $I_m \cup I_n \subseteq T$. Since T is connected in R^1, T itself is an interval. So, $int(T)$ is still an interval which is open and containing $I_m \cup I_n$. It contradicts the definition of component interval. Hence, each of these open intervals is a component of the metric subspace S.

Since these open intervals is open relative to R, not S, this does not contradict Exercise 4.40.

4.42 Given a compact S in R^m with the following property: For every pair of points a and b in S and for every $\varepsilon > 0$ there exists a finite set of points $\{x_0, x_1, \ldots, x_n\}$ in S with $x_0 = a$ and $x_n = b$ such that

$$\|x_k - x_{k-1}\| < \varepsilon \text{ for } k = 1, 2, .., n.$$

Prove or disprove: S is connected.

Proof: Suppose that S is disconnected, then there exist two subsets A and B such that

$$1.\ A,\ B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S. \qquad *$$

Since $A \neq \phi$ and $B \neq \phi$, we choose $a \in A$, and $b \in A$ and thus given $\varepsilon = 1$, then by hypothesis, we can find two points $a_1 \in A$, and $b_1 \in B$ such that $\|a_1 - b_1\| < 1$. For a_1, and b_1, given $\varepsilon = 1/2$, then by hypothesis, we can find two points $a_2 \in A$, and $b_2 \in B$ such that $\|a_2 - b_2\| < 1/2$. Continuous the steps, we finally have two sequence $\{a_n\} \subseteq A$ and $\{b_n\} \subseteq B$ such that $\|a_n - b_n\| < 1/n$ for all n. Since $\{a_n\} \subseteq A$, and $\{b_n\} \subseteq B$, we have $\{a_n\} \subseteq S$ and $\{b_n\} \subseteq S$ by $S = A \cup B$. Hence, there exist two subsequence $\{a_{n_k}\} \subseteq A$ and $\{b_{n_k}\} \subseteq B$ such that $a_{n_k} \to x$, and $b_{n_k} \to y$, where $x, y \in S$ since S is compact. In addition, since A is closed in S, and B is closed in S, we have $x \in A$ and $y \in B$. On the other hand, since $\|a_n - b_n\| < 1/n$ for all n, we have $x = y$. That is,

$A \cap B \neq \phi$ which contradicts (*)-3. Hence, we have prove that S is connected.

Remark: We given another proof by the method of two valued function as follows. Let f be a two valued function defined on S, and choose any two points $a, b \in S$. If we can show that $f(a) = f(b)$, we have proved that f is a constant which implies that S is connected. Since f is a continuous function defined on a compact set S, then f is uniformly on S. Thus, given $1 > \varepsilon > 0$, there exists a $\delta > 0$ such that as $\|x - y\| < \delta$, $x, y \in S$, we have $|f(x) - f(y)| < \varepsilon < 1 \Rightarrow f(x) = f(y)$. Hence, for this δ, there exists a finite set of points $\{x_0, x_1, \ldots, x_n\}$ in S with $x_0 = a$ and $x_n = b$ such that

$$\|x_k - x_{k-1}\| < \delta \text{ for } k = 1, 2, \ldots, n.$$

So, we have $f(a) = f(x_0) = f(x_1) = \ldots = f(x_n) = f(b)$.

4.43 Prove that a metric space S is connected if, and only if, every nonempty proper subset of S has a nonempty boundary.

Proof: ($\Rightarrow$) Suppose that S is connected, and if there exists a nonempty proper subset U of S such that $\partial U = \phi$, then let $B = cl(S - U)$, we have (define $cl(U) = A$)

$$1.\ A \neq \phi.\ B \neq \phi \text{ since } S - U \neq \phi,$$

$$2.\ A \cup B = cl(U) \cup cl(S - U) \supseteq U \cup (S - U) = S$$
$$\Rightarrow S = A \cup B,$$

$$3.\ A \cap B = cl(U) \cap cl(S - U) = \partial U = \phi,$$

and

$$4. \text{ Both } A \text{ and } B \text{ are closed in } S \Rightarrow \text{ Both } A \text{ and } B \text{ are open in } S.$$

Hence, S is disconnected. That is, if S is connected, then every nonempty proper subset of S has a nonempty boundary.

($\Leftarrow$) Suppose that every nonempty proper subset of S has a nonempty boundary. If S is disconnected, then there exist two subsets A and B such that

$$1.\ A, B \text{ are closed in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

Then for this A, A is a nonempty proper subset of S with ($cl(A) = A$, and $cl(B) = B$)

$$\partial A = cl(A) \cap cl(S - A) = cl(A) \cap cl(B) = A \cap B = \phi$$

which contradicts the hypothesis that every nonempty proper subset of S has a nonempty boundary. So, S is connected.

4.44. Prove that every convex subset of R^n is connected.

Proof: Given a convex subset S of R^n, and since for any pair of points a, b, the set $\{(1 - \theta)a + \theta b : 0 < \theta < 1\} := T \subseteq S$, i.e., $g : [0,1] \to T$ by $g(\theta) = (1 - \theta)a + \theta b$ is a continuous function such that $g(0) = a$, and $g(1) = b$. So, S is path-connected. It implies that S is connected.

Remark: 1. In the exercise, it tells us that every n –ball is connected. (In fact, every n –ball is path-connected.) In particular, as $n = 1$, any interval (open, closed, half-open, or infinite) in R is connected. For $n = 2$, any disk (open, closed, or not) in R^2 is connected.

2. Here is a good exercise on the fact that a path-connected set is connected. Given $[0,1] \times [0,1] := S$, and if T is a countable subset of S. Prove that $S - T$ is connected. (In fact, $S - T$ is path-connected.)

Proof: Given any two points a and b in $S - T$, then consider the vertical line L passing through the middle point $(a + b)/2$. Let $A = \{x : x \in L \cap S\}$, and consider the lines form a to A, and from b to A. Note that A is uncountable, and two such lines (form a to A, and from b to A) are disjoint. So, if every line contains a point of T, then it leads us to get T is uncountable. However, T is countable. So, it has some line (form a to A, and from b to A) is in $S - T$. So, it means that $S - T$ is path-connected. So, $S - T$ is connected.

4.45 Given a function $f : R^n \to R^m$ which is 1-1 and continuous on R^n. If A is open and disconnected in R^n, prove that $f(A)$ is open and disconnected in $f(R^n)$.

Proof: The exercise is **wrong.** There is a counter-example. Let $f : R \to R^2$

$$f = \begin{cases} (\cos(\frac{2\pi x}{1+x} - \frac{\pi}{2}), 1 - \sin(\frac{2\pi x}{1+x} - \frac{\pi}{2})) \text{ if } x \geq 0 \\ (\cos(\frac{2\pi x}{1-x} - \frac{\pi}{2}), -1 + \sin(\frac{2\pi x}{1-x} - \frac{\pi}{2})) \text{ if } x < 0 \end{cases}$$

Remark: If we restrict $n, m = 1$, the conclusion holds. That is, Let $f : R \to R$ be continuous and 1-1. If A is open and disconnected, then so is $f(A)$.

Proof: In order to show this, it suffices to show that f maps an open interval I to another open interval. Since f is continuous on I, and I is connected, $f(I)$ is connected. It implies that $f(I)$ is an interval. Trivially, there is no point x in I such that $f(x)$ equals the endpoints of $f(I)$. Hence, we know that $f(I)$ is an open interval.

Supplement: Here are two exercises on **Homeomorphism** to make the reader get more and feel something.

1. Let $f : E \subseteq R \to R$. If $\{(x, f(x)) : x \in E\}$ is compact, then f is uniformly continuous on E.

Proof: Let $\{(x, f(x)) : x \in E\} = S$, and thus define $g(x) = (I(x) = x, f(x)) : E \to S$. Claim that g is continuous on E. Consider $h : S \to E$ by $h(x, f(x)) = x$. Trivially, h is 1-1, continuous on a compact set S. So, its inverse function g is 1-1 and continuous on a compact set E. The claim has proved.

Since g is continuous on E, we know that f is continuous on a compact set E. Hence, f is uniformly continuous on E.

Note: The question in Supplement 1, there has another proof by the method of contradiction, and use the property of compactness. We omit it.

2. Let $f : (0,1) \to R$. If $\{(x, f(x)) : x \in (0,1)\}$ is path-connected, then f is continuous on $(0,1)$.

Proof: Let $a \in (0,1)$, then there is a compact interval $(a \in)[a_1, a_2] \subseteq (0,1)$. Claim that the set

$$\{(x, f(x)) : x \in [a_1, a_2]\} := S \text{ is compact.}$$

Since S is path-connected, there is a continuous function $g : [0,1] \to S$ such that $g(0) = (a_1, f(a_1))$ and $g(1) = (a_2, f(a_2))$. If we can show $g([0,1]) = S$, we have shown that S is compact. Consider $h : S \to R$ by $h(x, f(x)) = x$; h is clearly continuous on S. So, the composite function $h \circ g : [0,1] \to R$ is also continuous. Note that $h \circ g(0) = a_1$, and $h \circ g(1) = a_2$, and the range of $h \circ g$ is connected. So, $[a_1, a_2] \subseteq h(g([0,1]))$. Hence, $g([0,1]) = S$. We have proved the claim and by Supplement 1, we know that f is continuous at a. Since a is arbitrary, we know that f is continuous on $(0,1)$.

Note: The question in Supplement 2, there has another proof directly by definition of continuity. We omit the proof.

4.46 Let $A = \{(x, y) : 0 < x \leq 1, y = \sin 1/x\}$, $B = \{(x, y) : y = 0, -1 \leq x \leq 0\}$, and let $S = A \cup B$. Prove that S is connected but not arcwise conneceted. (See Fig. 4.5, Section 4.18.)

Proof: Let f be a two valued function defined on S. Since A, and B are connected in S, then we have

$$f(A) = a, \text{ and } f(B) = b, \text{ where } \{a,b\} = \{0,1\}.$$

Given a sequence $\{x_n\}(\subseteq A)$ with $x_n \to (0,0)$, then we have

$$\begin{aligned} a &= \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to 0} x_n\right) \text{ by continuity of } f \text{ at } 0 \\ &= f(0,0) \\ &= b. \end{aligned}$$

So, we have f is a constant. That is, S is connected.

Assume that S is arcwise connected, then there exists a continuous function $g : [0,1] \to S$ such that $g(0) = (0,0)$ and $g(1) = (1,\sin 1)$. Given $\varepsilon = 1/2$, there exists a $\delta > 0$ such that as $|t| < \delta$, we have

$$\|g(t) - g(0)\| = \|g(t)\| < 1/2. \qquad *$$

Let N be a positive integer so that $\frac{1}{2N\pi} < \delta$, thus let $(\frac{1}{2N\pi}, 0) := p$ and $\left(\frac{1}{2(N+1)\pi}, 0\right) := q$. Define two subsets U and V as follows:

$$U = \left\{(x,y) : x > \frac{p+q}{2}\right\} \cap g([q,p])$$

$$V = \left\{(x,y) : x < \frac{p+q}{2}\right\} \cap g([q,p])$$

Then we have

(1). $U \cup V = g([q,p])$, (2). $U \neq \phi$, since $p \in U$ and $V \neq \phi$, since $q \in V$,

(3). $U \cap V = \phi$ by the given set A, and (*)

Since $\{(x,y) : x > \frac{p+q}{2}\}$ and $\{(x,y) : x < \frac{p+q}{2}\}$ are open in R^2, then U and V are open in $g([q,p])$. So, we have

(4). U is open in $g([q,p])$ and V is open in $g([q,p])$.

From (1)-(4), we have $g([q,p])$ is disconnected which is absurb since a connected subset under a continuous function is connected. So, such g cannot exist. It means that S is not arcwise connected.

Remark: This exercise gives us an example to say that **connectedness does not imply path-connectedness.** And it is important example which is worth keeping in mind.

4.47 Let $F = \{F_1, F_2, \ldots\}$ be a countable collection of connected compact sets in R^n such that $F_{k+1} \subseteq F_k$ for each $k \geq 1$. Prove that the intersection $\bigcap_{k=1}^{\infty} F_k$ is connected and closed.

Proof: Since F_k is compact for each $k \geq 1$, F_k is closed for each $k \geq 1$. Hence, $\bigcap_{k=1}^{\infty} F_k := F$ is closed. Note that by **Theorem 3.39**, we know that F is compact. Assume that F is not connected. Then there are two subsets A and B with

$$1. A \neq \phi,\ B \neq \phi.\ 2.\ A \cap B = \phi.\ 3.\ A \cup B = F.\ 4.\ A, B \text{ are closed in } F.$$

Note that A, B are closed and disjoint in R^n. By exercise 4.57, there exist U and V which are open and disjoint in R^n such that $A \subseteq U$, and $B \subseteq V$. Claim that there exists F_k such that $F_k \subseteq U \cup V$. Suppose **NOT**, then there exists $x_k \in F_k - (U \cup V.)$ Without loss of generality, we may assume that $x_k \notin F_{k+1}$. So, we have a sequence $\{x_k\} \subseteq F_1$ which implies that there exists a convergent subsequence $\{x_{k(n)}\}$, say $\lim_{k(n)\to\infty} x_{k(n)} = x$. It is clear that $x \in F_k$ for all k since x is an accumulation point of each F_k. So, we have

$$x \in F = \cap_{k=1}^{\infty} F_k = A \cup B \subseteq U \cup V$$

which implies that x is an interior point of $U \cup V$ since U and V are open. So, $B(x;\delta) \subseteq U \cup V$ for some $\delta > 0$, which contradicts to the choice of x_k. Hence, we have proved that there exists F_k such that $F_k \subseteq U \cup V$. Let $C = U \cap F_k$, and $D = V \cap F_k$, then we have

1. $C \neq \phi$ since $A \subseteq U$ and $A \subseteq F_k$, and $D \neq \phi$ since $B \subseteq V$ and $B \subseteq F_k$.
2. $C \cap D = (U \cap F_k) \cap (V \cap F_k) \subseteq U \cap V = \phi$.
3. $C \cup D = (U \cap F_k) \cup (V \cap F_k) = F_k$.
4. C is open in F_k and D is open in F_k by C, D are open in R^n.

Hence, we have F_k is disconnected which is absurb. So, we know that $F = \cap_{k=1}^{\infty} F_k$ is connected.

4.48 Let S be an open connected set in R^n. Let T be a component of $R^n - S$. Prove that $R^n - T$ is connected.

Proof: If S is empty, there is nothing to proved. Hence, we assume that S is nonempty. Write $R^n - S = \cup_{x \in R^n - S} U(x)$, where $U(x)$ is a component of $R^n - S$. So, we have

$$R^n = S \cup (\cup_{x \in R^n - S} U(x)).$$

Say $T = U(p)$, for some p. Then

$$R^n - T = S \cup (\cup_{x \in R^n - S - T} U(x)).$$

Claim that $cl(S) \cap U(x) \neq \phi$ for all $x \in R^n - S - T$. If we can show the claim, given $a, b \in R^n - T$, and a two valued function on $R^n - T$. Note that $cl(S)$ is also connected. We consider three cases. (1) $a \in S$, $b \in U(x)$ for some x. (2) $a, b \in S$. (3) $a \in U(x)$, $b \in U(x')$.

For case (1), let $c \in cl(S) \cap U(x)$, then there are $\{s_n\} \subseteq S$ and $\{u_n\} \subseteq U(x)$ with $s_n \to c$ and $u_n \to c$, then we have

$$f(a) = \lim_{n\to\infty} f(s_n) = f\left(\lim_{n\to\infty} s_n\right) = f(c) = f\left(\lim_{n\to\infty} u_n\right) = \lim_{n\to\infty} f(u_n) = f(b)$$

which implies that $f(a) = f(b)$.

For case (2), it is clear $f(a) = f(b)$ since S itself is connected.

For case (3), we choose $s \in S$, and thus use case (1), we know that

$$f(a) = f(s) = f(b).$$

By case (1)-(3), we have f is constant on $R^n - T$. That is, $R^n - T$ is connected.

It remains to show the claim. To show $cl(S) \cap U(x) \neq \phi$ for all $x \in R^n - S - T$, i.e., to show that for all $x \in R^n - S - T$,

$$\begin{aligned} cl(S) \cap U(x) &= (S \cup S') \cap U(x) \\ &= S' \cap U(x) \\ &\neq \phi. \end{aligned}$$

Suppose **NOT,** i.e., for some x, $S' \cap U(x) = \phi$ which implies that $U(x) \subseteq R^n - cl(S)$ which is open. So, there is a component V of $R^n - cl(S)$ contains $U(x)$, where V is open by **Theorem 4.44.** However, $R^n - cl(S) \subseteq R^n - S$, so we have V is contained in $U(x)$. Therefore, we have $U(x) = V$. Note that $U(x) \subseteq R^n - S$, and $R^n - S$ is closed. So, $cl(U(x)) \subseteq R^n - S$. By definition of component, we have $cl(U(x)) = U(x)$, which is closed. So, we have proved that $U(x) = V$ is open and closed. It implies that $U(x) = R^n$ or ϕ which is absurb. Hence, the claim has proved.

4.49 Let (S, d) be a connected metric space which is not bounded. Prove that for every

a in S and every $r > 0$, the set $\{x : d(x,a) = r\}$ is nonempty.

Proof: Assume that $\{x : d(x,a) = r\}$ is empty. Denote two sets $\{x : d(x,a) < r\}$ by A and $\{x : d(x,a) > r\}$ by B. Then we have

1. $A \neq \phi$ since $a \in A$ and $B \neq \phi$ since S is unbounded,
2. $A \cap B = \phi$,
3. $A \cup B = S$,
4. $A = B(a;r)$ is open in S,

and consider B as follows. Since $\{x : d(x,a) \leq r\}$ is closed in S, $B = S - \{x : d(x,a) \leq r\}$ is open in S. So, we know that S is disconnected which is absurb. Hence, we know that the set $\{x : d(x,a) = r\}$ is nonempty.

Supplement on a connected metric space

Definition Two subsets A and B of a metric space X are said to be separated if both

$$A \cap cl(B) = \phi \text{ and } cl(A) \cap B = \phi.$$

A set $E \subseteq X$ is said to be connected if E is not a union of two nonempty separated sets.

We now prove the definition of connected metric space is **equivalent** to this definiton as follows.

Theorem A set E in a metric space X is connected if, and only if E is not the union of two nonempty disjoint subsets, each of which is open in E.

Proof: ($\Rightarrow$) Suppose that E is the union of two nonempty disjoint subsets, each of which is open in E, denote two sets, U and V. Claim that

$$U \cap cl(V) = cl(U) \cap V = \phi.$$

Suppose **NOT**, i.e., $x \in U \cap cl(V)$. That is, there is a $\delta > 0$ such that

$$B_X(x,\delta) \cap E = B_E(x,\delta) \subseteq U \text{ and } B_X(x,\delta) \cap V \neq \phi$$

which implies that

$$\begin{aligned} B_X(x,\delta) \cap V &= B_X(x,\delta) \cap (V \cap E) \\ &= (B_X(x,\delta) \cap E) \cap V \\ &\subseteq U \cap V = \phi, \end{aligned}$$

a contradiction. So, we have $U \cap cl(V) = \phi$. Similarly for $cl(U) \cap V = \phi$. So, X is disconnected. That is, we have shown that if a set E in a metric space X is connected, then E is not the union of two nonempty disjoint subsets, each of which is open in E.

($\Leftarrow$) Suppose that E is disconnected, then E is a union of two nonempty separated sets, denoted $E = A \cup B$, where $A \cap cl(B) = cl(A) \cap B = \phi$. Claim that A and B are open in E. Suppose **NOT**, it means that there is a point $x (\in A)$ which is not an interior point of A. So, for any ball $B_E(x,r)$, there is a correspounding $x_r \in B$, where $x_r \in B_E(x,r)$. It implies that $x \in cl(B)$ which is absurb with $A \cap cl(B) = \phi$. So, we proved that A is open in E. Similarly, B is open in E. Hence, we have proved that if E is not the union of two nonempty disjoint subsets, each of which is open in E, then E in a metric space X is connected.

Exercise Let A and B be connected sets in a metric space with $A - B$ not connected and suppose $A - B = C_1 \cup C_2$ where $cl(C_1) \cap C_2 = C_1 \cap cl(C_2) = \phi$. Show that $B \cup C_1$ is connected.

Proof: Assume that $B \cup C_1$ is disconnected, and thus we will prove that C_1 is disconnected. Consider, by $cl(C_1) \cap C_2 = C_1 \cap cl(C_2) = \phi$,

$$C_1 \cap cl[C_2 \cup (A \cap B)] = C_1 \cap cl(A \cap B) \ (\subseteq C_1 \cap cl(B)) \qquad *$$

and

$$cl(C_1) \cap [C_2 \cup (A \cap B)] = cl(C_1) \cap (A \cap B) \ (\subseteq cl(C_1) \cap B) \qquad **$$

we know that at least one of (*) and (**) is nonempty by the hypothesis A is connected. In addition, by (*) and (**), we know that at leaset one of

$$C_1 \cap cl(B)$$

and

$$cl(C_1) \cap B$$

is nonempty. So, we know that C_1 is disconnected by the hypothesis B is connected, and the concept of two valued function.

From above sayings and hypothesis, we now have

1. B is connected.
2. C_1 is disconnected.
3. $B \cup C_1$ is disconnected.

Let D be a component of $B \cup C_1$ so that $B \subseteq D$; we have, let $(B \cup C_1) - D = E(\subseteq C_1)$,

$$D \cap cl(E) = cl(D) \cap E = \phi$$

which implies that

$$cl(E) \cap (A - E) = \phi, \text{ and } cl(A - E) \cap E = \phi.$$

So, we have prove that A is disconnected wich is absurb. Hence, we know that $B \cup C_1$ is connected.

Remark We prove that $cl(A - E) \cap E = cl(E) \cap (A - E) = \phi$ as follows.

Proof: Since

$$D \cap cl(E) = \phi,$$

we obtain that

$$\begin{aligned} & cl(E) \cap (A - E) \\ = \ & cl(E) \cap [(D \cup C_2) \cup (A \cap B)] \\ \subseteq \ & cl(E) \cap [(D \cup C_2) \cup B] \\ = \ & cl(E) \cap (D \cup C_2) \text{ since } B \subseteq D \\ = \ & cl(E) \cap C_2 \text{ since } D \cap cl(E) = \phi \\ \subseteq \ & cl(C_1) \cap C_2 \text{ since } E \subseteq C_1 \\ = \ & \phi. \end{aligned}$$

And since

$$cl(D) \cap E = \phi,$$

we obtain that

$$\begin{aligned}
& cl(A-E)\cap E \\
=\ & cl[(D\cup C_2)\cup(A\cap B)]\cap E \\
\subseteq\ & cl[(D\cup C_2)\cup B]\cap E \\
=\ & cl(D\cup C_2)\cap E \text{ since } B\subseteq D \\
=\ & cl(C_2)\cap E \text{ since } cl(D)\cap E=\phi \\
\subseteq\ & cl(C_2)\cap C_1 \text{ since } E\subseteq C_1 \\
=\ & \phi.
\end{aligned}$$

Exercise Prove that every connected metric space with at least two points is uncountable.

Proof: Let X be a connected metric space with two points a and b, where $a\neq b$. Define a set $A_r=\{x : d(x,a)>r\}$ and $B_r=\{x : d(x,a)<r\}$. It is clear that both of sets are open and disjoint. Assume X is countable. Let $r\in\left[\frac{d(a,b)}{4},\frac{d(a,b)}{2}\right]$, it guarantee that both of sets are non-empty. Since $\left[\frac{d(a,b)}{4},\frac{d(a,b)}{2}\right]$ is uncountable, we know that there is a $\delta>0$ such that $A_\delta\cup B_\delta=X$. It implies that X is disconnected. So, we know that such X is countable.

Uniform continuity

4.50 Prove that a function which is uniformly continuous on S is also continuous on S.

Proof: Let f be uniformly continuous on S, then given $\varepsilon>0$, there exists a $\delta>0$ such that as $d(x,y)<\delta$, x and y in S, then we have

$$d(f(x),f(y))<\varepsilon.$$

Fix y, called a. Then given $\varepsilon>0$, there exists a $\delta>0$ such that as $d(x,a)<\delta$, x in S, then we have

$$d(f(x),f(a))<\varepsilon.$$

That is, f is continuous at a. Since a is arbitrary, we know that f is continuous on S.

4.51 If $f(x)=x^2$ for x in R, prove that f is not uniformly continuous on R.

Proof: Assume that f is uniformly continuous on R, then given $\varepsilon=1$, there exists a $\delta>0$ such that as $|x-y|<\delta$, we have

$$|f(x)-f(y)|<1.$$

Choose $x=y+\frac{\delta}{2}$, $(\Rightarrow |x-y|<\delta)$, then we have

$$|f(x)-f(y)|=\left|\delta y+\left(\frac{\delta}{2}\right)^2\right|<1.$$

When we choose $y=\frac{1}{\delta}$, then

$$\left|1+\left(\frac{\delta}{2}\right)^2\right|=1+\left(\frac{\delta}{2}\right)<1$$

which is absurb. Hence, we know that f is not uniformly continuous on R.

Remark: There are some similar questions written below.

1. Here is a useful lemma to make sure that a function is uniformly continuous on (a,b), but we need its differentiability.

(**Lemma**) Let $f:(a,b)\subseteq R\to R$ be differentiable and $|f'(x)|\leq M$ for all $x\in(a,b)$. Then f is uniformly continuous on (a,b), where a,b may be $\pm\infty$.

Proof: By **Mean Value Theorem**, we have

$$|f(x) - f(y)| = |f'(z)||x - y|, \text{ where } z \in (x,y) \text{ or } (y,x)$$
$$\leq M|x - y| \text{ by hypothesis.} \qquad *$$

Then given $\varepsilon > 0$, there is a $\delta = \varepsilon/M$ such that as $|x - y| < \delta$, $x,y \in (a,b)$, we have

$$|f(x) - f(y)| < \varepsilon, \text{ by } (*).$$

Hence, we know that f is uniformly continuous on (a,b).

Note: A standard example is written in Remark 2. But in Remark 2, we still use definition of uniform continuity to practice what it says.

2. $\sin x$ is uniformly continuous on R.

Proof: Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $|x - y| < \delta$, we have

$$|\sin x - \sin y| < \varepsilon.$$

Since $\sin x - \sin y = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$, $|\sin x| \leq |x|$, and $|\cos x| \leq 1$, we have

$$|\sin x - \sin y| \leq |x - y|$$

So, if we choose $\delta = \varepsilon$, then as $|x - y| < \delta$, it implies that

$$|\sin x - \sin y| < \varepsilon.$$

That is, $\sin x$ is uniformly continuous on R.

Note: $|\sin x - \sin y| \leq |x - y|$ for all $x,y \in R$, can be proved by **Mean Value Theorem** as follows.

proof: By **Mean Value Theorem,** $\sin x - \sin y = (\sin z)'(x - y)$; it implies that

$$|\sin x - \sin y| \leq |x - y|.$$

3. $\sin(x^2)$ is **NOT** uniformly continuous on R.

Proof: Assume that $\sin(x^2)$ is uniformly continuous on R. Then given $\varepsilon = 1$, there is a $\delta > 0$ such that as $|x - y| < \delta$, we have

$$|\sin(x^2) - \sin(y^2)| < 1. \qquad *$$

Consider

$$\sqrt{n\pi + \frac{\pi}{2}} - \sqrt{n\pi} = \frac{\frac{\pi}{2}}{\sqrt{n\pi + \frac{\pi}{2}} + \sqrt{n\pi}} < \frac{\pi}{4\sqrt{n\pi}} (\to 0),$$

and thus choose $N = \left[\frac{\pi}{(4\delta)^2}\right] + 1\left(> \frac{\pi}{(4\delta)^2}\right)$ which implies

$$\sqrt{N\pi + \frac{\pi}{2}} - \sqrt{N\pi} < \delta.$$

So, choose $x = \sqrt{N\pi + \frac{\pi}{2}}$ and $y = \sqrt{N\pi}$, then by (*), we have

$$|f(x) - f(y)| = \left|\sin\left(N\pi + \frac{\pi}{2}\right) - \sin(N\pi)\right| = \left|-\sin\frac{\pi}{2}\right| = 1 < 1$$

which is absurb. So, $\sin(x^2)$ is not uniformly continuous on R.

4. $\sqrt{x}$ is uniformly continuous on $[0,\infty)$.

Proof: Since $\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|}$ for all $x,y \in [0,\infty)$, then given $\varepsilon > 0$, there exists a $\delta = \varepsilon^2$ such that as $|x - y| < \delta$, $x,y \in [0,\infty)$, we have

$$\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|} < \sqrt{\delta} = \varepsilon.$$

So, we know that $\sqrt{x}$ is uniformly continuous on $[0,\infty)$.

Note: We have the following interesting results:. Prove that, for $x \geq 0,\ y \geq 0,$

$$|x^p - y^p| \leq \begin{cases} |x-y|^p \text{ if } 0 < p < 1, \\ p|x-y|(x^{p-1} + y^{p-1}) \text{ if } 1 \leq p < \infty. \end{cases}$$

Proof: (As $0 < p < 1$) Without loss of generality, let $x \geq y,$ consider $f(x) = (x-y)^p - x^p + y^p,$ then

$$f'(x) = p\left[(x-y)^{p-1} - x^{p-1}\right] \geq 0, \text{ note that } p-1 < 0.$$

So, we have f is an increasing function defined on $[0,\infty)$ for all given $y \geq 0.$ Hence, we have $f(x) \geq f(0) = 0.$ So,

$$x^p - y^p \leq (x-y)^p \text{ if } x \geq y \geq 0$$

which implies that

$$|x^p - y^p| \leq |x-y|^p$$

for $x \geq 0,\ y \geq 0.$

Ps: The inequality, we can prove the case $p = 1/2$ directly. Thus the inequality is not surprising for us.

(As $1 \leq p < \infty$) Without loss of generality, let $x \geq y,$ consider

$$\begin{aligned} x^p - y^p &= (pz^{p-1})(x-y), \text{ where } z \in (y,x), \text{ by } \textbf{Mean Value Theorem}. \\ &\leq px^{p-1}(x-y), \text{ note that } p-1 \geq 0, \\ &\leq p(x^{p-1} + y^{p-1})(x-y) \end{aligned}$$

which implies

$$|x^p - y^p| \leq p|x-y|(x^{p-1} + y^{p-1})$$

for $x \geq 0,\ y \geq 0.$

5. In general, we have

$$x^r \begin{cases} \text{is uniformly continuous on } [0,\infty), \text{ if } r \in [0,1], \\ \text{is } \textbf{NOT} \text{ uniformly continuous on } [0,\infty), \text{ if } r > 1, \end{cases}$$

and

$$\sin(x^r) = \begin{cases} \text{is uniformly continuous on } [0,\infty), \text{ if } r \in [0,1], \\ \text{is } \textbf{NOT} \text{ uniformly continuous on } [0,\infty), \text{ if } r > 1. \end{cases}$$

Proof: (x^r) As $r = 0,$ it means that x^r is a constant function. So, it is obviuos. As $r \in (0,1],$ then given $\varepsilon > 0,$ there is a $\delta = \varepsilon^{1/r} > 0$ such that as $|x-y| < \delta,\ x,y \in [0,\infty),$ we have

$$\begin{aligned} |x^r - y^r| &\leq |x-y|^r \text{ by note in the exercise} \\ &< \delta^r \\ &= \varepsilon. \end{aligned}$$

So, x^r is uniformly continuous on $[0,\infty),$ if $r \in [0,1].$

As $r > 1,$ assume that x^r is uniformly continuous on $[0,\infty),$ then given $\varepsilon = 1 > 0,$ there exists a $\delta > 0$ such that as $|x-y| < \delta,\ x,y \in [0,\infty),$ we have

$$|x^r - y^r| < 1. \qquad *$$

By **Mean Value Theorem,** we have (let $x = y + \delta/2,\ y > 0$)

$$\begin{aligned} x^r - y^r &= rz^{r-1}(x-y) \\ &\geq ry^{r-1}(\delta/2). \end{aligned}$$

So, if we choose $y \geq (\frac{2}{r\delta})^{\frac{1}{r-1}}$, then we have

$$x^r - y^r \geq 1$$

which is absurb with (*). Hence, x^r is not uniformly continuous on $[0,\infty)$.

Ps: The reader should try to realize why x^r is not uniformly continuous on $[0,\infty)$, for $r > 1$. The ruin of non-uniform continuity comes from that x is large enough. At the same time, compare it with theorem that a continuous function defined on a compact set K is uniflormly continuous on K.

($\sin x^r$) As $r = 0$, it means that x^r is a constant function. So, it is obviuos. As $r \in (0,1]$, given $\varepsilon > 0$, there is a $\delta = \varepsilon^{1/r} > 0$ such that as $|x-y| < \delta$, $x,y \in [0,\infty)$, we have

$$\begin{aligned} |\sin x^r - \sin y^r| &= \left|2\cos\left(\frac{x^r+y^r}{2}\right)\sin\left(\frac{x^r-y^r}{2}\right)\right| \\ &\leq |x^r - y^r| \\ &\leq |x-y|^r \text{ by the note in the Remark 4.} \\ &< \delta^r \\ &= \varepsilon. \end{aligned}$$

So, $\sin x^r$ is uniformly continuous on $[0,\infty)$, if $r \in [0,1]$.

As $r > 1$, assume that $\sin x^r$ is uniformly continuous on $[0,\infty)$, then given $\varepsilon = 1$, there is a $\delta > 0$ such that as $|x-y| < \delta$, $x,y \in [0,\infty)$, we have

$$|\sin x^r - \sin y^r| < 1. \qquad **$$

Consider a sequence $\left\{(n\pi + \frac{\pi}{2})^{1/r} - (n\pi)^{1/r}\right\}$, it is easy to show that the sequence tends to 0 as $n \to \infty$. So, there exists a positive integer N such that $|x-y| < \delta$, $x = (n\pi + \frac{\pi}{2})^{1/r}$, $y = (n\pi)^{1/r}$. Then

$$\sin x^r - \sin y^r = 1$$

which contradicts (**). So, we know that $\sin x^r$ is not uniformly continuous on $[0,\infty)$.

Ps: For $\left\{(n\pi + \frac{\pi}{2})^{1/r} - (n\pi)^{1/r}\right\} := x_n \to 0$ as $n \to 0$, here is a short proof by using **L-Hospital Rule.**

Proof: Write

$$\begin{aligned} x_n &= \left(n\pi + \frac{\pi}{2}\right)^{1/r} - (n\pi)^{1/r} \\ &= (n\pi)^{1/r}\left[\left(1 + \frac{1}{2n}\right)^{1/r} - 1\right] \\ &= \frac{\left[(1+\frac{1}{2n})^{1/r} - 1\right]}{(n\pi)^{-1/r}} \end{aligned}$$

and thus consider the following limit

$$\lim_{x\to\infty}\frac{\left[\left(1+\frac{1}{2x}\right)^{1/r}-1\right]}{(x\pi)^{-1/r}},\ \left(\frac{0}{0}\right)$$
$$=\lim_{x\to\infty}\frac{\pi^{1/r}}{2}x^{\frac{1}{r}-1}\left(1+\frac{1}{2x}\right)^{\frac{1}{r}-1} \text{ by } \textbf{L-Hospital Rule.}$$
$$=0.$$
Hence $x_n \to 0$ as $n \to \infty$.

6. Here is a useful criterion for a function which is **NOT** uniformly continuous defined a subset A in a metric space. We say a function f is not uniformly continuous on a subset A in a metric space if, and only if, there exists $\varepsilon_0 > 0$, and two sequences $\{x_n\}$ and $\{y_n\}$ such that as
$$\lim_{n\to\infty} x_n - y_n = 0$$
which implies that
$$|f(x_n) - f(y_n)| \geq \varepsilon_0 \text{ for } n \text{ is large enough.}$$
The criterion is directly from the definition on uniform continuity. So, we omit the proof.

4.52 Assume that f is uniformly continuous on a bounded set S in R^n. Prove that f must be bounded on S.

Proof: Since f is uniformly continuous on a bounded set S in R^n, given $\varepsilon = 1$, then there exists a $\delta > 0$ such that as $\|x-y\| < \delta$, $x,y \in S$, we have
$$d(f(x),f(y)) < 1.$$
Consider the closure of S, $cl(S)$ is closed and bounded. Hence $cl(S)$ is compact. Then for any open covering of $cl(S)$, there is a finite subcover. That is,
$$cl(S) \subseteq \cup_{x\in cl(S)} B(x;\delta/2),$$
$$\Rightarrow cl(S) \subseteq \cup_{k=1}^{k=n} B(x_k;\delta/2), \text{ where } x_k \in cl(S),$$
$$\Rightarrow S \subseteq \cup_{k=1}^{k=n} B(x_k;\delta/2), \text{ where } x_k \in cl(S).$$
Note that if $B(x_k;\delta/2) \cap S = \phi$ for some k, then we remove this ball. So, we choose $y_k \in B(x_k;\delta/2) \cap S$, $1 \leq k \leq n$ and thus we have
$$B(x_k;\delta/2) \subseteq B(y_k;\delta) \text{for } 1 \leq k \leq n,$$
since let $z \in B(x_k;\delta/2)$,
$$\|z-y_k\| \leq \|z-x_k\| + \|x_k-y_k\| < \delta/2 + \delta/2 = \delta.$$
Hence, we have
$$S \subseteq \cup_{k=1}^{k=n} B(y_k;\delta), \text{ where } y_k \in S.$$
Given $x \in S$, then there exists $B(y_k;\delta)$ for some k such that $x \in B(y_k;\delta)$. So,
$$d(f(x),f(x_k)) < 1 \Rightarrow f(x) \in B(f(y_k);1)$$
Note that $\cup_{k=1}^{k=n} B(f(y_k);1)$ is bounded since every $B(f(y_k);1)$ is bounded. So, let B be a bounded ball so that $\cup_{k=1}^{k=n} B(f(y_k);1) \subseteq B$. Hence, we have every $x \in S$, $f(x) \in B$. That is, f is bounded.

Remark: If we know that the codomain is complete, then we can reduce the above proof. See Exercise 4.55.

4.53 Let f be a function defined on a set S in R^n and assume that $f(S) \subseteq R^m$. Let g be defined on $f(S)$ with value in R^k, and let h denote the composite function defined by

$h(x) = g[f(x)]$ if $x \in S$. If f is uniformly continuous on S and if g is uniformly continuous on $f(S)$, show that h is uniformly continuous on S.

Proof: Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $\|x - y\|_{R^n} < \delta$, $x, y \in S$, we have

$$\|h(x) - h(y)\| = \|g(f(x)) - g(f(y))\| < \varepsilon.$$

For the same ε, since g is uniformly continuous on $f(S)$, then there exists a $\delta' > 0$ such that as $\|f(x) - f(y)\|_{R^m} < \delta'$, we have

$$\|g(f(x)) - g(f(y))\| < \varepsilon.$$

For this δ', since f is uniformly continuous on S, then there exists a $\delta > 0$ such that as $\|x - y\|_{R^n} < \delta$, $x, y \in S$, we have

$$\|f(x) - f(y)\|_{R^m} < \delta'.$$

So, given $\varepsilon > 0$, there is a $\delta > 0$ such that as $\|x - y\|_{R^n} < \delta$, $x, y \in S$, we have

$$\|h(x) - h(y)\| < \varepsilon.$$

That is, h is uniformly continuous on S.

Remark: It should be noted that (Assume that all functions written are continuous)

(1) (uniform continuity) $\circ$ (uniform continuity) $=$ uniform continuity.

(2) (uniform continuity) $\circ$ (**NOT** uniform continuity) $=$ (a) **NOT** uniform continuity, or (b) uniform continuity.

(3) (**NOT** uniform continuity) $\circ$ (uniform continuity) $=$ (a) **NOT** uniform continuity, or (b) uniform continuity.

(4) (**NOT** uniform continuity) $\circ$ (**NOT** uniform continuity) $=$ (a) **NOT** uniform continuity, or (b) uniform continuity.

For (1), it is from the exercise.

For (2), (a) let $f(x) = x$, and $g(x) = x^2$, $x \in R \Rightarrow f(g(x)) = f(x^2) = x^2$.
(b) let $f(x) = \sqrt{x}$, and $g(x) = x^2$, $x \in [0, \infty) \Rightarrow f(g(x)) = f(x^2) = x$.

For (3), (a) let $f(x) = x^2$, and $g(x) = x$, $x \in R \Rightarrow f(g(x)) = f(x) = x^2$.
(b) let $f(x) = x^2$, and $g(x) = \sqrt{x}$, $x \in [0, \infty) \Rightarrow f(g(x)) = f(\sqrt{x}) = x$.

For (4), (a) let $f(x) = x^2$, and $g(x) = x^3$, $x \in R \Rightarrow f(g(x)) = f(x^3) = x^6$.
(b) let $f(x) = 1/x$, and $g(x) = \frac{1}{\sqrt{x}}, x \in (0,1) \Rightarrow f(g(x)) = f\left(\frac{1}{\sqrt{x}}\right) = \sqrt{x}$.

Note. In (4), we have x^r is not uniformly continuous on $(0,1)$, for $r < 0$. Here is a proof.

Proof: Let $r < 0$, and assume that x^r is not uniformly continuous on $(0,1)$. Given $\varepsilon = 1$, there is a $\delta > 0$ such that as $|x - y| < \delta$, we have

$$|x^r - y^r| < 1. \quad *$$

Let $x_n = 2/n$, and $y_n = 1/n$. Then $x_n - y_n = 1/n$. Choose n large enough so that $1/n < \delta$. So, we have

$$|x^r - y^r| = \left|\left(\frac{2}{n}\right)^r - \left(\frac{1}{n}\right)^r\right|$$
$$= \left(\frac{1}{n}\right)^r |2^r - 1| \to \infty, \text{ as } n \to \infty \text{ since } r < 0,$$
which is absurb with (*). Hence, we know that x^r is not uniformly continuous on $(0,1)$, for $r < 0$.

Ps: The reader should try to realize why x^r is not uniformly continuous on $(0,1)$, for $r < 0$. The ruin of non-uniform continuity comes from that x is small enough.

4.54 Assume $f : S \to T$ is uniformly continuous on S, where S and T are metric spaces. If $\{x_n\}$ is any Cauchy sequence in S, prove that $\{f(x_n)\}$ is a Cauchy sequence in T. (Compare with Exercise 4.33.)

Proof: Given $\varepsilon > 0$, we want to find a positive integer N such that as $n, m \geq N$, we have
$$d(f(x_n), f(x_m)) < \varepsilon.$$
For the same ε, since f is uniformly continuous on S, then there is a $\delta > 0$ such that as $d(x,y) < \delta$, $x, y \in S$, we have
$$d(f(x), f(y)) < \varepsilon.$$
For this δ, since $\{x_n\}$ is a Cauchy sequence in S, then there is a positive integer N such that as $n, m \geq N$, we have
$$d(x_n, x_m) < \delta.$$
Hence, given $\varepsilon > 0$, there is a postive integer N such that as $n, m \geq N$, we have
$$d(f(x_n), f(x_m)) < \varepsilon.$$
That is, $\{f(x_n)\}$ is a Cauchy sequence in T.

Remark: The reader should compare with Exercise 4.33 and Exercise 4.55.

4.55 Let $f : S \to T$ be a function from a metric space S to another metric space T. Assume that f is uniformly continuous on a subset A of S and let T is complete. Prove that there is a unique extension of f to $cl(A)$ which is uniformly continuous on $cl(A)$.

Proof: Since $cl(A) = A \cup A'$, it suffices to consider the case $x \in A' - A$. Since $x \in A' - A$, then there is a sequence $\{x_n\} \subseteq A$ with $x_n \to x$. Note that this sequence is a Cauchy sequence, so we have by Exercise 4.54, $\{f(x_n)\}$ is a Cauchy sequence in T since f is uniformly on A. In addition, since T is complete, we know that $\{f(x_n)\}$ is a convergent sequence, say its limit L. Note that if there is another sequence $\{\tilde{x}_n\} \subseteq A$ with $\tilde{x}_n \to x$, then $\{f(\tilde{x}_n)\}$ is also a convergent sequence, say its limit L'. Note that $\{x_n\} \cup \{\tilde{x}_n\}$ is still a Cauchy sequence. So, we have
$$d(L, L') \leq d(L, f(x_n)) + d(f(x_n), f(\tilde{x}_n)) + d(f(\tilde{x}_n), L') \to 0 \text{ as } n \to \infty.$$
So, $L = L'$. That is, it is well-defined for $g : cl(A) \to T$ by the following
$$g(x) = \begin{cases} f(x) \text{ if } x \in A, \\ \lim_{n\to\infty} f(x_n) \text{ if } x \in A' - A, \text{ where } x_n \to x. \end{cases}$$
So, the function g is a extension of f to $cl(A)$.

Claim that this g is uniformly continuous on $cl(A)$. That is, given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $d(x,y) < \delta$, $x, y \in cl(A)$, we have
$$d(g(x), g(y)) < \varepsilon.$$
Since f is uniformly continuous on A, for $\varepsilon' = \varepsilon/3$, there is a $\delta' > 0$ such that as

$d(x,y) < \delta'$, $x,y \in A$, we have

$$d(f(x),f(y)) < \varepsilon'.$$

Let $x,y \in cl(A)$, and thus we have $\{x_n\} \subseteq A$ with $x_n \to x$, and $\{y_n\} \subseteq A$ with $y_n \to y$. Choose $\delta = \delta'/3$, then we have

$$d(x_n,x) < \delta'/3 \text{ and } d(y_n,y) < \delta'/3 \text{ as } n \geq N_1$$

So, as $d(x,y) < \delta = \delta'/3$, we have $(n \geq N_1)$

$$d(x_n,y_n) \leq d(x_n,x) + d(x,y) + d(y,y_n) < \delta'/3 + \delta'/3 + \delta'/3 = \delta'.$$

Hence, we have as $d(x,y) < \delta$, $(n \geq N_1)$

$$\begin{aligned} d(g(x),g(y)) &\leq d(g(x),f(x_n)) + d(f(x_n),f(y_n)) + d(f(y_n),f(y)) \\ &< d(g(x),f(x_n)) + \varepsilon' + d(f(y_n),g(y)) \end{aligned} \quad *$$

And since $\lim_{n\to\infty} f(x_n) = g(x)$, and $\lim_{n\to\infty} f(y_n) = g(y)$, we can choose $N \geq N_1$ such that

$$d(g(x),f(x_n)) < \varepsilon' \text{ and}$$
$$d(f(y_n),g(y)) < \varepsilon'.$$

So, as $d(x,y) < \delta$, $(n \geq N)$ we have

$$d(g(x),g(y)) < 3\varepsilon' = \varepsilon \text{ by } (*).$$

That is, g is uniformly on $cl(A)$.

It remains to show that g is a unique extension of f to $cl(A)$ which is uniformly continuous on $cl(A)$. If there is another extension h of f to $cl(A)$ which is uniformly continuous on $cl(A)$, then given $x \in A' - A$, we have, by continuity, (Say $x_n \to x$)

$$h(x) = h\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g\left(\lim_{n\to\infty} x_n\right) = g(x)$$

which implies that $h(x) = g(x)$ for all $x \in A' - A$. Hence, we have $h(x) = g(x)$ for all $x \in cl(A)$. That is, g is a unique extension of f to $cl(A)$ which is uniformly continuous on $cl(A)$.

Remark: 1. We do not require that A is bounded, in fact, A is any non-empty set in a metric space.

2. The exercise is a criterion for us to check that a given function is **NOT** uniformly continuous. For example, let $f : (0,1) \to R$ by $f(x) = 1/x$. Since $f(0+)$ does not exist, we know that f is not uniformly continuous. The reader should feel that a uniformly continuous is sometimes regarded as a **smooth** function. So, it is not surprising for us to know the exercise. Similarly to check $f(x) = x^2, x \in R$, and so on.

3. Here is an exercise to make us know that a uniformly continuous is a **smooth** function. Let $f : R \to R$ be uniformly continuous, then there exist $\alpha, \beta > 0$ such that

$$|f(x)| \leq \alpha|x| + \beta.$$

Proof: Since f is uniformly continuous on R, given $\varepsilon = 1$, there is a $\delta > 0$ such that as $|x - y| < \delta$, we have

$$|f(x) - f(y)| < 1. \quad *$$

Given any $x \in R$, then there is the positive integer N such that $N\delta > |x| > (N-1)\delta$. If $x > 0$, we consider

$$y_0 = 0,\ y_1 = \delta/2,\ y_2 = \delta, \ldots, y_{2N-1} = N\delta - \frac{\delta}{2}, y_{2N} = x.$$

Then we have

$$|f(x) - f(0)| \leq \sum_{k=1}^{N} |f(y_{2k}) - f(y_{2k-1})| + |f(y_{2k-1}) - f(y_{2k-2})|$$
$$\leq 2N \text{ by } (*)$$

which implies that

$$|f(x)| \leq 2N + |f(0)|$$
$$\leq 2\left(1 + \frac{|x|}{\delta}\right) + |f(0)| \text{ since } |x| > (N-1)\delta$$
$$\leq \frac{2}{\delta}|x| + (2 + |f(0)|).$$

Similarly for $x < 0$. So, we have proved that $|f(x)| \leq \alpha|x| + \beta$ for all x.

4.56 In a metric space (S, d), let A be a nonempty subset of S. Define a function $f_A : S \to R$ by the equation

$$f_A(x) = \inf\{d(x,y) : y \in A\}$$

for each x in S. The number $f_A(x)$ is called the distance from x to A.

(a) Prove that f_A is uniformly continuous on S.

(b) Prove that $cl(A) = \{x : x \in S \text{ and } f_A(x) = 0\}$.

Proof: (a) Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $d(x_1, x_2) < \delta$, $x_1, x_2 \in S$, we have

$$|f_A(x_1) - f_A(x_2)| < \varepsilon.$$

Consider $(x_1, x_2, y \in S)$

$$d(x_1, y) \leq d(x_1, x_2) + d(x_2, y), \text{ and } d(x_2, y) \leq d(x_1, x_2) + d(x_1, y)$$

So,

$$\inf\{d(x_1, y) : y \in A\} \leq d(x_1, x_2) + \inf\{d(x_2, y) : y \in A\} \text{ and}$$
$$\inf\{d(x_2, y) : y \in A\} \leq d(x_1, x_2) + \inf\{d(x_1, y) : y \in A\}$$

which implies that

$$f_A(x_1) - f_A(x_2) \leq d(x_1, x_2) \text{ and } f_A(x_2) - f_A(x_1) \leq d(x_1, x_2)$$

which implies that

$$|f_A(x_1) - f_A(x_2)| \leq d(x_1, x_2).$$

Hence, if we choose $\delta = \varepsilon$, then we have as $d(x_1, x_2) < \delta$, $x_1, x_2 \in S$, we have

$$|f_A(x_1) - f_A(x_2)| < \varepsilon.$$

That is, f_A is uniformly continuous on S.

(b) Define $K = \{x : x \in S \text{ and } f_A(x) = 0\}$, we want to show $cl(A) = K$. We prove it by two steps.

($\subseteq$) Let $x \in cl(A)$, then $B(x; r) \cap A \neq \phi$ for all $r > 0$. Choose $y_k \in B(x; 1/k) \cap A$, then we have

$$\inf\{d(x,y) : y \in A\} \leq d(x, y_k) \to 0 \text{ as } k \to \infty.$$

So, we have $f_A(x) = \inf\{d(x,y) : y \in A\} = 0$. So, $cl(A) \subseteq K$.

($\supseteq$) Let $x \in K$, then $f_A(x) = \inf\{d(x,y) : y \in A\} = 0$. That is, given any $\varepsilon > 0$, there is an element $y_\varepsilon \in A$ such that $d(x, y_\varepsilon) < \varepsilon$. That is, $y_\varepsilon \in B(x; \varepsilon) \cap A$. So, x is an adherent point of A. That is, $x \in cl(A)$. So, we have $K \subseteq cl(A)$.

From above saying, we know that $cl(A) = \{x : x \in S \text{ and } f_A(x) = 0\}$.

Remark: 1. The function f_A often appears in Analysis, so it is worth keeping it in mind.

In addition, part (b) comes from intuition. The reader may think it twice about distance 0.

2. Here is a good exercise to pratice. The statement is that suppose that K and F are disjoint subsets in a metric space X, K is compact, F is closed. Prove that there exists a $\delta > 0$ such that $d(p,q) > \delta$ if $p \in K$, $q \in F$. Show that the conclusion is may fail for two disjoint closed sets if neither is compact.

Proof: Suppose **NOT**, i.e., for any $\delta > 0$, there exist $p_\delta \in K$, and $q_\delta \in F$ such that $d(p_\delta, q_\delta) \leq \delta$. Let $\delta = 1/n$, then there exist two sequence $\{p_n\} \subseteq K$, and $\{q_n\} \subseteq F$ such that $d(p_n, q_n) \leq 1/n$. Note that $\{p_n\} \subseteq K$, and K is compact, then there exists a subsequence $\{p_{n_k}\}$ with $\lim_{n_k \to \infty} p_{n_k} = p \in K$. Hence, we consider $d(p_{n_k}, q_{n_k}) \leq \frac{1}{n_k}$ to get a contradiction. Since

$$d(p_{n_k}, p) + d(p, q_{n_k}) \leq d(p_{n_k}, q_{n_k}) \leq \frac{1}{n_k},$$

then let $n_k \to \infty$, we have $\lim_{n_k \to \infty} q_{n_k} = p$. That is, p is an accumulation point of F which implies that $p \in F$. So, we get a contradiction since $K \cap F = \phi$. That is, there exists a $\delta > 0$ such that $d(p,q) > \delta$ if $p \in K$, $q \in F$.

We give an example to show that the conclusion does not hold. Let $K = \{(x,0) : x \in R\}$ and $F = \{(x, 1/x) : x > 0\}$, then K and F are closd. It is clear that such δ cannot be found.

Note: Two disjoint closed sets may has the distance 0, however; if one of closed sets is compact, then we have a distance $\delta > 0$. The reader can think of them in R^n, and note that a bounded and closed subsets in R^n is compact. It is why the example is given.

4.57 In a metric space (S,d), let A and B be disjoint closed subsets of S. Prove that there exists disjoint open subsets U and V of S such that $A \subseteq U$ and $B \subseteq V$. Hint. Let $g(x) = f_A(x) - f_B(x)$, in the notation of Exercise 4.56, and consider $g^{-1}(-\infty, 0)$ and $g^{-1}(0, +\infty)$.

Proof: Let $g(x) = f_A(x) - f_B(x)$, then by Exercixe 4.56, we have $g(x)$ is uniformly continuous on S. So, $g(x)$ is continuous on S. Consider $g^{-1}(-\infty, 0)$ and $g^{-1}(0, +\infty)$, and note that A, B are disjoint and closed, then we have by part (b) in Exercise 4.56,

$$g(x) < 0 \text{ if } x \in A \text{ and}$$

$$g(x) > 0 \text{ if } x \in B.$$

So, we have $A \subseteq g^{-1}(-\infty, 0) := U$, and $B \subseteq g^{-1}(0, +\infty) := V$.

Discontinuities

4.58 Locate and classify the discontinuities of the functions f defined on R^1 by the following equations:

(a) $f(x) = \sin x / x$ if $x \neq 0$, $f(0) = 0$.

Solution: f is continuous on $R - \{0\}$, and since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, we know that f has a removable discontinuity at 0.

(b) $f(x) = e^{1/x}$ if $x \neq 0$, $f(0) = 0$.

Solution: f is continuous on $R - \{0\}$, and since $\lim_{x \to 0^+} e^{1/x} = \infty$ and $\lim_{x \to 0^-} e^{1/x} = 0$, we know that f has an irremovable discontinuity at 0.

(c) $f(x) = e^{1/x} + \sin 1/x$ if $x \neq 0$, $f(0) = 0$.

Solution: f is continuous on $R - \{0\}$, and since the limit $f(x)$ does not exist as $x \to 0$, we know that f has a irremovable discontinuity at 0.

(d) $f(x) = 1/(1-e^{1/x})$ if $x \neq 0$, $f(0) = 0$.

Solution: f is continuous on $R - \{0\}$, and since $\lim_{x\to 0^+} e^{1/x} = \infty$ and $\lim_{x\to 0^-} e^{1/x} = 0$, we know that f has an irremovabel discontinuity at 0. In addition, $f(0+) = 0$ and $f(0-) = 1$, we know that f has the lefthand jump at 0, $f(0) - f(0-) = -1$, and f is continuous from the right at 0.

4.59 Locate the points in R^2 at which each of the functions in Exercise 4.11 is not continuous.

(a) By Exercise 4.11, we know that $f(x,y)$ is discontinuous at $(0,0)$, where

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2} \text{ if } (x,y) \neq (0,0), \text{ and } f(0,0) = 0.$$

Let $g(x,y) = x^2 - y^2$, and $h(x,y) = x^2 + y^2$ both defined on $R^2 - \{(0,0)\}$, we know that g and h are continuous on $R^2 - \{(0,0)\}$. Note that $h \neq 0$ on $R^2 - \{(0,0)\}$. Hence, $f = g/h$ is continuous on $R^2 - \{(0,0)\}$.

(b) By Exercise 4.11, we know that $f(x,y)$ is discontinuous at $(0,0)$, where

$$f(x,y) = \frac{(xy)^2}{(xy)^2 + (x-y)^2} \text{ if } (x,y) \neq (0,0), \text{ and } f(0,0) = 0.$$

Let $g(x,y) = (xy)^2$, and $h(x,y) = (xy)^2 + (x-y)^2$ both defined on $R^2 - \{(0,0)\}$, we know that g and h are continuous on $R^2 - \{(0,0)\}$. Note that $h \neq 0$ on $R^2 - \{(0,0)\}$. Hence, $f = g/h$ is continuous on $R^2 - \{(0,0)\}$.

(c) By Exercise 4.11, we know that $f(x,y)$ is continuous at $(0,0)$, where

$$f(x,y) = \frac{1}{x}\sin(xy) \text{ if } x \neq 0, \text{ and } f(0,y) = y,$$

since $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. Let $g(x,y) = 1/x$ and $h(x,y) = \sin(xy)$ both defined on $R^2 - \{(0,0)\}$, we know that g and h are continuous on $R^2 - \{(0,0)\}$. Note that $h \neq 0$ on $R^2 - \{(0,0)\}$. Hence, $f = g/h$ is continuous on $R^2 - \{(0,0)\}$. Hence, f is continuous on R^2.

(d) By Exercise 4.11, we know that $f(x,y)$ is continuous at $(0,0)$, where

$$f(x,y) = \begin{cases} (x+y)\sin(1/x)\sin(1/y) & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

since $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. It is the same method as in Exercise 4.11, we know that f is discontinuous at $(x,0)$ for $x \neq 0$ and f is discontinuous at $(0,y)$ for $y \neq 0$. And it is clearly that f is continuous at (x,y), where $x \neq 0$ and $y \neq 0$.

(e) By Exercise 4.11, Since

$$f(x,y) = \begin{cases} \frac{\sin x - \sin y}{\tan x - \tan y}, & \text{if } \tan x \neq \tan y, \\ \cos^3 x & \text{if } \tan x = \tan y. \end{cases},$$

we rewrite

$$f(x,y) = \begin{cases} \frac{\cos\left(\frac{x+y}{2}\right)\cos x \cos y}{\cos\left(\frac{x-y}{2}\right)} & \text{if } \tan x \neq \tan y \\ \cos^3 x & \text{if } \tan x = \tan y. \end{cases}.$$

We consider $(x,y) \in (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$, others are similar. Consider two cases (1) $x = y$, and (2) $x \neq y$, we have

(1) ($x = y$) Since $\lim_{(x,y)\to(a,a)} f(x,y) = \cos^3 a = f(a,a)$. Hence, we know that f is

continuous at (a,a).

(2) $(x \neq y)$ Since $x \neq y$, it implies that $\tan x \neq \tan y$. Note that the denominator is not 0 since $(x,y) \in (-\pi/2,\pi/2) \times (-\pi/2,\pi/2)$. So, we know that f is continuous at (a,b), $a \neq b$.

So, we know that f is continuous on $(-\pi/2,\pi/2) \times (-\pi/2,\pi/2)$.

Monotonic functions

4.60 Let f be defined in the open interval (a,b) and assume that for each interior point x of (a,b) there exists a 1 $-$ball $B(x)$ in which f is increasing. Prove that f is an increasing function throughout (a,b).

Proof: Suppose **NOT**, i.e., there exist p,q with $p < q$ such that $f(p) > f(q)$. Consider $[p,q](\subseteq (a,b))$, and since for each interior point x of (a,b) there exists a 1 $-$ball $B(x)$ in which f is increasing. Then $[p,q] \subseteq \cup_{x\in[p,q]} B(x;\delta_x)$, (The choice of balls comes from the hypothesis). It implies that $[p,q] \subseteq \cup_{k=1}^{n} B(x_n;\delta_n) := B_n$. Note that if $B_i \subseteq B_j$, we remove such B_i and make one left. Without loss of generality, we assume that $x_1 \leq .. \leq x_n$.

$$f(p) \leq f(x_1) \leq \ldots \leq f(x_n) \leq f(q)$$

which is absurb. So, we know that f is an increasing function throughout (a,b).

4.61 Let f be continuous on a compact interval $[a,b]$ and assume that f does not have a loacal maximum or a local minimum at any interior point. (See the note following Exercise 4.25.) Prove that f must be monotonic on $[a,b]$.

Proof: Since f is continuous on $[a,b]$, we have

$$\max_{x\in[a,b]} f(x) = f(p), \text{ where } p \in [a,b] \text{ and}$$

$$\min_{x\in[a,b]} f(x) = f(q), \text{ where } q \in [a,b].$$

So, we have $\{p,q\} = \{a,b\}$ by hypothesis that f does not have a local maximum or a local minimum at any interior point. Without loss of generality, we assume that $p = a$, and $q = b$. Claim that f is decreasing on $[a,b]$ as follows.

Suppose **NOT**, then there exist $x,y \in [a,b]$ with $x < y$ such that $f(x) < f(y)$. Consider $[x,y]$ and by hyothesis, we know that $f|_{[x,y]}$ has the maximum at y, and $f|_{[a,y]}$ has the minimum at y. Then it implies that there exists $B(y;\delta) \cap [x,y]$ such that f is constant on $B(y;\delta) \cap [x,y]$, which contradicts to the hypothesis. Hence, we have proved that f is decreasing on $[a,b]$.

4.62 If f is one-to one and continuous on $[a,b]$, prove that f must be strictly monotonic on $[a,b]$. That is, prove that **every topological mapping of** $[a,b]$ **onto an interval** $[c,d]$ **must be strictly monotonic.**

Proof: Since f is continuous on $[a,b]$, we have

$$\max_{x\in[a,b]} f(x) = f(p), \text{ where } p \in [a,b] \text{ and}$$

$$\min_{x\in[a,b]} f(x) = f(q), \text{ where } q \in [a,b].$$

Assume that $p \in (a,b)$, then there exists a $\delta > 0$ such that $f(y) \leq f(p)$ for all $y \in (p-\delta,p+\delta) \subseteq [a,b]$. Choose $y_1 \in (x-\delta,x)$ and $y_2 \in (x,x+\delta)$, then we have by 1-1, $f(y_1) < f(x)$ and $f(y_2) < f(x)$. And thus choose r so that

$f(y_1) < r < f(x) \Rightarrow f(z_1) = r$, where $z_1 \in (y_1,x)$ by **Intermediate Value Theorem,**

$f(y_2) < r < f(x) \Rightarrow f(z_2) = r$, where $z_2 \in (x,y_2)$ by **Intermediate Value Theorem,**

which contradicts to 1-1. So, we know that $p \in \{a,b\}$. Similarly, we have $q \in \{a,b\}$.

Without loss of generality, we assume that $p = a$ and $q = b$. Claim that f is strictly decreasing on $[a,b]$.

Suppose **NOT**, then there exist $x,y \in [a,b]$, with $x < y$ such that $f(x) < f(y)$. ("=" does not hold since f is 1-1.) Consider $[x,y]$ and by above method, we know that $f|_{[x,y]}$ has the maximum at y, and $f|_{[a,y]}$ has the minimum at y. Then it implies that there exists $B(y;\delta) \cap [x,y]$ such that f is constant on $B(y;\delta) \cap [x,y]$, which contradicts to 1-1. Hence, for any $x < y (\in [a,b])$, we have $f(x) > f(y)$. ("=" does not hold since f is 1-1.) So, we have proved that f is strictly decreasing on $[a,b]$.

Reamrk: 1. Here is another proof by Exercise 4.61. It suffices to show that 1-1 and continuity imply that f does not have a local maximum or a local minimum at any interior point.

Proof: Suppose **NOT**, it means that f has a local extremum at some interior point x. Without loss of generality, we assume that f has a local minimum at the interior point x. Since x is an interior point of $[a,b]$, then there exists an open interval $(x-\delta, x+\delta) \subseteq [a,b]$ such that $f(y) \geq f(x)$ for all $y \in (x-\delta, x+\delta)$. Note that f is 1-1, so we have $f(y) > f(x)$ for all $y \in (x-\delta, x+\delta) - \{x\}$. Choose $y_1 \in (x-\delta)$ and $y_2 \in (x, x+\delta)$, then we have $f(y_1) > f(x)$ and $f(y_2) > f(x)$. And thus choose r so that

$f(y_1) > r > f(x) \Rightarrow f(p) = r$, where $p \in (y_1, x)$ by **Intermediate Value Theorem,**

$f(y_2) > r > f(x) \Rightarrow f(q) = r$, where $q \in (x, y_2)$ by **Intermediate Value Theorem,**

which contradicts to the hypothesis that f is 1-1. Hence, we have proved that 1-1 and continuity imply that f does not have a local maximum or a local minimum at any interior point.

2. **Under the assumption of continuity on a compact interval, one-to-one is equivalent to being strictly monotonic.**

Proof: By the exercise, we know that an one-to-one and continuous function defined on a compact interval implies that a strictly monotonic function. So, it remains to show that a strictly monotonic function implies that an one-to-one function. Without loss of generality, let f be increasing on $[a,b]$, then as $f(x) = f(y)$, we must have $x = y$ since if $x < y$, then $f(x) < f(y)$ and if $x > y$, then $f(x) > f(y)$. So, we have proved that a strictly monotonic function implies that an one-to-one function. Hence, we get that under the assumption of continuity on a compact interval, one-to-one is equivalent to being strictly monotonic.

4.63 Let f be an increasing function defined on $[a,b]$ and let $x_1, .., x_n$ be n points in the interior such that $a < x_1 < x_2 < ... < x_n < b$.

(a) Show that $\sum_{k=1}^{n} [f(x_k+) - f(x_k-)] \leq f(b-) - f(a+)$.

Proof: Let $a = x_0$ and $b = x_{n+1}$; since f is an increasing function defined on $[a,b]$, we know that both $f(x_k+)$ and $f(x_k-)$ exist for $1 \leq k \leq n$. Assume that $y_k \in (x_k, x_{k+1})$, then we have $f(y_k) \geq f(x_k+)$ and $f(x_{k-1}) \geq f(y_{k-1})$. Hence,

$$\begin{aligned}\sum_{k=1}^{n} [f(x_k+) - f(x_k-)] &\leq \sum_{k=1}^{n} [f(y_k) - f(y_{k-1})] \\ &\leq f(y_n) - f(y_0) \\ &\leq f(b-) - f(a+).\end{aligned}$$

(b) Deduce from part (a) that the set of dicontinuities of f is countable.

Proof: Let D denote the set of dicontinuities of f. Consider $D_m = \{x \in [a,b] : f(x+) - f(x-) \geq \frac{1}{m}\}$, then $D = \cup_{m=1}^{\infty} D_m$. Note that $\#(D_m) < \infty$, so we have D is countable. That is, the set of dicontinuities of f is countable.

(c) Prove that f has points of continuity in every open subintervals of $[a,b]$.

Proof: By (b), f has points of continuity in every open subintervals of $[a,b]$, since every open subinterval is uncountable.

Remark: (1) Here is another proof about (b). Denote $Q = \{x_1,\ldots,x_n,\ldots\}$, and let x be a point at which f is not continuous. Then we have $f(x+) - f(x-) > 0$. (If x is the end point, we consider $f(x+) - f(x) > 0$ or $f(x) - f(x-) > 0$) So, we have an open interval I_x such that $I_x \cap f([a,b]) = \{f(x)\}$. The interval I_x contains infinite many rational numbers, we choose the smallest index, say $m = m(x)$. Then the number of the set of discontinuities of f on $[a,b]$ is a subset of N. Hence, the number of the set of discontinuities of f on $[a,b]$ is countable.

(2) There is a similar exercise; we write it as a reference. Let f be a real valued function defined on $[0,1]$. Suppose that there is a positive number M having the following condition: for every choice of a finite number of points $x_1,..,x_n$ in $[0,1]$, we have $-M \leq \sum_{i=1}^{n} x_i \leq M$. Prove that $S : \{x \in [0,1] : f(x) \neq 0\}$ is countable.

Proof: Consider $S_n = \{x \in [0,1] : |f(x)| \geq 1/n\}$, then it is clear that every S_n is countable. Since $S = \cup_{n=1}^{\infty} S_n$, we know that S is countable.

4.64 Give an example of a function f, defined and strictly increasing on a set S in R, such that f^{-1} is not continuous on $f(S)$.

Solution: Let

$$f(x) = \begin{cases} x \text{ if } x \in [0,1), \\ 1 \text{ if } x = 2. \end{cases}$$

Then it is clear that f is strictly increasing on $[0,1]$, so f has the incerse function

$$f^{-1}(x) = \begin{cases} x \text{ if } x \in [0,1), \\ 2 \text{ if } x = 1. \end{cases}$$

which is not continuous on $f(S) = [0,1]$.

Remark: Compare with Exercise 4.65.

4.65 Let f be strictly increasing on a subset S of R. Assume that the image $f(S)$ has one of the following properties: (a) $f(S)$ is open; (b) $f(S)$ is connected; (c) $f(S)$ is closed. Prove that f must be continuous on S.

Proof: (a) Given $a \in S$, then $f(a) \in f(S)$. Given $\varepsilon > 0$, we wan to find a $\delta > 0$ such that as $x \in B(a;\delta) \cap S$, we have $|f(x) - f(a)| < \varepsilon$. Since $f(S)$ is open, then there exists $B(f(a),\varepsilon') \subseteq f(S)$, where $\varepsilon' < \varepsilon$.

Claim that there exists a $\delta > 0$ such that $f(B(a;\delta) \cap S) \subseteq B(f(a),\varepsilon')$. Choose $y_1 = f(a) - \varepsilon'/2$ and $y_2 = f(a) + \varepsilon'/2$, then $y_1 = f(x_1)$ and $y_2 = f(x_2)$, we have $x_1 < a < x_2$ since f is strictly increasing on S. Hence, for $x \in (x_1,x_2) \cap S$, we have $f(x_1) < f(x) < f(x_2)$ since f is strictly increasing on S. So, $f(x) \in B(f(a),\varepsilon')$. Let $\delta = \min(a - x_1, x_2 - a)$, then $B(a;\delta) \cap S = (a-\delta, a+\delta) \cap S \subseteq (x_1,x_2) \cap S$ which implis that $f(B(a;\delta) \cap S) \subseteq B(f(a),\varepsilon')$. $(\subseteq B(f(a),\varepsilon))$

Hence we have prove the claim, and the claim tells us that f is continuous at a. Since a is arbitrary, we know that f is continuous on S.

(b) Note that since $f(S) \subseteq R$, and $f(S)$ is connected, we know that $f(S)$ is an interval I. Given $a \in S$, then $f(a) \in I$. We discuss 2 cases as follows. (1) $f(a)$ is an interior point of I. (2) $f(a)$ is the endpoint of I.

For case (1), it is similar to (a). We omit the proof.

For case (2), it is similar to (a). We omit the proof.

So, we have proved that f is continuous on S.

(c) Given $a \in S$, then $f(a) \in f(S)$. Since $f(S)$ is closed, we consider two cases. (1) $f(a)$ is an isolated point and (2) $f(a)$ is an accumulation point.

For case (1), claim that a is an isolated point. Suppose **NOT**, there is a sequence $\{x_n\} \subseteq S$ with $x_n \to a$. Consider $\{x_n\}_{n=1}^{\infty} = \{x : x_n < a\} \cup \{x : x_n > a\}$, and thus we may assume that $\{x : x_n < a\} := \{a_n\}$ is a infinite subset of $\{x_n\}_{n=1}^{\infty}$. Since f is monotonic, we have $\lim_{n\to\infty} f(x_n) = f(a-)$. Since $f(S)$ is closed, we have $f(a-) \in f(S)$. Therefore, there exists $b \in f(S)$ such that $f(a-) = f(b) \leq f(a)$.

If $f(b) = f(a)$, then $b = a$ since f is strictly increasing. But is contradicts to that $f(a)$ is isolated. On the other hand, if $f(b) < f(a)$, then $b < a$ since f is strictly increasing. In addition, $f(a_n) \leq f(a-) = f(b)$ implies that $a_n \leq b$. But is contradicts to that $a_n \to a$.

Hence, we have proved that a is an isolated point. So, f is sutomatically continuous at a.

For case (2), suppose that $f(a)$ is an accumulation point. Then $B(f(a);\varepsilon) \cap f(S) \neq \phi$ and $B(f(a);\varepsilon)$ has infinite many numbers of points in $f(S)$. Choose $y_1, y_2 \in B(f(a);\varepsilon) \cap f(S)$ with $y_1 < y_2$, then $f(x_1) = y_1$, and $f(x_2) = y_2$. And thus it is similar to (a), we omit the proof.

So, we have proved that f is continuous on S by (1) and (2).

Remark: In (b), when we say f is monotonic on a subset of R, its image is also in R.

Supplement.

It should be noted that the discontinuities of a monotonic function need not be isolated. In fact, given any countable subset E of (a,b), which may even be dense, **we can construct a function** f, **monotonic on** (a,b), **discontinuous at every point of** E, **and at no other point of** (a,b). To show this, let the points of E be arranged in a sequence $\{x_n\}$, $n = 1,2,\ldots$ Let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n$ converges. Define

$$f(x) = \sum_{x_n < x} c_n \ (a < x < b)$$

Note: The summation is to be understood as follows: Sum over those indices n for whcih $x_n < x$. If there are no points x_n to the left of x, the sum is empty; following the usual convention, we define it to be zero. Since absolute convergence, the order in which the terms are arranged is immaterial.

Then $f(x)$ is desired.

The proof that we omit; the reader should see the book, **Principles of Mathematical Analysis written by Walter Rudin**, pp 97.

Metric space and fixed points

4.66 Let $B(S)$ denote the set of all real-valued functions which are defined and bounded on a nonempty set S. If $f \in B(S)$, let

$$\|f\| = \sup_{x\in S} |f(x)|.$$

The number $\|f\|$ is called the " sup norm " of f.

(a) Provet that the formula $d(f,g) = \|f - g\|$ defines a metric d on $B(S)$.

Proof: We prove that d is a metric on $B(S)$ as follows.

(1) If $d(f,g) = 0$, i.e., $\|f - g\| = \sup_{x \in S}|f(x) - g(x)| = 0 \geq |f(x) - g(x)|$ for all $x \in S$. So, we have $f = g$ on S.

(2) If $f = g$ on S, then $|f(x) - g(x)| = 0$ for all $x \in S$. That is, $\|f - g\| = 0 = d(f,g)$.

(3) Given $f, g \in B(S)$, then

$$\begin{aligned} d(f,g) &= \|f - g\| \\ &= \sup_{x \in S}|f(x) - g(x)| \\ &= \sup_{x \in S}|g(x) - f(x)| \\ &= \|g - f\| \\ &= d(g,f). \end{aligned}$$

(4) Given $f, g, h \in B(S)$, then since

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|,$$

we have

$$\begin{aligned} |f(x) - g(x)| &\leq \left(\sup_{x \in S}|f(x) - h(x)|\right) + \left(\sup_{x \in S}|h(x) - g(x)|\right) \\ &\leq \|f - h\| + \|h - g\| \end{aligned}$$

which implies that

$$\|f - g\| = \sup_{x \in S}|f(x) - g(x)| \leq \|f - h\| + \|h - g\|.$$

So, we have prove that d is a metric on $B(S)$.

(b) Prove that the metric space $(B(S), d)$ is complete. Hint: If $\{f_n\}$ is a Cauchy sequence in $B(S)$, show that $\{f_n(x)\}$ is a Cauchy sequence of real numbers for each x in S.

Proof: Let $\{f_n\}$ be a Cauchy sequence on $(B(S), d)$, That is, given $\varepsilon > 0$, there is a positive integer N such that as $m, n \geq N$, we have

$$d(f,g) = \|f_n - f_m\| = \sup_{x \in S}|f_n(x) - f_m(x)| < \varepsilon. \qquad *$$

So, for every point $x \in S$, the sequence $\{f_n(x)\} (\subseteq R)$ is a Cauchy sequence. Hence, the sequence $\{f_n(x)\}$ is a convergent sequence, say its limit $f(x)$. It is clear that the function $f(x)$ is well-defined. Let $\varepsilon = 1$ in (*), then there is a positive integer N such that as $m, n \geq N$, we have

$$|f_n(x) - f_m(x)| < 1, \text{ for all } x \in S. \qquad **$$

Let $m \to \infty$, and $n = N$, we have by (**)

$$|f_N(x) - f(x)| \leq 1, \text{ for all } x \in S$$

which implies that

$$|f(x)| \leq 1 + |f_N(x)|, \text{ for all } x \in S.$$

Since $|f_N(x)| \in B(S)$, say its bound M, and thus we have

$$|f(x)| \leq 1 + M, \text{ for all } x \in S$$

which implies that $f(x)$ is bounded. That is, $f(x) \in (B(S), d)$. Hence, we have proved that $(B(S), d)$ is a complete metric space.

Remark: 1. We do not require that S is bounded.

2. The boundedness of a function f cannot be remove since sup norm of f is finite.

3. The sup norm of f, often appears and is important; the reader should keep it in mind. And we will encounter it when we discuss on sequences of functions. Also, see Exercise 4.67.

4. Here is an important theorem, the reader can see the definition of uniform convergence in the text book, page 221.

4.67 Refer to Exercise 4.66 and let $C(S)$ denote the subset of $B(S)$ consisting of all funtions **continuous** and bounded on S, where now S is a metric space.

(a) Prove that $C(S)$ is a closed subset of $B(S)$.

Proof: Let f be an adherent point of $C(S)$, then $B(f;r) \cap C(S) \neq \phi$ for all $r > 0$. So, there exists a sequence $\{f_n(x)\}$ such that $f_n \to f$ as $n \to \infty$. So, given $\varepsilon' > 0$, there is a positive integer N such that as $n \geq N$, we have

$$d(f_n,f) = \|f_n - f\| = \sup_{x\in S}|f_n(x) - f(x)| < \varepsilon'.$$

So, we have

$$|f_N(x) - f(x)| < \varepsilon'. \text{ for all } x \in S. \qquad *$$

Given $s \in S$, and note that $f_N(x) \in C(S)$, so for this ε', there exists a $\delta > 0$ such that as $|x - s| < \delta$, $x,s \in S$, we have

$$|f_N(x) - f_N(s)| < \varepsilon'. \qquad **$$

We now prove that f is continuous at s as follows. Given $\varepsilon > 0$, and let $\varepsilon' = \varepsilon/3$, then there is a $\delta > 0$ such that as $|x - s| < \delta$, $x,s \in S$, we have

$$\begin{aligned} |f(x) - f(s)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(s)| + |f_N(s) - f(s)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \text{ by (*) and (**)} \\ &= \varepsilon. \end{aligned}$$

Hence, we know that f is continuous at s, and since s is arbitrary, we know that f is continuous on S.

(b) Prove that the metric subspace $C(S)$ is complete.

Proof: By (a), we know that $C(S)$ is complete since a closed subset of a complete metric space is complete.

Remark: 1. In (b), we can see Exercise 4.9.

2. The reader should see the text book in Charpter 9, and note that **Theorem 9.2** and **Theorem 9.3.**

4.68 Refer to the proof of the **fixed points theorem (Theorem 4.48)** for notation.

(a) Prove that $d(p,p_n) \leq d(x,f(x))\alpha^n/(1-\alpha)$.

Proof: The statement is that a contraction f of a complete metric space S has a unique fixed point p. Take any point $x \in S$, and consider the sequence of iterates:

$$x,\ f(x),\ f(f(x)),\ldots$$

That is, define a sequence $\{p_n\}$ inductively as follows:

$$p_0 = x,\ p_{n+1} = f(p_n)\ n = 0,1,2,\ldots$$

We will prove that $\{p_n\}$ converges to a fixed point of f. First we show that $\{p_n\}$ is a Cauchy sequence. Since f is a contraction ($d(f(x),f(y)) \leq \alpha d(x,y)$, $0 < \alpha < 1$ for all $x,y \in S$), we have

$$d(p_{n+1},p_n) = d(f(p_n),f(p_{n-1})) \leq \alpha d(p_n,p_{n-1}),$$

so, by induction, we find

$$d(p_{n+1},p_n) \le \alpha^n d(p_1,p_0) = \alpha^n d(x,f(x)).$$

Use the triangel inequality we find, for $m > n$,

$$\begin{aligned} d(p_m,p_n) &\le \sum_{k=n}^{m-1} d(p_{k+1},p_k) \\ &\le d(x,f(x)) \sum_{k=n}^{m-1} \alpha^k \\ &= d(x,f(x)) \frac{\alpha^n - \alpha^m}{1-\alpha} \\ &< d(x,f(x)) \frac{\alpha^n}{1-\alpha}. \end{aligned} \qquad *$$

Since $\alpha^n \to 0$ as $n \to \infty$, we know that $\{p_n\}$ is a Cauchy sequence. And since S is complete, we have $p_n \to p \in S$. The uniqueness is from the inequality, $d(f(x),f(y)) \le \alpha d(x,y)$.

From (*), we know that (let $m \to \infty$)

$$d(p,p_n) \le d(x,f(x)) \frac{\alpha^n}{1-\alpha}.$$

This inequality, which is **useful in numberical work**, provides an estimate for the distance from p_n to the fixed point p. An example is given in (b)

(b) Take $f(x) = \frac{1}{2}(x + 2/x)$, $S = [1,+\infty)$. Prove that f is contraction of S with contraction constant $\alpha = 1/2$ and fixed point $p = \sqrt{2}$. Form the sequence $\{p_n\}$ starting wth $x = p_0 = 1$ and show that $\left|p_n - \sqrt{2}\right| \le 2^{-n}$.

Proof: First, $f(x) - f(y) = \frac{1}{2}(x + 2/x) - \frac{1}{2}(y + 2/y) = \frac{1}{2}[(x-y) + 2(\frac{y-x}{xy})]$, then we have

$$\begin{aligned} |f(x) - f(y)| &= \left|\frac{1}{2}\left[(x-y) + 2\left(\frac{y-x}{xy}\right)\right]\right| \\ &= \left|\frac{1}{2}(x-y)\left(1 - \frac{2}{xy}\right)\right| \\ &\le \frac{1}{2}|x-y| \text{ since } \left|1 - \frac{2}{xy}\right| \le 1. \end{aligned}$$

So, f is a contraction of S with contraction constant $\alpha = 1/2$. By **Fixed Point Theorem**, we know that there is a unique p such that $f(p) = p$. That is,

$$\frac{1}{2}\left(p + \frac{2}{p}\right) = p \Rightarrow p = \sqrt{2}. \ (-\sqrt{2} \text{ is not our choice since } S = [1,+\infty).)$$

By (a), it is easy to know that

$$\left|p_n - \sqrt{2}\right| \le 2^{-n}.$$

Remark: Here is a modefied Fixed Point Theorem: Let f be function defined on a complete metric space S. If there exists a N such that $d(f^N(x) - f^N(y)) \le \alpha d(x,y)$ for all $x,y \in S$, where $0 < \alpha < 1$. Then f has a unique fixed point $p \in S$.

Proof: Since f^N is a contraction defined on a complete metric space, with the contraction constant α, with $0 < \alpha < 1$, by Fixed Point Theorem, we know that there exists a unique point $p \in S$, such that

$$\begin{aligned} f^N(p) &= p \\ &\Rightarrow f(f^N(p)) = f(p) \\ &\Rightarrow f^N(f(p)) = f(p). \end{aligned}$$

That is, $f(p)$ is also a fixed point of f^N. By uniqueness, we know that $f(p) = p$. In addition, if there is $p' \in S$ such that $f(p') = p'$. Then we have $f^2(p') = f(p') = p', \ldots, f^N(p') = .. = p'$. Hence, we have $p = p'$. That is, f has a unique fixed point $p \in S$.

4.69 Show by counterexample that the fixed-point theorem for contractions need not hold if either (a) the underlying metric space is not complete, or (b) the contraction constant $\alpha \geq 1$.

Solution: (a) Let $f = \frac{1}{2}(1+x) : (0,1) \to R$, then $|f(x) - f(y)| = \frac{1}{2}|x-y|$. So, f is a contraction on $(0,1)$. However, it has no any fixed point since if it has, say this point p, we get $\frac{1}{2}(1+p) = p \Rightarrow p = 1 \notin (0,1)$.

(b) Let $f = (1+x) : [0,1] \to R$, then $|f(x) - f(y)| = |x-y|$. So, f is a contraction with the contraction constant 1. However, it has no any fixed point since if it has, say this point p, we get $1 + p = p \Rightarrow 1 = 0$, a contradiction.

4.70 Let $f : S \to S$ be a function from a complete metric space (S,d) into itself. Assume there is a real sequence $\{a_n\}$ which converges to 0 such that $d(f^n(x), f^n(y)) \leq \alpha_n d(x,y)$ for all $n \geq 1$ and all x,y in S, where f^n is the nth iterate of f; that is,

$$f^1(x) = f(x),\ f^{n+1}(x) = f(f^n(x)) \text{ for } n \geq 1.$$

Prove that f has a unique point. Hint. Apply the fixed point theorem to f^m for a suitable m.

Proof: Since $a_n \to 0$, given $\varepsilon = 1/2$, then there is a positive integer N such that as $n \geq N$, we have

$$|a_n| < 1/2.$$

Note that $a_n \geq 0$ for all n. Hence, we have

$$d(f^N(x), f^N(y)) \leq \frac{1}{2} d(x,y) \text{ for } x,y \text{ in } S.$$

That is, $f^N(x)$ is a contraction defined on a complete metric space, with the contraction constant 1/2. By Fixed Point Theorem, we know that there exists a unique point $p \in S$, such that

$$\begin{aligned} f^N(p) &= p \\ &\Rightarrow f(f^N(p)) = f(p) \\ &\Rightarrow f^N(f(p)) = f(p). \end{aligned}$$

That is, $f(p)$ is also a fixed point of f^N. By uniqueness, we know that $f(p) = p$. In addition, if there is $p' \in S$ such that $f(p') = p'$. Then we have $f^2(p') = f(p') = p', \ldots, f^N(p') = .. = p'$. Hence, we have $p = p'$. That is, f has a unique fixed point $p \in S$.

4.71 Let $f : S \to S$ be a function from a metric space (S,d) into itself such that

$$d(f(x), f(y)) < d(x,y)$$

where $x \neq y$.

(a) Prove that f has at most one fixed point, and give an example of such an f with no fixed point.

Proof: If p and p' are fixed points of f where $p \neq p'$, then by hypothesis, we have

$$d(p,p') = d(f(p),f(p')) < d(p,p')$$

which is absurb. So, f has at most one fixed point.

Let $f : (0,1/2) \to (0,1/2)$ by $f(x) = x^2$. Then we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| < |x-y|.$$

However, f has no fixed point since if it had, say its fixed point p, then $p^2 = p \Rightarrow p = 1 \notin (0,1/2)$ or $p = 0 \notin (0,1/2)$.

(b) If S is compact, prove that f has exactly one fixed point. Hint. Show that $g(x) = d(x,f(x))$ attains its minimum on S.

Proof: Let $g = d(x,f(x))$, and thus show that g is continuous on a compact set S as follows. Since

$$\begin{aligned} d(x,f(x)) &\leq d(x,y) + d(y,f(y)) + d(f(y),f(x)) \\ &< d(x,y) + d(y,f(y)) + d(x,y) \\ &= 2d(x,y) + d(y,f(y)) \\ &\Rightarrow d(x,f(x)) - d(y,f(y)) < 2d(x,y) \end{aligned} \qquad *$$

and change the roles of x, and y, we have

$$d(y,f(y)) - d(x,f(x)) < 2d(x,y) \qquad **$$

Hence, by (*) and (**), we have

$$|g(x) - g(y)| = |d(x,f(x)) - d(y,f(y))| < 2d(x,y) \text{ for all } x,y \in S. \qquad ***$$

Given $\varepsilon > 0$, there exists a $\delta = \varepsilon/2$ such that as $d(x,y) < \delta$, $x,y \in S$, we have

$$|g(x) - g(y)| < 2d(x,y) < \varepsilon \text{ by (***)}.$$

So, we have proved that g is uniformly continuous on S.

So, consider $\min_{x \in S} g(x) = g(p)$, $p \in S$. We show that $g(p) = 0 = d(p,f(p))$. Suppose **NOT**, i.e., $f(p) \neq p$. Consider

$$d(f^2(p),f(p)) < d(f(p),p) = g(p)$$

which contradicts to $g(p)$ is the absolute maximum. Hence, $g(p) = 0 \Leftrightarrow p = f(p)$. That is, f has a unique fixed point in S by (a).

(c)Give an example with S compact in which f is not a contraction.

Solution: Let $S = [0,1/2]$, and $f = x^2 : S \to S$. Then we have

$$|x^2 - y^2| = |x+y||x-y| \leq |x-y|.$$

So, this f is not contraction.

Remark: 1. In (b), the Choice of g is natural, since we want to get a fixed point. That is, $f(x) = x$. Hence, we consider the function $g = d(x,f(x))$.

2. Here is a exercise that makes us know more about Remark 1. Let $f : [0,1] \to [0,1]$ be a continuous function, show that there is a point p such that $f(p) = p$.

Proof: Consider $g(x) = f(x) - x$, then g is a continuous function defined on $[0,1]$. Assume that there is no point p such that $g(p) = 0$, that is, no such p so that $f(p) = p$. So, by **Intermediate Value Theorem,** we know that $g(x) > 0$ for all $x \in [0,1]$, or $g(x) < 0$ for all $x \in [0,1]$. Without loss of generality, suppose that $g(x) > 0$ for all $x \in [0,1]$ which is absurb since $g(1) = f(1) - 1 \leq 0$. Hence, we know that there is a point p such that $f(p) = p$.

3. Here is another proof on (b).

Proof: Given any point $x \in S$, and thus consider $\{f^n(x)\} \subseteq S$. Then there is a convergent subsequence $\{f^{n(k)}(x)\}$, say its limit p, since S is compact. Consider

$$\begin{aligned} d(f(p),p) &= d\left(f\left[\lim_{k\to\infty} f^{n(k)}(x)\right], \lim_{k\to\infty} f^{n(k)}(x)\right) \\ &= d\left(\lim_{k\to\infty} f[f^{n(k)}(x)], \lim_{k\to\infty} f^{n(k)}(x)\right) \text{ by continuity of } f \text{ at } p \\ &= \lim_{k\to\infty} d(f^{n(k)+1}(x), f^{n(k)}(x)) \end{aligned} \quad (1)$$

and

$$d(f^{n(k)+1}(x), f^{n(k)}(x)) \leq \ldots \leq d(f^2[f^{n(k-1)}(x)], f[f^{n(k-1)}(x)]). \quad (2)$$

Note that

$$\begin{aligned} &\lim_{k\to\infty} d(f^2[f^{n(k-1)}(x)], f[f^{n(k-1)}(x)]) \\ &= d\left(\lim_{k\to\infty} f^2[f^{n(k-1)}(x)], \lim_{k\to\infty} f[f^{n(k-1)}(x)]\right) \\ &= d\left(f^2\left[\lim_{k\to\infty} f^{n(k-1)}(x)\right], f\left[\lim_{k\to\infty} f^{n(k-1)}(x)\right]\right) \text{ by continuity of } f^2 \text{ and } f \text{ at } p \\ &= d(f^2(p), f(p)). \end{aligned} \quad (3)$$

So, by (1)-(3), we know that

$$f(p, f(p)) \leq d(f^2(p), f(p)) \Rightarrow p = f(p)$$

by hypothesis

$$d(f(x), f(y)) < d(x,y)$$

where $x \neq y$. Hence, f has a unique fixed point p by (a) in Exercise.

Note. 1. If $x_n \to x$, and $y_n \to y$, then $d(x_n, y_n) \to d(x,y)$. That is,

$$\lim_{n\to\infty} d(x_n, y_n) = d\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right).$$

Proof: Consider

$$d(x_n, y_n) \leq d(x_n, x) + d(x,y) + d(y, y_n) \text{ and}$$
$$d(x,y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y),$$

then

$$|d(x_n, y_n) - d(x,y)| \leq d(x, x_n) + d(y, y_n) \to 0.$$

So, we have prove it.

2. The reader should compare the method with Exercise 4.72.

4.72 Assume that f satisfies the condition in Exercise 4.71. If $x \in S$, let $p_0 = x$, $p_{n+1} = f(p_n)$, and $c_n = d(p_n, p_{n+1})$ for $n \geq 0$.

(a) Prove that $\{c_n\}$ is a decreasing sequence, and let $c = \lim c_n$.

Proof: Consider

$$\begin{aligned} c_{n+1} - c_n &= d(p_{n+1}, p_{n+2}) - d(p_n, p_{n+1}) \\ &= d(f(p_n), f(p_{n+1})) - d(p_n, p_{n+1}) \\ &\leq d(p_n, p_{n+1}) - d(p_n, p_{n+1}) \\ &= 0, \end{aligned}$$

so $\{c_n\}$ is a decreasing sequence. And $\{c_n\}$ has a lower bound 0, by **Completeness of** R, we know that $\{c_n\}$ is a convergent sequence, say $c = \lim c_n$.

(b) Assume there is a subsequence $\{p_{k(n)}\}$ which converges to a point q in $S.$ Prove that

$$c = d(q,f(q)) = d(f(q),f[f(q)]).$$

Deduce that q is a fixed point of f and that $p_n \to q.$

Proof: Since $\lim_{n\to\infty} p_{k(n)} = q,$ and $\lim_{n\to\infty} c_n = c,$ we have $\lim_{n\to\infty} c_{k(n)} = c.$ So, we consider

$$\begin{aligned} c &= \lim_{n\to\infty} c_{k(n)} \\ &= \lim_{n\to\infty} d(p_{k(n)},p_{k(n)+1}) \\ &= \lim_{n\to\infty} d(p_{k(n)},f(p_{k(n)})) \\ &= d(q,f(q)) \end{aligned}$$

and

$$d(p_{k(n)},p_{k(n)+1}) \leq d(p_{k(n)-1},p_{k(n)}) \leq \ldots \leq d(f(p_{k(n-1)}),f^2(p_{k(n-1)})),$$

we have

$$c = d(q,f(q)) \leq \lim_{n\to\infty} d(f(p_{k(n-1)}),f^2(p_{k(n-1)})) = d(f(q),f^2(q)). \qquad *$$

So, by (*) and hypoethesis

$$d(f(x),f(y)) < d(x,y)$$

where $x \neq y,$ we know that $q = f(q)\big(\Rightarrow c = 0,$ in fact, this q is a unique fixed point.$\big).$

In order to show that $p_n \to p,$ we consider (let $m \geq k(n)$)

$$d(p_m,q) = d(p_m,f(q)) \leq d(p_{m-1},q) \leq .. \leq d(p_{k(n)},q)$$

So, given $\varepsilon > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$d(p_{k(n)},q) < \varepsilon.$$

Hence, as $m \geq k(N),$ we have

$$d(p_m,q) < \varepsilon.$$

That is, $p_n \to p.$