

# Chapter 3 Elements of Point set Topology

## Open and closed sets in $R^1$ and $R^2$

3.1 Prove that an open interval in  $R^1$  is an open set and that a closed interval is a closed set.

**proof:** 1. Let  $(a, b)$  be an open interval in  $R^1$ , and let  $x \in (a, b)$ . Consider  $\min(x - a, b - x) := L$ . Then we have  $B(x, L) = (x - L, x + L) \subseteq (a, b)$ . That is,  $x$  is an interior point of  $(a, b)$ . Since  $x$  is arbitrary, we have every point of  $(a, b)$  is interior. So,  $(a, b)$  is open in  $R^1$ .

2. Let  $[a, b]$  be a closed interval in  $R^1$ , and let  $x$  be an adherent point of  $[a, b]$ . We want to show  $x \in [a, b]$ . If  $x \notin [a, b]$ , then we have  $x < a$  or  $x > b$ . Consider  $x < a$ , then

$$B(x, \frac{a-x}{2}) \cap [a, b] = (\frac{3x-a}{2}, \frac{x+a}{2}) \cap [a, b] = \phi$$

which contradicts the definition of an adherent point. Similarly for  $x > b$ .

Therefore, we have  $x \in [a, b]$  if  $x$  is an adherent point of  $[a, b]$ . That is,  $[a, b]$  contains its all adherent points. It implies that  $[a, b]$  is closed in  $R^1$ .

3.2 Determine all the accumulation points of the following sets in  $R^1$  and decide whether the sets are open or closed (or neither).

(a) All integers.

**Solution:** Denote the set of all integers by  $Z$ . Let  $x \in Z$ , and consider  $(B(x, \frac{x+1}{2}) - \{x\}) \cap Z = \phi$ . So,  $Z$  has no accumulation points.

However,  $B(x, \frac{x+1}{2}) \cap Z = \{x\} \neq \phi$ . So  $Z$  contains its all adherent points. It means that  $Z$  is closed. Trivially,  $Z$  is not open since  $B(x, r)$  is not contained in  $Z$  for all  $r > 0$ .

**Remark:** 1. Definition of an adherent point: Let  $S$  be a subset of  $R^n$ , and  $x$  a point in  $R^n$ ,  $x$  is not necessarily in  $S$ . Then  $x$  is said to be adherent to  $S$  if every  $n$ -ball  $B(x)$  contains at least one point of  $S$ . To be roughly,  $B(x) \cap S \neq \phi$ .

2. Definition of an accumulation point: Let  $S$  be a subset of  $R^n$ , and  $x$  a point in  $R^n$ , then  $x$  is called an accumulation point of  $S$  if every  $n$ -ball  $B(x)$  contains at least one point of  $S$  distinct from  $x$ . To be roughly,  $(B(x) - \{x\}) \cap S \neq \phi$ . That is,  $x$  is an accumulation point if, and only if,  $x$  adheres to  $S - \{x\}$ . Note that in this sense,  $(B(x) - \{x\}) \cap S = B(x) \cap (S - \{x\})$ .

3. Definition of an isolated point: If  $x \in S$ , but  $x$  is not an accumulation point of  $S$ , then  $x$  is called an isolated point.

4. Another solution for  $Z$  is closed: Since  $R - Z = \cup_{n \in Z} (n, n + 1)$ , we know that  $R - Z$  is open. So,  $Z$  is closed.

5. In logics, if there does not exist any accumulation point of a set  $S$ , then  $S$  is automatically a closed set.

(b) The interval  $(a, b]$ .

**solution:** In order to find all accumulation points of  $(a, b]$ , we consider 2 cases as follows.

1.  $(a, b]$  : Let  $x \in (a, b]$ , then  $(B(x, r) - \{x\}) \cap (a, b] \neq \phi$  for any  $r > 0$ . So, every point of  $(a, b]$  is an accumulation point.

2.  $R^1 - (a, b] = (-\infty, a] \cup (b, \infty)$  : For points in  $(b, \infty)$  and  $(-\infty, a)$ , it is easy to know that these points cannot be accumulation points since  $x \in (b, \infty)$  or  $x \in (-\infty, a)$ , there

exists an  $n$ -ball  $B(x, r_x)$  such that  $(B(x, r_x) - \{x\}) \cap (a, b] = \emptyset$ . For the point  $a$ , it is easy to know that  $(B(a, r) - \{a\}) \cap (a, b] \neq \emptyset$ . That is, in this case, there is only one accumulation point  $a$  of  $(a, b]$ .

So, from 1 and 2, we know that the set of the accumulation points of  $(a, b]$  is  $[a, b]$ .

Since  $a \notin (a, b]$ , we know that  $(a, b]$  cannot contain its all accumulation points. So,  $(a, b]$  is not closed.

Since an  $n$ -ball  $B(b, r)$  is not contained in  $(a, b]$  for any  $r > 0$ , we know that the point  $b$  is not interior to  $(a, b]$ . So,  $(a, b]$  is not open.

(c) All numbers of the form  $1/n$ , ( $n = 1, 2, 3, \dots$ ).

**Solution:** Write the set  $\{1/n : n = 1, 2, \dots\} = \{1, 1/2, 1/3, \dots, 1/n, \dots\} := S$ .

Obviously, 0 is the only one accumulation point of  $S$ . So,  $S$  is not closed since  $S$  does not contain the accumulation point 0. Since  $1 \in S$ , and  $B(1, r)$  is not contained in  $S$  for any  $r > 0$ ,  $S$  is not open.

**Remark:** Every point of  $\{1/n : n = 1, 2, 3, \dots\}$  is isolated.

(d) All rational numbers.

**Solutions:** Denote all rational numbers by  $\mathbb{Q}$ . It is trivially seen that the set of accumulation points is  $\mathbb{R}^1$ .

So,  $\mathbb{Q}$  is not closed. Consider  $x \in \mathbb{Q}$ , any  $n$ -ball  $B(x)$  is not contained in  $\mathbb{Q}$ . That is,  $x$  is not an interior point of  $\mathbb{Q}$ . (In fact, every point of  $\mathbb{Q}$  is not an interior point of  $\mathbb{Q}$ .) So,  $\mathbb{Q}$  is not open.

(e) All numbers of the form  $2^{-n} + 5^{-m}$ , ( $m, n = 1, 2, \dots$ ).

**Solution:** Write the set

$$\begin{aligned} \{2^{-n} + 5^{-m} : m, n = 1, 2, \dots\} &= \bigcup_{m=1}^{m=\infty} \left\{ \frac{1}{2} + 5^{-m}, \frac{1}{4} + 5^{-m}, \dots, \frac{1}{2^n} + 5^{-m}, \dots \right\} := S \\ &= \left\{ \frac{1}{2} + \frac{1}{5}, \frac{1}{2} + \frac{1}{5^2}, \dots, \frac{1}{2} + \frac{1}{5^m}, \dots \right\} \cup & 1 \\ &\quad \left\{ \frac{1}{4} + \frac{1}{5}, \frac{1}{4} + \frac{1}{5^2}, \dots, \frac{1}{4} + \frac{1}{5^m}, \dots \right\} \cup & 2 \\ &\quad \dots \cup \\ &\quad \left\{ \frac{1}{2^n} + \frac{1}{5}, \frac{1}{2^n} + \frac{1}{5^2}, \dots, \frac{1}{2^n} + \frac{1}{5^m} + \dots \right\} \cup & 3 \\ &\quad \dots \end{aligned}$$

So, we find that  $S' = \{ \frac{1}{2^n} : n = 1, 2, \dots \} \cup \{ \frac{1}{5^m} : m = 1, 2, \dots \} \cup \{0\}$ . So,  $S$  is not closed since it does not contain 0. Since  $\frac{1}{2} \in S$ , and  $B(\frac{1}{2}, r)$  is not contained in  $S$  for any  $r > 0$ ,  $S$  is not open.

**Remark:** By (1)-(3), we can regard them as three sequences

$$\left\{ \frac{1}{2} + 5^{-m} \right\}_{m=1}^{m=\infty}, \left\{ \frac{1}{4} + 5^{-m} \right\}_{m=1}^{m=\infty} \text{ and } \left\{ \frac{1}{2^n} + 5^{-m} \right\}_{m=1}^{m=\infty}, \text{ respectively.}$$

And it means that for (1), the sequence  $\{5^{-m}\}_{m=1}^{m=\infty}$  moves  $\frac{1}{2}$ . Similarly for others. So, it is easy to see why  $\frac{1}{2}$  is an accumulation point of  $\{ \frac{1}{2} + 5^{-m} \}_{m=1}^{m=\infty}$ . And thus get the set of all accumulation points of  $\{2^{-n} + 5^{-m} : m, n = 1, 2, \dots\}$ .

(f) All numbers of the form  $(-1)^n + (1/m)$ , ( $m, n = 1, 2, \dots$ ).

**Solution:** Write the set of all numbers  $(-1)^n + (1/m)$ , ( $m, n = 1, 2, \dots$ ) as

$$\left\{ 1 + \frac{1}{m} \right\}_{m=1}^{m=\infty} \cup \left\{ -1 + \frac{1}{m} \right\}_{m=1}^{m=\infty} := S.$$

And thus by the remark in (e), it is easy to know that  $S' = \{1, -1\}$ . So,  $S$  is not closed since  $S' \subsetneq S$ . Since  $2 \in S$ , and  $B(2, r)$  is not contained in  $S$  for any  $r > 0$ ,  $S$  is not open.

(g) All numbers of the form  $(1/n) + (1/m)$ ,  $(m, n = 1, 2, \dots)$ .

**Solution:** Write the set of all numbers  $(1/n) + (1/m)$ ,  $(m, n = 1, 2, \dots)$  as

$$\left\{1 + \frac{1}{m}\right\}_{m=1}^{m=\infty} \cup \left\{\frac{1}{2} + \frac{1}{m}\right\}_{m=1}^{m=\infty} \cup \dots \cup \left\{\frac{1}{n} + \frac{1}{m}\right\}_{m=1}^{m=\infty} \cup \dots := S.$$

We find that  $S' = \{1/n : n \in \mathbb{N}\} \cup \{1/m : m \in \mathbb{N}\} \cup \{0\} = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ . So,  $S$  is not closed since  $S' \subsetneq S$ . Since  $1 \in S$ , and  $B(1, r)$  is not contained in  $S$  for any  $r > 0$ ,  $S$  is not open.

(h) All numbers of the form  $(-1)^n/[1 + (1/n)]$ ,  $(n = 1, 2, \dots)$ .

**Soluton:** Write the set of all numbers  $(-1)^n/[1 + (1/n)]$ ,  $(n = 1, 2, \dots)$  as

$$\left\{\frac{1}{1 + \frac{1}{2k}}\right\}_{k=1}^{k=\infty} \cup \left\{\frac{-1}{1 + \frac{1}{2k-1}}\right\}_{k=1}^{k=\infty} := S.$$

We find that  $S' = \{-1, 1\}$ . So,  $S$  is not closed since  $S' \subsetneq S$ . Since  $\frac{-1}{2} \in S$ , and  $B(\frac{-1}{2}, r)$  is not contained in  $S$  for any  $r > 0$ ,  $S$  is not open.

3.3 The same as Exercise 3.2 for the following sets in  $\mathbb{R}^2$ .

(a) All complex  $z$  such that  $|z| > 1$ .

**Solution:** Denote  $\{z \in \mathbb{C} : |z| > 1\}$  by  $S$ . It is easy to know that  $S' = \{z \in \mathbb{C} : |z| \geq 1\}$ . So,  $S$  is not closed since  $S' \subsetneq S$ . Let  $z \in S$ , then  $|z| > 1$ . Consider  $B(z, \frac{|z|-1}{2}) \subseteq S$ , so every point of  $S$  is interior. That is,  $S$  is open.

(b) All complex  $z$  such that  $|z| \geq 1$ .

**Solution:** Denote  $\{z \in \mathbb{C} : |z| \geq 1\}$  by  $S$ . It is easy to know that  $S' = \{z \in \mathbb{C} : |z| \geq 1\}$ . So,  $S$  is closed since  $S' \subseteq S$ . Since  $1 \in S$ , and  $B(1, r)$  is not contained in  $S$  for any  $r > 0$ ,  $S$  is not open.

(c) All complex numbers of the form  $(1/n) + (i/m)$ ,  $(m, n = 1, 2, \dots)$ .

**Solution:** Write the set of all complex numbers of the form  $(1/n) + (i/m)$ ,  $(m, n = 1, 2, \dots)$  as

$$\left\{1 + \frac{i}{m}\right\}_{m=1}^{m=\infty} \cup \left\{\frac{1}{2} + \frac{i}{m}\right\}_{m=1}^{m=\infty} \cup \dots \cup \left\{\frac{1}{n} + \frac{i}{m}\right\}_{m=1}^{m=\infty} \cup \dots := S.$$

We know that  $S' = \{1/n : n = 1, 2, \dots\} \cup \{i/m : m = 1, 2, \dots\} \cup \{0\}$ . So,  $S$  is not closed since  $S' \subsetneq S$ . Since  $1 + i \in S$ , and  $B(1 + i, r)$  is not contained in  $S$  for any  $r > 0$ ,  $S$  is not open.

(d) All points  $(x, y)$  such that  $x^2 + y^2 < 1$ .

**Solution:** Denote  $\{(x, y) : x^2 + y^2 < 1\}$  by  $S$ . We know that  $S' = \{(x, y) : x^2 + y^2 \leq 1\}$ . So,  $S$  is not closed since  $S' \subsetneq S$ . Let  $p = (x, y) \in S$ , then  $x^2 + y^2 < 1$ . It is easy to find that  $r > 0$  such that  $B(p, r) \subseteq S$ . So,  $S$  is open.

(e) All points  $(x, y)$  such that  $x > 0$ .

**Solution:** Write all points  $(x, y)$  such that  $x > 0$  as  $\{(x, y) : x > 0\} := S$ . It is easy to know that  $S' = \{(x, y) : x \geq 0\}$ . So,  $S$  is not closed since  $S' \subsetneq S$ . Let  $x \in S$ , then it is easy to find  $r_x > 0$  such that  $B(x, r_x) \subseteq S$ . So,  $S$  is open.

(f) All points  $(x, y)$  such that  $x \geq 0$ .

**Solution:** Write all points  $(x, y)$  such that  $x \geq 0$  as  $\{(x, y) : x \geq 0\} := S$ . It is easy to

know that  $S' = \{(x, y) : x \geq 0\}$ . So,  $S$  is closed since  $S' \subseteq S$ . Since  $(0, 0) \in S$ , and  $B((0, 0), r)$  is not contained in  $S$  for any  $r > 0$ ,  $S$  is not open.

3.4 Prove that every nonempty open set  $S$  in  $R^1$  contains both rational and irrational numbers.

**proof:** Given a nonempty open set  $S$  in  $R^1$ . Let  $x \in S$ , then there exists  $r > 0$  such that  $B(x, r) \subseteq S$  since  $S$  is open. And in  $R^1$ , the open ball  $B(x, r) = (x - r, x + r)$ . Since any interval contains both rational and irrational numbers, we have  $S$  contains both rational and irrational numbers.

3.5 Prove that the only set in  $R^1$  which are both open and closed are the empty set and  $R^1$  itself. Is a similar statement true for  $R^2$ ?

**Proof:** Let  $S$  be the set in  $R^1$ , and thus consider its complement  $T = R^1 - S$ . Then we have both  $S$  and  $T$  are open and closed. Suppose that  $S \neq R^1$  and  $S \neq \phi$ , we will show that it is **impossible** as follows.

Since  $S \neq R^1$ , and  $S \neq \phi$ , then  $T \neq \phi$  and  $T \neq R^1$ . Choose  $s_0 \in S$  and  $t_0 \in T$ , then we consider the **new** point  $\frac{s_0+t_0}{2}$  which is in  $S$  or  $T$  since  $R = S \cup T$ . If  $\frac{s_0+t_0}{2} \in S$ , we say  $\frac{s_0+t_0}{2} = s_1$ , otherwise, we say  $\frac{s_0+t_0}{2} = t_1$ .

Continue these steps, we finally have **two sequences** named  $\{s_n\} \subseteq S$  and  $\{t_m\} \subseteq T$ . In addition, the two sequences are **convergent** to the same point, say  $p$  by our construction. So, we get  $p \in S$  and  $p \in T$  since both  $S$  and  $T$  are closed.

However, it leads us to get a contradiction since  $p \in S \cap T = \phi$ . Hence  $S = R^1$  or  $S = \phi$ .

**Remark:** 1. In the proof, the statement is true for  $R^n$ .

2. The construction is not strange for us since the process is called *Bolzano Process*.

3.6 Prove that every closed set in  $R^1$  is the intersection of a countable collection of open sets.

**proof:** Given a closed set  $S$ , and consider its complement  $R^1 - S$  which is open. If  $R^1 - S = \phi$ , there is nothing to prove. So, we can assume that  $R^1 - S \neq \phi$ .

Let  $x \in R^1 - S$ , then  $x$  is an interior point of  $R^1 - S$ . So, there exists an open interval  $(a, b)$  such that  $x \in (a, b) \subseteq R^1 - S$ . In order to show our statement, we choose a smaller interval  $(a_x, b_x)$  so that  $x \in (a_x, b_x)$  and  $[a_x, b_x] \subseteq (a, b) \subseteq R^1 - S$ . Hence, we have

$$R^1 - S = \bigcup_{x \in R^1 - S} [a_x, b_x]$$

which implies that

$$\begin{aligned} S &= R^1 - \bigcup_{x \in R^1 - S} [a_x, b_x] \\ &= \bigcap_{x \in R^1 - S} (R^1 - [a_x, b_x]) \\ &= \bigcap_{n=1}^{n=\infty} (R^1 - [a_n, b_n]) \text{ (by Lindelof Convering Theorem).} \end{aligned}$$

**Remark:** 1. There exists another proof by **Representation Theorem for Open Sets on The Real Line**.

2. Note that it is true for that every closed set in  $R^1$  is the intersection of a countable collection of **closed** sets.

3. The proof is suitable for  $R^n$  if the statement is that every closed set in  $R^n$  is the intersection of a countable collection of open sets. All we need is to change intervals into disks.

**3.7** Prove that a nonempty, bounded closed set  $S$  in  $R^1$  is either a closed interval, or that  $S$  can be obtained from a closed interval by removing a countable disjoint collection of open intervals whose endpoints belong to  $S$ .

**proof:** If  $S$  is an interval, then it is clear that  $S$  is a closed interval. Suppose that  $S$  is not an interval. Since  $S (\neq \phi)$  is bounded and closed, both  $\sup S$  and  $\inf S$  are in  $S$ . So,  $R^1 - S = [\inf S, \sup S] - S$ . Denote  $[\inf S, \sup S]$  by  $I$ . Consider  $R^1 - S$  is open, then by **Representation Theorem for Open Sets on The Real Line**, we have

$$\begin{aligned} R^1 - S &= \bigcup_{m=1}^{m=\infty} I_m \\ &= I - S \end{aligned}$$

which implies that

$$S = I - \bigcup_{m=1}^{m=\infty} I_m.$$

That is,  $S$  can be obtained from a closed interval by removing a countable disjoint collection of open intervals whose endpoints belong to  $S$ .

### Open and closed sets in $R^n$

**3.8** Prove that open  $n$ -balls and  $n$ -dimensional open intervals are open sets in  $R^n$ .

**proof:** Given an open  $n$ -ball  $B(x, r)$ . Choose  $y \in B(x, r)$  and thus consider  $B(y, d) \subseteq B(x, r)$ , where  $d = \min(|x - y|, r - |x - y|)$ . Then  $y$  is an interior point of  $B(x, r)$ . Since  $y$  is arbitrary, we have all points of  $B(x, r)$  are interior. So, the open  $n$ -ball  $B(x, r)$  is open.

Given an  $n$ -dimensional open interval  $(a_1, b_1) \times (a_2, b_2) \times \dots \times (a_n, b_n) := I$ . Choose  $x = (x_1, x_2, \dots, x_n) \in I$  and thus consider  $r = \min_{i=1}^n (r_i)$ , where  $r_i = \min(x_i - a_i, b_i - x_i)$ . Then  $B(x, r) \subseteq I$ . That is,  $x$  is an interior point of  $I$ . Since  $x$  is arbitrary, we have all points of  $I$  are interior. So, the  $n$ -dimensional open interval  $I$  is open.

**3.9** Prove that the interior of a set in  $R^n$  is open in  $R^n$ .

**Proof:** Let  $x \in \text{int}S$ , then there exists  $r > 0$  such that  $B(x, r) \subseteq S$ . Choose any point of  $B(x, r)$ , say  $y$ . Then  $y$  is an interior point of  $B(x, r)$  since  $B(x, r)$  is open. So, there exists  $d > 0$  such that  $B(y, d) \subseteq B(x, r) \subseteq S$ . So  $y$  is also an interior point of  $S$ . Since  $y$  is arbitrary, we find that every point of  $B(x, r)$  is interior to  $S$ . That is,  $B(x, r) \subseteq \text{int}S$ . Since  $x$  is arbitrary, we have all points of  $\text{int}S$  are interior. So,  $\text{int}S$  is open.

**Remark:** 1 It should be noted that  $S$  is open if, and only if  $S = \text{int}S$ .

2.  $\text{int}(\text{int}S) = \text{int}S$ .

3. If  $S \subseteq T$ , then  $\text{int}S \subseteq \text{int}T$ .

**3.10** If  $S \subseteq R^n$ , prove that  $\text{int}S$  is the union of all open subsets of  $R^n$  which are contained in  $S$ . This is described by saying that  $\text{int}S$  is **the largest open subset** of  $S$ .

**proof:** It suffices to show that  $\text{int}S = \bigcup_{A \subseteq S} A$ , where  $A$  is open. To show the statement, we consider two steps as follows.

1. ( $\subseteq$ ) Let  $x \in \text{int}S$ , then there exists  $r > 0$  such that  $B(x, r) \subseteq S$ . So,  $x \in B(x, r) \subseteq \bigcup_{A \subseteq S} A$ . That is,  $\text{int}S \subseteq \bigcup_{A \subseteq S} A$ .

2. ( $\supseteq$ ) Let  $x \in \bigcup_{A \subseteq S} A$ , then  $x \in A$  for some open set  $A (\subseteq S)$ . Since  $A$  is open,  $x$  is an interior point of  $A$ . There exists  $r > 0$  such that  $B(x, r) \subseteq A \subseteq S$ . So  $x$  is an interior point of  $S$ , i.e.,  $x \in \text{int}S$ . That is,  $\bigcup_{A \subseteq S} A \subseteq \text{int}S$ .

From 1 and 2, we know that  $\text{int}S = \bigcup_{A \subseteq S} A$ , where  $A$  is open.

Let  $T$  be an open subset of  $S$  such that  $\text{int}S \subseteq T$ . Since  $\text{int}S = \bigcup_{A \subseteq S} A$ , where  $A$  is open,

we have  $\text{int}S \subseteq T \subseteq \bigcup_{A \subseteq S} A$  which implies  $\text{int}S = T$  by  $\text{int}S = \bigcup_{A \subseteq S} A$ . Hence,  $\text{int}S$  is the largest open subset of  $S$ .

3.11 If  $S$  and  $T$  are subsets of  $R^n$ , prove that

$$(\text{int}S) \cap (\text{int}T) = \text{int}(S \cap T) \text{ and } (\text{int}S) \cup (\text{int}T) \subseteq \text{int}(S \cup T).$$

**Proof:** For the part  $(\text{int}S) \cap (\text{int}T) = \text{int}(S \cap T)$ , we consider two steps as follows.

1. ( $\subseteq$ ) Since  $\text{int}S \subseteq S$  and  $\text{int}T \subseteq T$ , we have  $(\text{int}S) \cap (\text{int}T) \subseteq S \cap T$  which implies that (Note that  $(\text{int}S) \cap (\text{int}T)$  is open.)

$$(\text{int}S) \cap (\text{int}T) = \text{int}((\text{int}S) \cap (\text{int}T)) \subseteq \text{int}(S \cap T).$$

2. ( $\supseteq$ ) Since  $S \cap T \subseteq S$  and  $S \cap T \subseteq T$ , we have  $\text{int}(S \cap T) \subseteq \text{int}S$  and  $\text{int}(S \cap T) \subseteq \text{int}T$ . So,

$$\text{int}(S \cap T) \subseteq (\text{int}S) \cap (\text{int}T).$$

From 1 and 2, we know that  $(\text{int}S) \cap (\text{int}T) = \text{int}(S \cap T)$ .

For the part  $(\text{int}S) \cup (\text{int}T) \subseteq \text{int}(S \cup T)$ , we consider  $(\text{int}S) \subseteq S$  and  $(\text{int}T) \subseteq T$ . So,

$$(\text{int}S) \cup (\text{int}T) \subseteq S \cup T$$

which implies that (Note that  $(\text{int}S) \cup (\text{int}T)$  is open.)

$$\text{int}((\text{int}S) \cup (\text{int}T)) = (\text{int}S) \cup (\text{int}T) \subseteq \text{int}(S \cup T).$$

**Remark:** It is not necessary that  $(\text{int}S) \cup (\text{int}T) = \text{int}(S \cup T)$ . For example, let  $S = Q$ , and  $T = Q^c$ , then  $\text{int}S = \phi$ , and  $\text{int}T = \phi$ . However,  $\text{int}(S \cup T) = \text{int}R^1 = R$ .

3.12 Let  $S'$  denote the derived set and  $\bar{S}$  the closure of a set  $S$  in  $R^n$ . Prove that

(a)  $S'$  is closed in  $R^n$ ; that is  $(S')' \subseteq S'$ .

**proof:** Let  $x$  be an adherent point of  $S'$ . In order to show  $S'$  is closed, it suffices to show that  $x$  is an accumulation point of  $S$ . Assume  $x$  is not an accumulation point of  $S$ , i.e., there exists  $d > 0$  such that

$$(B(x, d) - \{x\}) \cap S = \phi. \quad *$$

Since  $x$  adheres to  $S'$ , then  $B(x, d) \cap S' \neq \phi$ . So, there exists  $y \in B(x, d)$  such that  $y$  is an accumulation point of  $S$ . Note that  $x \neq y$ , by assumption. Choose a smaller radius  $\tilde{d}$  so that

$$B(y, \tilde{d}) \subseteq B(x, d) - \{x\} \text{ and } B(y, \tilde{d}) \cap S \neq \phi.$$

It implies

$$\phi \neq B(y, \tilde{d}) \cap S \subseteq (B(x, d) - \{x\}) \cap S = \phi \text{ by } (*)$$

which is absurd. So,  $x$  is an accumulation point of  $S$ . That is,  $S'$  contains all its adherent points. Hence  $S'$  is closed.

(b) If  $S \subseteq T$ , then  $S' \subseteq T'$ .

**Proof:** Let  $x \in S'$ , then  $(B(x, r) - \{x\}) \cap S \neq \phi$  for any  $r > 0$ . It implies that  $(B(x, r) - \{x\}) \cap T \neq \phi$  for any  $r > 0$  since  $S \subseteq T$ . Hence,  $x$  is an accumulation point of  $T$ . That is,  $x \in T'$ . So,  $S' \subseteq T'$ .

(c)  $(S \cup T)' = S' \cup T'$

**Proof:** For the part  $(S \cup T)' = S' \cup T'$ , we show it by two steps.

1. Since  $S \subseteq S \cup T$  and  $T \subseteq S \cup T$ , we have  $S' \subseteq (S \cup T)'$  and  $T' \subseteq (S \cup T)'$  by (b). So,

$$S' \cup T' \subseteq (S \cup T)'$$

2. Let  $x \in (S \cup T)'$ , then  $(B(x, r) - \{x\}) \cap (S \cup T) \neq \phi$ . That is,

$$((B(x, r) - \{x\}) \cap S) \cup ((B(x, r) - \{x\}) \cap T) \neq \phi.$$

So, at least one of  $(B(x, r) - \{x\}) \cap S$  and  $(B(x, r) - \{x\}) \cap T$  is not empty. If  $(B(x, r) - \{x\}) \cap S \neq \phi$ , then  $x \in S'$ . And if  $(B(x, r) - \{x\}) \cap T \neq \phi$ , then  $x \in T'$ . So,

$$(S \cup T)' \subseteq S' \cup T'.$$

From 1 and 2, we have  $(S \cup T)' = S' \cup T'$ .

**Remark:** Note that since  $(S \cup T)' = S' \cup T'$ , we have  $cl(S \cup T) = cl(S) \cup cl(T)$ , where  $cl(S)$  is the closure of  $S$ .

$$(d) (\bar{S})' = S'.$$

**Proof:** Since  $\bar{S} = S \cup S'$ , then  $(\bar{S})' = (S \cup S')' = S' \cup (S')' = S'$  since  $(S')' \subseteq S'$  by (a).

(e)  $\bar{S}$  is closed in  $R^n$ .

**Proof:** Since  $(\bar{S})' = S' \subseteq \bar{S}$  by (d), then  $\bar{S}$  contains all its accumulation points. Hence,  $\bar{S}$  is closed.

**Remark:** There is another proof which is like (a). But it is too tedious to write.

(f)  $\bar{S}$  is the intersection of all closed subsets of  $R^n$  containing  $S$ . That is,  $\bar{S}$  is the smallest closed set containing  $S$ .

**Proof:** It suffices to show that  $\bar{S} = \bigcap_{A \supseteq S} A$ , where  $A$  is closed. To show the statement, we consider two steps as follows.

1. ( $\subseteq$ ) Since  $\bar{S}$  is closed and  $S \subseteq \bar{S}$ , then  $\bigcap_{A \supseteq S} A \subseteq \bar{S}$ .

2. ( $\supseteq$ ) Let  $x \in \bar{S}$ , then  $B(x, r) \cap S \neq \phi$  for any  $r > 0$ . So, if  $A \supseteq S$ , then  $B(x, r) \cap A \neq \phi$  for any  $r > 0$ . It implies that  $x$  is an adherent point of  $A$ . Hence if  $A \supseteq S$ , and  $A$  is closed, we have  $x \in A$ . That is,  $x \in \bigcap_{A \supseteq S} A$ . So,  $\bar{S} \subseteq \bigcap_{A \supseteq S} A$ .

From 1 and 2, we have  $\bar{S} = \bigcap_{A \supseteq S} A$ .

Let  $S \subseteq T \subseteq \bar{S}$ , where  $T$  is closed. Then  $\bar{S} = \bigcap_{A \supseteq S} A \subseteq T$ . It leads us to get  $T = \bar{S}$ . That is,  $\bar{S}$  is the smallest closed set containing  $S$ .

**Remark:** In the exercise, there has something to remeber. We list them below.

Remark 1. If  $S \subseteq T$ , then  $S' \subseteq T'$ .

2. If  $S \subseteq T$ , then  $\bar{S} \subseteq \bar{T}$ .

3.  $\bar{S} = S \cup S'$ .

4.  $S$  is closed if, and only if  $S' \subseteq S$ .

5.  $\bar{S}$  is closed.

6.  $\bar{S}$  is the smallest closed set containing  $S$ .

3.13 Let  $S$  and  $T$  be subsets of  $R^n$ . Prove that  $cl(S \cap T) \subseteq cl(S) \cap cl(T)$  and that  $S \cap cl(T) \subseteq cl(S \cap T)$  if  $S$  is open, where  $cl(S)$  is the closure of  $S$ .

**Proof:** Since  $S \cap T \subseteq S$  and  $S \cap T \subseteq T$ , then  $cl(S \cap T) \subseteq cl(S)$  and,  $cl(S \cap T) \subseteq cl(T)$ . So,  $cl(S \cap T) \subseteq cl(S) \cap cl(T)$ .

Given an open set  $S$ , and let  $x \in S \cap cl(T)$ , then we have

1.  $x \in S$  and  $S$  is open.  
 $\Rightarrow B(x, d) \subseteq S$  for some  $d > 0$ .  
 $\Rightarrow B(x, r) \cap S \supseteq B(x, r)$  if  $r \leq d$ .  
 $\Rightarrow B(x, r) \cap S \supseteq B(x, d)$  if  $r > d$ .

and

2.  $x \in cl(T)$   
 $\Rightarrow B(x, r) \cap T \neq \emptyset$  for any  $r > 0$ .

From 1 and 2, we know

$$B(x, r) \cap (S \cap T) = (B(x, r) \cap S) \cap T = B(x, r) \cap T \neq \emptyset \text{ if } r \leq d.$$

$$B(x, r) \cap (S \cap T) = (B(x, r) \cap S) \cap T = B(x, d) \cap T \neq \emptyset \text{ if } r > d.$$

So, it means that  $x$  is an adherent point of  $S \cap T$ . That is,  $x \in cl(S \cap T)$ . Hence,  $S \cap cl(T) \subseteq cl(S \cap T)$ .

**Remark:** It is not necessary that  $cl(S \cap T) = cl(S) \cap cl(T)$ . For example,  $S = Q$  and  $T = Q^c$ , then  $cl(S \cap T) = \emptyset$  and  $cl(S) \cap cl(T) = R^1$ .

**Note.** The statements in Exercises 3.9 through 3.13 are true in any metric space.

**3.14** A set  $S$  in  $R^n$  is called **convex** if, for every pair of points  $x$  and  $y$  in  $S$  and every real  $\theta$  satisfying  $0 < \theta < 1$ , we have  $\theta x + (1 - \theta)y \in S$ . Interpret this statement geometrically (in  $R^2$  and  $R^3$ ) and prove that

(a) Every  $n$ -ball in  $R^n$  is convex.

**Proof:** Given an  $n$ -ball  $B(p, r)$ , and let  $x, y \in B(p, r)$ . Consider  $\theta x + (1 - \theta)y$ , where  $0 < \theta < 1$ .

Then

$$\begin{aligned} \|\theta x + (1 - \theta)y - p\| &= \|\theta(x - p) + (1 - \theta)(y - p)\| \\ &\leq \theta\|x - p\| + (1 - \theta)\|y - p\| \\ &< \theta r + (1 - \theta)r \\ &= r. \end{aligned}$$

So, we have  $\theta x + (1 - \theta)y \in B(p, r)$  for  $0 < \theta < 1$ . Hence, by the definition of convex, we know that every  $n$ -ball in  $R^n$  is convex.

(b) Every  $n$ -dimensional open interval is convex.

**Proof:** Given an  $n$ -dimensional open interval  $I = (a_1, b_1) \times \dots \times (a_n, b_n)$ . Let  $x, y \in I$ , and thus write  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$ . Consider

$\theta x + (1 - \theta)y = (\theta x_1 + (1 - \theta)y_1, \theta x_2 + (1 - \theta)y_2, \dots, \theta x_n + (1 - \theta)y_n)$  where  $0 < \theta < 1$ .

Then

$$a_i < \theta x_i + (1 - \theta)y_i < b_i, \text{ where } i = 1, 2, \dots, n.$$

So, we have  $\theta x + (1 - \theta)y \in I$  for  $0 < \theta < 1$ . Hence, by the definition of convex, we know that every  $n$ -dimensional open interval is convex.

(c) The interior of a convex is convex.

**Proof:** Given a convex set  $S$ , and let  $x, y \in intS$ . Then there exists  $r > 0$  such that  $B(x, r) \subseteq S$ , and  $B(y, r) \subseteq S$ . Consider  $\theta x + (1 - \theta)y := p \in S$ , where  $0 < \theta < 1$ , since  $S$  is convex.



Claim that  $B(p, r) \subseteq S$  as follows.

Let  $q \in B(p, r)$ . We want to find two special points  $\tilde{x} \in B(x, r)$ , and  $\tilde{y} \in B(y, r)$  such that  $q = \theta\tilde{x} + (1 - \theta)\tilde{y}$ .

Since the three  $n$ -balls  $B(x, r)$ ,  $B(y, r)$ , and  $B(p, r)$  have the same radius. By parallelogram principle, we let  $\tilde{x} = q + (x - p)$ , and  $\tilde{y} = q + (y - p)$ , then  $\|x - \tilde{x}\| = \|q - p\| < r$ , and  $\|\tilde{y} - y\| = \|q - p\| < r$ .

It implies that  $\tilde{x} \in B(x, r)$ , and  $\tilde{y} \in B(y, r)$ . In addition,

$$\begin{aligned} & \theta\tilde{x} + (1 - \theta)\tilde{y} \\ &= \theta(q + (x - p)) + (1 - \theta)(q + (y - p)) \\ &= q. \end{aligned}$$

Since  $\tilde{x}, \tilde{y} \in S$ , and  $S$  is convex, then  $q = \theta\tilde{x} + (1 - \theta)\tilde{y} \in S$ . It implies that  $B(p, r) \subseteq S$  since  $q$  is arbitrary. So, we have proved the claim. That is, for  $0 < \theta < 1$ ,  $\theta x + (1 - \theta)y = p \in \text{int}S$  if  $x, y \in \text{int}S$ , and  $S$  is convex. Hence, by the definition of convex, we know that the interior of a convex is convex.

(d) *The closure of a convex is convex.*

**Proof:** Given a convex set  $S$ , and let  $x, y \in \bar{S}$ . Consider  $\theta x + (1 - \theta)y := p$ , where  $0 < \theta < 1$ , and claim that  $p \in \bar{S}$ , i.e., we want to show that  $B(p, r) \cap S \neq \emptyset$ .

Suppose **NOT**, there exists  $r > 0$  such that

$$B(p, r) \cap S = \emptyset. \quad *$$

Since  $x, y \in \bar{S}$ , then  $B(x, \frac{r}{2}) \cap S \neq \emptyset$  and  $B(y, \frac{r}{2}) \cap S \neq \emptyset$ . And let  $\tilde{x} \in B(x, \frac{r}{2}) \cap S$  and  $\tilde{y} \in B(y, \frac{r}{2}) \cap S$ . Consider

$$\begin{aligned} \|(\tilde{\theta}\tilde{x} + (1 - \tilde{\theta})\tilde{y}) - p\| &= \|(\tilde{\theta}\tilde{x} + (1 - \tilde{\theta})\tilde{y}) - (\theta x + (1 - \theta)y)\| \\ &\leq \|\tilde{\theta}\tilde{x} - \theta x\| + \|(1 - \tilde{\theta})\tilde{y} - (1 - \theta)y\| \\ &= \|\tilde{\theta}\tilde{x} - \tilde{\theta}x + \tilde{\theta}x - \theta x\| + \\ &= \|(1 - \tilde{\theta})\tilde{y} - (1 - \tilde{\theta})y + (1 - \tilde{\theta})y - (1 - \theta)y\| \\ &\leq \tilde{\theta}\|\tilde{x} - x\| + (1 - \tilde{\theta})\|\tilde{y} - y\| + |\tilde{\theta} - \theta|(\|x\| + \|y\|) \\ &< \frac{r}{2} + |\tilde{\theta} - \theta|(\|x\| + \|y\|) \\ &< r \end{aligned}$$

if we choose a suitable number  $\tilde{\theta}$ , where  $0 < \tilde{\theta} < 1$ .

Hence, we have the point  $\tilde{\theta}\tilde{x} + (1 - \tilde{\theta})\tilde{y} \in B(p, r)$ . Note that  $\tilde{x}, \tilde{y} \in S$  and  $S$  is convex, we have  $\tilde{\theta}\tilde{x} + (1 - \tilde{\theta})\tilde{y} \in S$ . It leads us to get a contradiction by (\*). Hence, we have proved the claim. That is, for  $0 < \theta < 1$ ,  $\theta x + (1 - \theta)y = p \in \bar{S}$  if  $x, y \in \bar{S}$ . Hence, by the definition of convex, we know that the closure of a convex is convex.

3.15 Let  $F$  be a collection of sets in  $R^n$ , and let  $S = \cup_{A \in F} A$  and  $T = \cap_{A \in F} A$ . For each of the following statements, either give a proof or exhibit a counterexample.

(a) *If  $x$  is an accumulation point of  $T$ , then  $x$  is an accumulation point of each set  $A$  in  $F$ .*

**Proof:** Let  $x$  be an accumulation point of  $T$ , then  $(B(x, r) - \{x\}) \cap T \neq \emptyset$  for any  $r > 0$ . Note that for any  $A \in F$ , we have  $T \subseteq A$ . Hence  $(B(x, r) - \{x\}) \cap A \neq \emptyset$  for any  $r > 0$ . That is,  $x$  is an accumulation point of  $A$  for any  $A \in F$ .

**The conclusion is that** If  $x$  is an accumulation point of  $T = \cap_{A \in F} A$ , then  $x$  is an accumulation point of each set  $A$  in  $F$ .

(b) If  $x$  is an accumulation point of  $S$ , then  $x$  is an accumulation point of at least one set  $A$  in  $F$ .

**Proof: No!** For example, Let  $S = \mathbb{R}^n$ , and  $F$  be the collection of sets consisting of a single point  $x \in \mathbb{R}^n$ . Then it is trivially seen that  $S = \bigcup_{A \in F} A$ . And if  $x$  is an accumulation point of  $S$ , then  $x$  is not an accumulation point of each set  $A$  in  $F$ .

**3.16** Prove that the set  $S$  of rational numbers in the interval  $(0, 1)$  cannot be expressed as the intersection of a countable collection of open sets. Hint: Write  $S = \{x_1, x_2, \dots\}$ , assume that  $S = \bigcap_{k=1}^{\infty} S_k$ , where each  $S_k$  is open, and construct a sequence  $\{Q_n\}$  of closed intervals such that  $Q_{n+1} \subseteq Q_n \subseteq S_n$  and such that  $x_n \notin Q_n$ . Then use the Cantor intersection theorem to obtain a contradiction.

**Proof:** We prove the statement by method of contradiction. Write  $S = \{x_1, x_2, \dots\}$ , and assume that  $S = \bigcap_{k=1}^{\infty} S_k$ , where each  $S_k$  is open.

Since  $x_1 \in S_1$ , there exists a bounded and open interval  $I_1 \subseteq S_1$  such that  $x_1 \in I_1$ . Choose a closed interval  $Q_1 \subseteq I_1$  such that  $x_1 \notin Q_1$ . Since  $Q_1$  is an interval, it contains infinite rationals, call one of these,  $x_2$ . Since  $x_2 \in S_2$ , there exists an open interval  $I_2 \subseteq S_2$  and  $I_2 \subseteq Q_1$ . Choose a closed interval  $Q_2 \subseteq I_2$  such that  $x_2 \notin Q_2$ . Suppose  $Q_n$  has been constructed so that

1.  $Q_n$  is a closed interval.
2.  $Q_n \subseteq Q_{n-1} \subseteq S_{n-1}$ .
3.  $x_n \notin Q_n$ .

Since  $Q_n$  is an interval, it contains infinite rationals, call one of these,  $x_{n+1}$ . Since  $x_{n+1} \in S_{n+1}$ , there exists an open interval  $I_{n+1} \subseteq S_{n+1}$  and  $I_{n+1} \subseteq Q_n$ . Choose a closed interval  $Q_{n+1} \subseteq I_{n+1}$  such that  $x_{n+1} \notin Q_{n+1}$ . So,  $Q_{n+1}$  satisfies our induction hypothesis, and the construction can process.

Note that

1. For all  $n$ ,  $Q_n$  is not empty.
2. For all  $n$ ,  $Q_n$  is bounded since  $I_1$  is bounded.
3.  $Q_{n+1} \subseteq Q_n$ .
4.  $x_n \notin Q_n$ .

Then  $\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$  by **Cantor Intersection Theorem**.

Since  $Q_n \subseteq S_n$ ,  $\bigcap_{n=1}^{\infty} Q_n \subseteq \bigcap_{n=1}^{\infty} S_n = S$ . So, we have

$$S \cap \left(\bigcap_{n=1}^{\infty} Q_n\right) = \bigcap_{n=1}^{\infty} Q_n \neq \emptyset$$

which is absurd since  $S \cap \left(\bigcap_{n=1}^{\infty} Q_n\right) = \emptyset$  by the fact  $x_n \notin Q_n$ . Hence, we have proved that our assumption does not hold. That is,  $S$  the set of rational numbers in the interval  $(0, 1)$  cannot be expressed as the intersection of a countable collection of open sets.

**Remark:** 1. Often, the property is described by saying  $Q$  is not an  $G_\delta$ -set.

2. It should be noted that  $Q^c$  is an  $G_\delta$ -set.

3. For the famous Theorem called **Cantor Intersection Theorem**, the reader should see another classical text book, Principles of Mathematical Analysis written by Walter Rudin, Theorem 3.10 in page 53.

4. For the method of proof, the reader should see another classical text book, Principles of Mathematical Analysis written by Walter Rudin, Theorem 2.43, in page 41.

## Covering theorems in $R^n$

3.17 If  $S \subseteq R^n$ , prove that the collection of isolated points of  $S$  is countable.

**Proof:** Denote the collection of isolated points of  $S$  by  $F$ . Let  $x \in F$ , there exists an  $n$ -ball  $(B(x, r_x) - \{x\}) \cap S = \emptyset$ . Write  $Q^n = \{x_1, x_2, \dots\}$ , then there are many numbers in  $Q^n$  lying on  $B(x, r_x) - \{x\}$ . We choose the smallest index, say  $m = m(x)$ , and denote  $x$  by  $x_m$ .

So,  $F = \{x_m : m \in P\}$ , where  $P(\subseteq N)$ , a subset of positive integers. Hence,  $F$  is countable.

3.18 Prove that the set of open disks in the  $xy$ -plane with center  $(x, x)$  and radius  $x > 0$ ,  $x$  rational, is a countable covering of the set  $\{(x, y) : x > 0, y > 0\}$ .

**Proof:** Denote the set of open disks in the  $xy$ -plane with center  $(x, x)$  and radius  $x > 0$  by  $S$ . Choose any point  $(a, b)$ , where  $a > 0$ , and  $b > 0$ . We want to find an  $2$ -ball  $B((x, x), x) (\in S)$  which contains  $(a, b)$ . It suffices to find  $x \in Q$  such that  $\|(x, x) - (a, b)\| < x$ . Since

$$\|(x, x) - (a, b)\| < x \Leftrightarrow \|(x, x) - (a, b)\|^2 < x^2 \Leftrightarrow x^2 - 2(a+b)x + (a^2 + b^2) < 0.$$

Since  $x^2 - 2(a+b)x + (a^2 + b^2) = [x - (a+b)]^2 - 2ab$ , we can choose a suitable rational number  $x$  such that  $x^2 - 2(a+b)x + (a^2 + b^2) < 0$  since  $a > 0$ , and  $b > 0$ . Hence, for any point  $(a, b)$ , where  $a > 0$ , and  $b > 0$ , we can find an  $2$ -ball  $B((x, x), x) (\in S)$  which contains  $(a, b)$ .

That is,  $S$  is a countable covering of the set  $\{(x, y) : x > 0, y > 0\}$ .

**Remark:** The reader should give a geometric appearance or draw a graph.

3.19 The collection  $F$  of open intervals of the form  $(1/n, 2/n)$ , where  $n = 2, 3, \dots$ , is an open covering of the open interval  $(0, 1)$ . Prove (without using Theorem 3.31) that no finite subcollection of  $F$  covers  $(0, 1)$ .

**Proof:** Write  $F$  as  $\{(\frac{1}{2}, 1), (\frac{1}{3}, \frac{2}{3}), \dots, (\frac{1}{n}, \frac{2}{n}), \dots\}$ . Obviously,  $F$  is an open covering of  $(0, 1)$ . Assume that there exists a finite subcollection of  $F$  covers  $(0, 1)$ , and thus write them as  $F' = \{(\frac{1}{n_1}, \frac{1}{m_1}), \dots, (\frac{1}{n_k}, \frac{1}{m_k})\}$ . Choose  $p \in (0, 1)$  so that  $p < \min_{1 \leq i \leq k} (\frac{1}{n_i})$ . Then  $p \notin (\frac{1}{n_i}, \frac{1}{m_i})$ , where  $1 \leq i \leq k$ . It contradicts the fact  $F'$  covers  $(0, 1)$ .

**Remark:** The reader should be noted that if we use Theorem 3.31, then we cannot get the correct proof. In other words, the author T. M. Apostol mistakes the statement.

3.20 Give an example of a set  $S$  which is closed but not bounded and exhibit a countable open covering  $F$  such that no finite subset of  $F$  covers  $S$ .

**Solution:** Let  $S = R^1$ , then  $R^1$  is closed but not bounded. And let  $F = \{(n, n+2) : n \in Z\}$ , then  $F$  is a countable open covering of  $S$ . In addition, it is trivially seen that **no finite subset of  $F$  covers  $S$** .

3.21 Given a set  $S$  in  $R^n$  with the property that for every  $x$  in  $S$  there is an  $n$ -ball  $B(x)$  such that  $B(x) \cap S$  is countable. Prove that  $S$  is countable.

**Proof:** Note that  $F = \{B(x) : x \in S\}$  forms an open covering of  $S$ . Since  $S \subseteq R^n$ , then there exists a countable subcover  $F'(\subseteq F)$  of  $S$  by **Lindelof Covering Theorem**. Write  $F' = \{B(x_n) : n \in N\}$ . Since

$$S = S \cap (\cup_{n \in N} B(x_n)) = \cup_{n \in N} (S \cap B(x_n)),$$

and

$S \cap B(x_n)$  is countable by hypothesis.

Then  $S$  is countable.

**Remark:** The reader should be noted that exercise 3.21 is equivalent to exercise 3.23.

3.22 Prove that a collection of disjoint open sets in  $R^n$  is necessarily countable. Give an example of a collection of disjoint closed sets which is not countable.

**Proof:** Let  $F$  be a collection of disjoint open sets in  $R^n$ , and write  $Q^n = \{x_1, x_2, \dots\}$ . Choose an open set  $S (\neq \phi)$  in  $F$ , then there exists an  $n$ -ball  $B(y, r) \subseteq S$ . In this ball, there are infinite numbers in  $Q^n$ . We choose the smallest index, say  $m = m(y)$ . Then we have  $F = \{S_m : m \in P \subseteq N\}$  which is countable.

For the example that a collection of disjoint closed sets which is not countable, we give it as follows. Let  $G = \{\{x\} : x \in R^n\}$ , then we complete it.

3.23 Assume that  $S \subseteq R^n$ . A point  $x$  in  $R^n$  is said to be **condensation** point of  $S$  if every  $n$ -ball  $B(x)$  has the property that  $B(x) \cap S$  is not countable. Prove that if  $S$  is not countable, then there exists a point  $x$  in  $S$  such that  $x$  is a condensation point of  $S$ .

**Proof:** It is equivalent to exercise 3.21.

**Remark:** Compare with two definitions on a condensation point and an accumulation point, it is easy to know that a condensation point is an accumulation point. However, an accumulation point is not a condensation point, for example,  $S = \{1/n : n \in N\}$ . We have 0 is an accumulation point of  $S$ , but not a condensation point of  $S$ .

3.24 Assume that  $S \subseteq R^n$  and assume that  $S$  is not countable. Let  $T$  denote the set of condensation points of  $S$ . Prove that

(a)  $S - T$  is countable.

**Proof:** If  $S - T$  is uncountable, then there exists a point  $x$  in  $S - T$  such that  $x$  is a condensation point of  $S - T$  by exercise 3.23. Obviously,  $x (\in S)$  is also a condensation point of  $S$ . It implies  $x \in T$ . So, we have  $x \in S \cap T$  which is absurd since  $x \in S - T$ .

**Remark:** The reader should regard  $T$  as a special part of  $S$ , and the advantage of  $T$  helps us realize the uncountable set  $S (\subseteq R^n)$ . Compare with **Cantor-Bendixon Theorem** in exercise 3.25.

(b)  $S \cap T$  is not countable.

**Proof:** Suppose  $S \cap T$  is countable, then  $S = (S \cap T) \cup (S - T)$  is countable by (a) which is absurd. So,  $S \cap T$  is not countable.

(c)  $T$  is a closed set.

**Proof:** Let  $x$  be an adherent point of  $T$ , then  $B(x, r) \cap T \neq \phi$  for any  $r > 0$ . We want to show  $x \in T$ . That is to show  $x$  is a condensation point of  $S$ . Claim that  $B(x, r) \cap S$  is uncountable for any  $r > 0$ .

Suppose **NOT**, then there exists an  $n$ -ball  $B(x, d) \cap S$  which is countable. Since  $x$  is an adherent point of  $T$ , then  $B(x, d) \cap T \neq \phi$ . Choose  $y \in B(x, d) \cap T$  so that  $B(y, \delta) \subseteq B(x, d)$  and  $B(y, \delta) \cap S$  is uncountable. However, we get a contradiction since

$$B(y, \delta) \cap S \text{ (is uncountable)} \subseteq B(x, d) \cap S \text{ (is countable)}.$$

Hence,  $B(x, r) \cap S$  is uncountable for any  $r > 0$ . That is,  $x \in T$ . Since  $T$  contains its all adherent points,  $T$  is closed.

(d)  $T$  contains no isolated points.

**Proof:** Let  $x \in T$ , and if  $x$  is an isolated point of  $T$ , then there exists an  $n$ -ball  $B(x, d)$  such that  $B(x, d) \cap T = \{x\}$ . On the other hand,  $x \in T$  means that  $(B(x, d) - \{x\}) \cap S$  is

uncountable. Hence, by exercise 3.23, we know that there exists  $y \in (B(x, d) - \{x\}) \cap S$  such that  $y$  is a condensation point of  $(B(x, d) - \{x\}) \cap S$ . So,  $y$  is a condensation point of  $S$ . It implies  $y \in T$ . It is impossible since

1.  $y(\neq x) \in T$ .
2.  $y \in B(x, d)$ .
3.  $B(x, d) \cap T = \{x\}$ .

Hence,  $x$  is not an isolated point of  $T$ , if  $x \in T$ . That is,  $T$  contains no isolated points.

**Remark:** Use exercise 3.25, by (c) and (d) we know that  $T$  is perfect.

**Note that Exercise 3.23 is a special case of (b).**

**3.25** A set in  $R^n$  is called **perfect** if  $S' = S$ , that is, if  $S$  is a closed set which contains no isolated points. Prove that every uncountable closed set  $F$  in  $R^n$  can be expressed in the form  $F = A \cup B$ , where  $A$  is perfect and  $B$  is countable (**Cantor-Bendixon theorem**).

*Hint. Use Exercise 3.24.*

**Proof:** Let  $F$  be a uncountable closed set in  $R^n$ . Then by exercise 3.24,  $F = (F \cap T) \cup (F - T)$ , where  $T$  is the set of condensation points of  $F$ . Note that since  $F$  is closed,  $T \subseteq F$  by the fact, a condensation point is an accumulation point. Define  $F \cap T = A$  and  $F - T = B$ , then  $B$  is countable and  $A(= T)$  is perfect.

**Remark:** 1. The reader should see another classical text book, Principles of Mathematical Analysis written by Walter Rudin, Theorem 2.43, in page 41. Since the theorem is famous, we list it below.

**Theorem 2.43** Let  $P$  be a nonempty perfect set in  $R^k$ . Then  $P$  is uncountable.

**Theorem (Modified 2.43)** Let  $P$  be a nonempty perfect set in a **complete separable metric space**. Then  $P$  is uncountable.

2. Let  $S$  has measure zero in  $R^1$ . Prove that there is a nonempty perfect set  $P$  in  $R^1$  such that  $P \cap S = \phi$ .

**Proof:** Since  $S$  has measure zero, there exists a collection of open intervals  $\{I_k\}$  such that

$$S \subseteq \cup I_k \text{ and } \sum |I_k| < 1.$$

Consider its complement  $(\cup I_k)^c$  which is closed with positive measure. Since the complement has a positive measure, we know that it is uncountable. Hence, by Cantor-Bendixon Theorem, we know that

$$(\cup I_k)^c = A \cup B, \text{ where } A \text{ is perfect and } B \text{ is countable.}$$

So, let  $A = P$ , we have prove it.

**Note:** From the similar method, we can show that given any set  $S$  in  $R^1$  with measure  $0 \leq d < \infty$ , there is a non-empty perfect set  $P$  such that  $P \cap S = \phi$ . In particular,  $S = \mathbb{Q}$ ,  $S =$ the set of algebraic numbers, and so on. In addition, even for cases in  $R^k$ , it still holds.

## Metric Spaces

**3.26** In any metric space  $(M, d)$  prove that the empty set  $\phi$  and the whole set  $M$  are both open and closed.

**proof:** In order to show the statement, it suffices to show that  $M$  is open and closed since  $M - M = \phi$ . Let  $x \in M$ , then for any  $r > 0$ ,  $B_M(x, r) \subseteq M$ . That is,  $x$  is an interior

point of  $M$ . Since  $x$  is arbitrary, we know that every point of  $M$  is interior. So,  $M$  is open.

Let  $x$  be an adherent point of  $M$ , it is clearly  $x \in M$  since we consider all points lie in  $M$ . Hence,  $M$  contains its all adherent points. It implies that  $M$  is closed.

**Remark:** The reader should regard the statement as a common sense.

3.27 Consider the following two metrics in  $R^n$  :

$$d_1(x,y) = \max_{1 \leq i \leq n} |x_i - y_i|, \quad d_2(x,y) = \sum_{i=1}^{i=n} |x_i - y_i|.$$

In each of the following metric spaces prove that the ball  $B(a;r)$  has the geometric appearance indicated:

(a) In  $(R^2, d_1)$ , a square with sides parallel to the coordinate axes.

**Solution:** It suffices to consider the case  $B((0,0), 1)$ . Let  $x = (x_1, x_2) \in B((0,0), 1)$ , then we have

$$|x_1| < 1, \text{ and } |x_2| < 1.$$

So, it means that the ball  $B((0,0), 1)$  is a square with sides lying on the coordinate axes. Hence, we know that  $B(a;r)$  is a square with sides parallel to the coordinate axes.

(b) In  $(R^2, d_2)$ , a square with diagonals parallel to the axes.

**Solution:** It suffices to consider the case  $B((0,0), 1)$ . Let  $x = (x_1, x_2) \in B((0,0), 1)$ , then we have

$$|x_1 + x_2| < 1.$$

So, it means that the ball  $B((0,0), 1)$  is a square with diagonals lying on the coordinate axes. Hence, we know that  $B(a;r)$  is a square with diagonals parallel to the coordinate axes.

(c) A cube in  $(R^3, d_1)$ .

**Solution:** It suffices to consider the case  $B((0,0,0), 1)$ . Let  $x = (x_1, x_2, x_3) \in B((0,0,0), 1)$ , then we have

$$|x_1| < 1, |x_2| < 1, \text{ and } |x_3| < 1.$$

So, it means that the ball  $B((0,0,0), 1)$  is a cube with length 2. Hence, we know that  $B(a;r)$  is a cube with length  $2a$ .

(d) An octahedron in  $(R^3, d_2)$ .

**Solution:** It suffices to consider the case  $B((0,0,0), 1)$ . Let  $x = (x_1, x_2, x_3) \in B((0,0,0), 1)$ , then we have

$$|x_1 + x_2 + x_3| < 1.$$

It means that the ball  $B((0,0,0), 1)$  is an octahedron. Hence,  $B(a;r)$  is an octahedron.

**Remark:** The exercise tells us one thing that  $B(a;r)$  may not be an  $n$ -ball if we consider some different matrices.

3.28 Let  $d_1$  and  $d_2$  be the metrics of Exercise 3.27 and let  $\|x - y\|$  denote the usual Euclidean metric. Prove that the following inequalities for all  $x$  and  $y$  in  $R^n$  :

$$d_1(x,y) \leq \|x - y\| \leq d_2(x,y) \text{ and } d_2(x,y) \leq \sqrt{n} \|x - y\| \leq nd_1(x,y).$$

**Proof:** List the definitions of the three metrics, and compare with them as follows.

1.  $d_1(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ .
2.  $\|x - y\| = \left( \sum_{i=1}^{i=n} (x_i - y_i)^2 \right)^{1/2}$ .
3.  $d_2(x, y) = \sum_{i=1}^{i=n} |x_i - y_i|$ .

Then we have

(a)

$$\begin{aligned} d_1(x, y) &= \max_{1 \leq i \leq n} |x_i - y_i| = \left( \max_{1 \leq i \leq n} |x_i - y_i|^2 \right)^{1/2} \\ &\leq \left( \sum_{i=1}^{i=n} (x_i - y_i)^2 \right)^{1/2} = \|x - y\|. \end{aligned}$$

(b)

$$\begin{aligned} \|x - y\| &= \left( \sum_{i=1}^{i=n} (x_i - y_i)^2 \right)^{1/2} \\ &\leq \left[ \left( \sum_{i=1}^{i=n} |x_i - y_i| \right)^2 \right]^{1/2} = \sum_{i=1}^{i=n} |x_i - y_i| = d_2(x, y). \end{aligned}$$

(c)

$$\begin{aligned} \sqrt{n} \|x - y\| &= \sqrt{n} \left( \sum_{i=1}^{i=n} (x_i - y_i)^2 \right)^{1/2} = \left( n \sum_{i=1}^{i=n} (x_i - y_i)^2 \right)^{1/2} \\ &\leq \left\{ n \left( n \left[ \max_{1 \leq i \leq n} |x_i - y_i| \right]^2 \right) \right\}^{1/2} = n \max_{1 \leq i \leq n} |x_i - y_i| \\ &= d_1(x, y). \end{aligned}$$

(d)

$$\begin{aligned} [d_2(x, y)]^2 &= \left( \sum_{i=1}^{i=n} |x_i - y_i| \right)^2 = \sum_{i=1}^{i=n} (x_i - y_i)^2 + \sum_{1 \leq i < j \leq n} 2|x_i - y_i||x_j - y_j| \\ &\leq \sum_{i=1}^{i=n} (x_i - y_i)^2 + (n-1) \sum_{i=1}^{i=n} (x_i - y_i)^2 \text{ by } A. P. \geq G. P. \\ &= n \sum_{i=1}^{i=n} (x_i - y_i)^2 \\ &= n \|x - y\|^2. \end{aligned}$$

So,

$$d_2(x, y) \leq \sqrt{n} \|x - y\|.$$

From (a)-(d), we have proved these inequalities.

**Remark:** 1. Let  $M$  be a given set and suppose that  $(M, d)$  and  $(M, \bar{d})$  are metric spaces. We define the metrics  $d$  and  $\bar{d}$  are **equivalent** if, and only if, there exist positive constants  $\alpha, \beta$  such that

$$\alpha d(x, y) \leq \bar{d}(x, y) \leq \beta d(x, y).$$

The concept is much important for us to consider the same set with different metrics. For

example, in this exercise, Since three metrics are equivalent, it is easy to know that  $(R^k, d_1)$ ,  $(R^k, d_2)$ , and  $(R^k, \|\cdot\|)$  are **complete**. (For definition of complete metric space, the reader can see this text book, page 74.)

2. It should be noted that on a finite dimensional vector space  $X$ , any two norms are equivalent.

3.29 If  $(M, d)$  is a metric space, define  $d'(x, y) = \frac{d(x, y)}{1+d(x, y)}$ . Prove that  $d'$  is also a metric for  $M$ . Note that  $0 \leq d'(x, y) < 1$  for all  $x, y$  in  $M$ .

**Proof:** In order to show that  $d'$  is a metric for  $M$ , we consider the following four steps.

(1) For  $x \in M$ ,  $d'(x, x) = 0$  since  $d(x, x) = 0$ .

(2) For  $x \neq y$ ,  $d'(x, y) = \frac{d(x, y)}{1+d(x, y)} > 0$  since  $d(x, y) > 0$ .

(3) For  $x, y \in M$ ,  $d'(x, y) = \frac{d(x, y)}{1+d(x, y)} = \frac{d(y, x)}{1+d(y, x)} = d'(y, x)$

(4) For  $x, y, z \in M$ ,

$$\begin{aligned} d'(x, y) &= \frac{d(x, y)}{1+d(x, y)} = 1 - \frac{1}{1+d(x, y)} \\ &\leq 1 - \frac{1}{1+d(x, z) + d(z, y)} \text{ since } d(x, y) \leq d(x, z) + d(z, y) \\ &= \frac{d(x, z) + d(z, y)}{1+d(x, z) + d(z, y)} \\ &\leq \frac{d(x, z)}{1+d(x, z) + d(z, y)} + \frac{d(z, y)}{1+d(x, z) + d(z, y)} \\ &\leq \frac{d(x, z)}{1+d(x, z)} + \frac{d(z, y)}{1+d(z, y)} \\ &= d'(x, z) + d'(z, y) \end{aligned}$$

Hence, from (1)-(4), we know that  $d'$  is also a metric for  $M$ . Obviously,  $0 \leq d'(x, y) < 1$  for all  $x, y$  in  $M$ .

**Remark:** 1. The exercise tells us how to form a new metric from an old metric. Also, the reader should compare with exercise 3.37. This is another construction.

2. Recall **Discrete metric**  $d$ , we find that given any set nonempty  $S$ ,  $(S, d)$  is a metric space, and thus use the exercise, we get another metric space  $(S, d')$ , and so on. Hence, here is a common sense that given any nonempty set, we can use discrete metric to form many and many metric spaces.

3.30 Prove that every finite subset of a metric space is closed.

**Proof:** Let  $x$  be an adherent point of a finite subset  $S = \{x_i : i = 1, 2, \dots, n\}$  of a metric space  $(M, d)$ . Then for any  $r > 0$ ,  $B(x, r) \cap S \neq \emptyset$ . If  $x \notin S$ , then  $B_M(x, \delta) \cap S = \emptyset$  where  $\delta = \min_{1 \leq i \leq n} d(x_i, x)$ . It is impossible. Hence,  $x \in S$ . That is,  $S$  contains its all adherent points. So,  $S$  is closed.

3.31 In a metric space  $(M, d)$  the closed ball of radius  $r > 0$  about a point  $a$  in  $M$  is the set  $\bar{B}(a; r) = \{x : d(x, a) \leq r\}$ .

(a) Prove that  $\bar{B}(a; r)$  is a closed set.

**Proof:** Let  $x \in M - \bar{B}(a; r)$ , then  $d(x, a) > r$ . Consider  $B(x, \delta)$ , where  $\delta = \frac{d(x, a) - r}{2}$ , then if  $y \in B(x, \delta)$ , we have  $d(y, a) \geq d(x, a) - d(x, y) > d(x, a) - \delta = \frac{d(x, a) + r}{2} > r$ . Hence,  $B(x, \delta) \subseteq M - \bar{B}(a; r)$ . That is, every point of  $M - \bar{B}(a; r)$  is interior. So,  $M - \bar{B}(a; r)$  is open, or equivalently,  $\bar{B}(a; r)$  is a closed set.



(b) Give an example of a metric space in which  $\bar{B}(a; r)$  is not the closure of the open ball  $B(a; r)$ .

**Solution:** Consider discrete metric space  $M$ , then we have (let  $x \in M$ )

$$\text{The closure of } B(a; 1) = \{a\}$$

and

$$\bar{B}(a; 1) = M.$$

Hence, if we let  $\{a\}$  is a proper subset of  $M$ , then  $\bar{B}(a; 1)$  is not the closure of the open ball  $B(a; 1)$ .

3.32 In a metric space  $M$ , if subsets satisfy  $A \subseteq S \subseteq \bar{A}$ , where  $\bar{A}$  is the closure of  $A$ , then  $A$  is said to be **dense** in  $S$ . For example, the set  $Q$  of rational numbers is dense in  $R$ . If  $A$  is dense in  $S$  and if  $S$  is dense in  $T$ , prove that  $A$  is dense in  $T$ .

**Proof:** Since  $A$  is dense in  $S$  and  $S$  is dense in  $T$ , we have  $\bar{A} \supseteq S$  and  $\bar{S} \supseteq T$ . Then  $\bar{A} \supseteq T$ . That is,  $A$  is dense in  $T$ .

3.33 Refer to exercise 3.32. A metric space  $M$  is said to be separable if there is a countable subset  $A$  which is dense in  $M$ . For example,  $R^1$  is separable because the set  $Q$  of rational numbers is a countable dense subset. Prove that every Euclidean space  $R^k$  is separable.

**Proof:** Since  $Q^k$  is a countable subset of  $R^k$ , and  $\bar{Q}^k = R^k$ , then we know that  $R^k$  is separable.

3.34 Refer to exercise 3.33. Prove that the Lindelof covering theorem (Theorem 3.28) is valid in any separable metric space.

**Proof:** Let  $(M, d)$  be a separable metric space. Then there exists a countable subset  $S = \{x_n : n \in N\} (\subseteq M)$  which is dense in  $M$ . Given a set  $A \subseteq M$ , and an open covering  $F$  of  $A$ . Write  $P = \{B(x_n, r_m) : x_n \in S, r_m \in Q\}$ .

Claim that if  $x \in M$ , and  $G$  is an open set in  $M$  which contains  $x$ . Then  $x \in B(x_n, r_m) \subseteq G$  for some  $B(x_n, r_m) \subseteq P$ .

Since  $x \in G$ , there exists  $B(x, r_x) \subseteq G$  for some  $r_x > 0$ . Note that  $x \in cl(S)$  since  $S$  is dense in  $M$ . Then,  $B(x, r_x/2) \cap S \neq \emptyset$ . So, if we choose  $x_n \in B(x, r_x/2) \cap S$  and  $r_m \in Q$  with  $r_x/2 < r_m < r_x/3$ , then we have

$$x \in B(x_n, r_m)$$

and

$$B(x_n, r_m) \subseteq B(x, r_x)$$

since if  $y \in B(x_n, r_m)$ , then

$$\begin{aligned} d(y, x) &\leq d(y, x_n) + d(x_n, x) \\ &< r_m + \frac{r_x}{2} \\ &< \frac{r_x}{3} + \frac{r_x}{2} \\ &< r_x \end{aligned}$$

So, we have proved the claim  $x \in B(x_n, r_m) \subseteq B(x, r_x) \subseteq G$  or some  $B(x_n, r_m) \in P$ .

Use the claim to show the statement as follows. Write  $A \subseteq \cup_{G \in F} G$ , and let  $x \in A$ , then there is an open set  $G$  in  $F$  such that  $x \in G$ . By the claim, there is  $B(x_n, r_m) := B_{n+m}$  in  $P$  such that  $x \in B_{n+m} \subseteq G$ . There are, of course, infinitely many such  $B_{n+m}$  corresponding to each  $G$ , but we choose only one of these, for example, the one of smallest index, say  $q = q(x)$ . Then we have  $x \in B_{q(x)} \subseteq G$ .

The set of all  $B_{q(x)}$  obtained as  $x$  varies over all elements of  $A$  is a countable collection of open sets which covers  $A$ . To get a countable subcollection of  $F$  which covers  $A$ , we simply correlate to each set  $B_{q(x)}$  one of the sets  $G$  of  $F$  which contained  $B_{q(x)}$ . This complete the proof.

3.35 Refer to exercise 3.32. If  $A$  is dense in  $S$  and  $B$  is open in  $S$ , prove that  $B \subseteq cl(A \cap B)$ , where  $cl(A \cap B)$  means the closure of  $A \cap B$ .

Hint. Exercise 3.13.

**Proof:** Since  $A$  is dense in  $S$  and  $B$  is open in  $S$ ,  $\bar{A} \supseteq S$  and  $S \cap B = B$ . Then

$$\begin{aligned} B &= S \cap B \\ &\subseteq \bar{A} \cap B, B \text{ is open in } S \\ &\subseteq cl(A \cap B) \end{aligned}$$

by exercise 3.13.

3.36 Refer to exercise 3.32. If each of  $A$  and  $B$  is dense in  $S$  and if  $B$  is open in  $S$ , prove that  $A \cap B$  is dense in  $S$ .

**Proof:** Since

$$\begin{aligned} &cl(A \cap B), B \text{ is open} \\ &\supseteq cl(A) \cap B \text{ by exercise 3.13} \\ &\supseteq S \cap B \text{ since } A \text{ is dense in } S \\ &= B \text{ since } B \text{ is open in } S \end{aligned}$$

then

$$cl(A \cap B) \supseteq B$$

which implies

$$cl(A \cap B) \supseteq S$$

since  $B$  is dense in  $S$ .

3.37 Given two metric spaces  $(S_1, d_1)$  and  $(S_2, d_2)$ , a metric  $\rho$  for the Cartesian product  $S_1 \times S_2$  can be constructed from  $d_1 \times d_2$  in many ways. For example, if  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  are in  $S_1 \times S_2$ , let  $\rho(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$ . Prove that  $\rho$  is a metric for  $S_1 \times S_2$  and construct further examples.

**Proof:** In order to show that  $\rho$  is a metric for  $S_1 \times S_2$ , we consider the following four steps.

(1) For  $x = (x_1, x_2) \in S_1 \times S_2$ ,  $\rho(x, x) = d_1(x_1, x_1) + d_2(x_2, x_2) = 0 + 0 = 0$ .

(2) For  $x \neq y$ ,  $\rho(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) > 0$  since if  $\rho(x, y) = 0$ , then  $x_1 = y_1$  and  $x_2 = y_2$ .

(3) For  $x, y \in S_1 \times S_2$ ,

$$\begin{aligned} \rho(x, y) &= d_1(x_1, y_1) + d_2(x_2, y_2) \\ &= d_1(y_1, x_1) + d_2(y_2, x_2) \\ &= \rho(y, x). \end{aligned}$$

(4) For  $x, y, z \in S_1 \times S_2$ ,

$$\begin{aligned} \rho(x, y) &= d_1(x_1, y_1) + d_2(x_2, y_2) \\ &\leq d_1(x_1, z_1) + d_1(z_1, y_1) + d_2(x_2, z_2) + d_2(z_2, y_2) \\ &= [d_1(x_1, z_1) + d_2(x_2, z_2)] + [d_1(z_1, y_1) + d_2(z_2, y_2)] \\ &\leq \rho(x, z) + \rho(z, y). \end{aligned}$$

Hence from (1)-(4), we know that  $\rho$  is a metric for  $S_1 \times S_2$ .

For other metrics, we define

$$\rho_1(x, y) := \alpha d_1(x_1, y_1) + \beta d_2(x_2, y_2) \text{ for } \alpha, \beta > 0.$$

$$\rho_2(x, y) := d_1(x_1, y_1) + \frac{d_2(x_2, y_2)}{1 + d_2(x_2, y_2)}$$

and so on. (The proof is similar with us by above exercises.)

### Compact subsets of a metric space

**3.38** Assume  $S \subseteq T \subseteq M$ . Then  $S$  is compact in  $(M, d)$  if, and only if,  $S$  is compact in the metric subspace  $(T, d)$ .

**Proof:** Suppose that  $S$  is compact in  $(M, d)$ . Let  $F = \{O_\alpha : O_\alpha \text{ is open in } T\}$  be an open covering of  $S$ . Since  $O_\alpha$  is open in  $T$ , there exists the corresponding  $G_\alpha$  which is open in  $M$  such that  $G_\alpha \cap T = O_\alpha$ . It is clear that  $\{G_\alpha\}$  forms an open covering of  $S$ . So there is a finite subcovering  $\{G_1, \dots, G_n\}$  of  $S$  since  $S$  is compact in  $(M, d)$ . That is,  $S \subseteq \bigcup_{k=1}^{k=n} G_k$ . It implies that

$$\begin{aligned} S &= T \cap S \\ &\subseteq T \cap (\bigcup_{k=1}^{k=n} G_k) \\ &= \bigcup_{k=1}^{k=n} (T \cap G_k) \\ &= \bigcup_{k=1}^{k=n} O_k (\in F). \end{aligned}$$

So, we find a finite subcovering  $\{O_1, \dots, O_n\}$  of  $S$ . That is,  $S$  is compact in  $(T, d)$ .

Suppose that  $S$  is compact in  $(T, d)$ . Let  $G = \{G_\alpha : G_\alpha \text{ is open in } M\}$  be an open covering of  $S$ . Since  $G_\alpha \cap T := O_\alpha$  is open in  $T$ , the collection  $\{O_\alpha\}$  forms an open covering of  $S$ . So, there is a finite subcovering  $\{O_1, \dots, O_n\}$  of  $S$  since  $S$  is compact in  $(T, d)$ . That is,  $S \subseteq \bigcup_{k=1}^{k=n} O_k$ . It implies that

$$S \subseteq \bigcup_{k=1}^{k=n} O_k \subseteq \bigcup_{k=1}^{k=n} G_k.$$

So, we find a finite subcovering  $\{G_1, \dots, G_n\}$  of  $S$ . That is,  $S$  is compact in  $(M, d)$ .

**Remark:** The exercise tells us one thing that the property of compact is not changed, but we should note the property of being open may be changed. For example, in the 2-dimensional Euclidean space, an open interval  $(a, b)$  is not open since  $(a, b)$  cannot contain any 2-ball.

**3.39** If  $S$  is a closed and  $T$  is compact, then  $S \cap T$  is compact.

**Proof:** Since  $T$  is compact,  $T$  is closed. We have  $S \cap T$  is closed. Since  $S \cap T \subseteq T$ , by **Theorem 3.39**, we know that  $S \cap T$  is compact.

**3.40** The intersection of an arbitrary collection of compact subsets of  $M$  is compact.

**Proof:** Let  $F = \{T : T \text{ is compact in } M\}$ , and thus consider  $\bigcap_{T \in F'} T$ , where  $F' \subseteq F$ . We have  $\bigcap_{T \in F'} T$  is closed. Choose  $S \in F'$ . then we have  $\bigcap_{T \in F'} T \subseteq S$ . Hence, by **Theorem 3.39**  $\bigcap_{T \in F'} T$  is compact.

**3.41** The union of a finite number of compact subsets of  $M$  is compact.

**Proof:** Denote  $\{T_k \text{ is a compact subset of } M : k = 1, 2, \dots, n\}$  by  $S$ . Let  $F$  be an open covering of  $\bigcup_{k=1}^{k=n} T_k$ . If there does **NOT** exist a finite subcovering of  $\bigcup_{k=1}^{k=n} T_k$ , then there does not exist a finite subcovering of  $T_m$  for some  $T_m \in S$ . Since  $F$  is also an open covering of  $T_m$ , it leads us to get  $T_m$  is not compact which is absurd. Hence, if  $F$  is an open covering of  $\bigcup_{k=1}^{k=n} T_k$ , then there exists a finite subcovering of  $\bigcup_{k=1}^{k=n} T_k$ . So,  $\bigcup_{k=1}^{k=n} T_k$  is

compact.

**3.42** Consider the metric space  $Q$  of rational numbers with the Euclidean metric of  $R^1$ . Let  $S$  consists of all rational numbers in the open interval  $(a, b)$ , where  $a$  and  $b$  are irrational. Then  $S$  is a closed and bounded subset of  $Q$  which is not compact.

**Proof:** Obviously,  $S$  is bounded. Let  $x \in Q - S$ , then  $x < a$ , or  $x > b$ . If  $x < a$ , then  $B_Q(x, d) = (x - d, x + d) \cap Q \subseteq Q - S$ , where  $d = a - x$ . Similarly,  $x > b$ . Hence,  $x$  is an interior point of  $Q - S$ . That is,  $Q - S$  is open, or equivalently,  $S$  is closed.

**Remark:** 1. The exercise tells us an counterexample about that in a metric space, a closed and bounded subset is not necessary to be compact.

2. Here is another counterexample. Let  $M$  be an infinite set, and thus consider the metric space  $(M, d)$  with discrete metric  $d$ . Then by the fact  $B(x, 1/2) = \{x\}$  for any  $x \in M$ , we know that  $F = \{B(x, 1/2) : x \in M\}$  forms an open covering of  $M$ . It is clear that there does not exist a finite subcovering of  $M$ . Hence,  $M$  is not compact.

3. In any metric space  $(M, d)$ , we have three equivalent conditions on compact which list them below. Let  $S \subseteq M$ .

- (a) Given any open covering of  $S$ , there exists a finite subcovering of  $S$ .
- (b) Every infinite subset of  $S$  has an accumulation point in  $S$ .
- (c)  $S$  is **totally bounded** and **complete**.

4. It should be note that if we consider the Euclidean space  $(R^n, d)$ , we have four equivalent conditions on compact which list them below. Let  $S \subseteq R^n$ .

- Remark
- (a) Given any open covering of  $S$ , there exists a finite subcovering of  $S$ .
  - (b) Every infinite subset of  $S$  has an accumulation point in  $S$ .
  - (c)  $S$  is **totally bounded** and **complete**.
  - (d)  $S$  is bounded and closed.

5. The concept of compact is familiar with us since it can be regarded as a extension of **Bolzano – Weierstrass Theorem**.

### Miscellaneous properties of the interior and the boundary

If  $A$  and  $B$  denote arbitrary subsets of a metric space  $M$ , prove that:

**3.43**  $intA = M - cl(M - A)$ .

**Proof:** In order to show the statement, it suffices to show that  $M - intA = cl(M - A)$ .

1. ( $\subseteq$ ) Let  $x \in M - intA$ , we want to show that  $x \in cl(M - A)$ , i.e.,  $B(x, r) \cap (M - A) \neq \emptyset$  for all  $r > 0$ . Suppose  $B(x, d) \cap (M - A) = \emptyset$  for some  $d > 0$ . Then  $B(x, d) \subseteq A$  which implies that  $x \in intA$ . It leads us to get a contradiction since  $x \in M - intA$ . Hence, if  $x \in M - intA$ , then  $x \in cl(M - A)$ . That is,  $M - intA \subseteq cl(M - A)$ .

2. ( $\supseteq$ ) Let  $x \in cl(M - A)$ , we want to show that  $x \in M - intA$ , i.e.,  $x$  is not an interior point of  $A$ . Suppose  $x$  is an interior point of  $A$ , then  $B(x, d) \subseteq A$  for some  $d > 0$ . However, since  $x \in cl(M - A)$ , then  $B(x, d) \cap (M - A) \neq \emptyset$ . It leads us to get a contradiction since  $B(x, d) \subseteq A$ . Hence, if  $x \in cl(M - A)$ , then  $x \in M - intA$ . That is,  $cl(M - A) \supseteq M - intA$ .

From 1 and 2, we know that  $M - intA = cl(M - A)$ , or equivalently,  $intA = M - cl(M - A)$ .

$$3.44 \text{ int}(M - A) = M - \bar{A}.$$

**Proof:** Let  $B = M - A$ , and by exercise 3.33, we know that

$$M - \text{int}B = \text{cl}(M - B)$$

which implies that

$$\text{int}B = M - \text{cl}(M - B)$$

which implies that

$$\text{int}(M - A) = M - \text{cl}(A).$$

$$3.45 \text{ int}(\text{int}A) = \text{int}A.$$

**Proof:** Since  $S$  is open if, and only if,  $S = \text{int}S$ . Hence, Let  $S = \text{int}A$ , we have the equality  $\text{int}(\text{int}A) = \text{int}A$ .

$$3.46$$

(a)  $\text{int}(\bigcap_{i=1}^n A_i) = \bigcap_{i=1}^n (\text{int}A_i)$ , where each  $A_i \subseteq M$ .

**Proof:** We prove the equality by considering two steps.

(1) ( $\subseteq$ ) Since  $\bigcap_{i=1}^n A_i \subseteq A_i$  for all  $i = 1, 2, \dots, n$ , then  $\text{int}(\bigcap_{i=1}^n A_i) \subseteq \text{int}A_i$  for all  $i = 1, 2, \dots, n$ . Hence,  $\text{int}(\bigcap_{i=1}^n A_i) \subseteq \bigcap_{i=1}^n (\text{int}A_i)$ .

(2) ( $\supseteq$ ) Since  $\text{int}A_i \subseteq A_i$ , then  $\bigcap_{i=1}^n (\text{int}A_i) \subseteq \bigcap_{i=1}^n A_i$ . Since  $\bigcap_{i=1}^n (\text{int}A_i)$  is open, we have

$$\bigcap_{i=1}^n (\text{int}A_i) \subseteq \text{int}(\bigcap_{i=1}^n A_i).$$

From (1) and (2), we know that  $\text{int}(\bigcap_{i=1}^n A_i) = \bigcap_{i=1}^n (\text{int}A_i)$ .

**Remark:** Note (2), we use the theorem, a finite intersection of an open sets is open. Hence, we ask whether an infinite intersection has the same conclusion or not. Unfortunately, the answer is **NO!** Just see (b) and (c) in this exercise.

(b)  $\text{int}(\bigcap_{A \in F} A) \subseteq \bigcap_{A \in F} (\text{int}A)$ , if  $F$  is an infinite collection of subsets of  $M$ .

**Proof:** Since  $\bigcap_{A \in F} A \subseteq A$  for all  $A \in F$ . Then  $\text{int}(\bigcap_{A \in F} A) \subseteq \text{int}A$  for all  $A \in F$ . Hence,  $\text{int}(\bigcap_{A \in F} A) \subseteq \bigcap_{A \in F} (\text{int}A)$ .

(c) Give an example where equality does not hold in (b).

**Proof:** Let  $F = \{(\frac{-1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ , then  $\text{int}(\bigcap_{A \in F} A) = \emptyset$ , and  $\bigcap_{A \in F} (\text{int}A) = \{0\}$ . So, we can see that in this case,  $\text{int}(\bigcap_{A \in F} A)$  is a proper subset of  $\bigcap_{A \in F} (\text{int}A)$ . Hence, the equality does not hold in (b).

**Remark:** The key to find the counterexample, it is similar to find an example that an infinite intersection of opens set is not open.

$$3.47$$

(a)  $\bigcup_{A \in F} (\text{int}A) \subseteq \text{int}(\bigcup_{A \in F} A)$ .

**Proof:** Since  $\text{int}A \subseteq A$ ,  $\bigcup_{A \in F} (\text{int}A) \subseteq \bigcup_{A \in F} A$ . We have  $\bigcup_{A \in F} (\text{int}A) \subseteq \text{int}(\bigcup_{A \in F} A)$  since  $\bigcup_{A \in F} (\text{int}A)$  is open.

(b) Give an example of a finite collection  $F$  in which equality does not hold in (a).

**Solution:** Consider  $F = \{Q, Q^c\}$ , then we have  $\text{int}Q \cup \text{int}Q^c = \emptyset$  and  $\text{int}(Q \cup Q^c) = \text{int}R^1 = R^1$ . Hence,  $(\text{int}Q) \cup (\text{int}Q^c) = \emptyset$  is a proper subset of  $\text{int}(Q \cup Q^c) = R^1$ . That is, the equality does not hold in (a).

$$3.48$$

(a)  $\text{int}(\partial A) = \phi$  if  $A$  is open or if  $A$  is closed in  $M$ .

**Proof:** (1) Suppose that  $A$  is open. We prove it by the method of contradiction. Assume that  $\text{int}(\partial A) \neq \phi$ , and thus choose

$$\begin{aligned} x &\in \text{int}(\partial A) \\ &= \text{int}(cl(A) \cap cl(M - A)) \\ &= \text{int}(cl(A) \cap (M - A)) \\ &= \text{int}(cl(A)) \cap \text{int}(M - A) \text{ since } \text{int}(S \cap T) = \text{int}(S) \cap \text{int}(T). \end{aligned}$$

Since

$$x \in \text{int}(cl(A)) \Rightarrow B(x, r_1) \subseteq cl(A) = A \cup A'$$

and

$$x \in \text{int}(M - A) \Rightarrow B(x, r_2) \subseteq M - A = A^c \quad *$$

we choose  $r = \min(r_1, r_2)$ , then  $B(x, r) \subseteq (A \cup A') \cap A^c = A' \cap A^c$ . However,

$$x \in A' \text{ and } x \notin A \Rightarrow B(x, r) \cap A \neq \phi \text{ for this } r. \quad **$$

Hence, we get a contradiction since

$$B(x, r) \cap A = \phi \text{ by } (*)$$

and

$$B(x, r) \cap A \neq \phi \text{ by } (**).$$

That is,  $\text{int}(\partial A) = \phi$  if  $A$  is open.

(2) Suppose that  $A$  is closed, then we have  $M - A$  is open. By (1), we have

$$\text{int}(\partial(M - A)) = \phi.$$

Note that

$$\begin{aligned} \partial(M - A) &= cl(M - A) \cap cl(M - (M - A)) \\ &= cl(M - A) \cap cl(A) \\ &= \partial A \end{aligned}$$

. Hence,  $\text{int}(\partial A) = \phi$  if  $A$  is closed.

(b) Give an example in which  $\text{int}(\partial A) = M$ .

**Solution:** Let  $M = R^1$ , and  $A = Q$ , then  $\partial A = cl(A) \cap cl(M - A) = cl(Q) \cap cl(Q^c) = R^1$ . Hence, we have  $R^1 = \text{int}(\partial A) = M$ .

3.49 If  $\text{int}A = \text{int}B = \phi$  and if  $A$  is closed in  $M$ , then  $\text{int}(A \cup B) = \phi$ .

**Proof:** Assume that  $\text{int}(A \cup B) \neq \phi$ , then choose  $x \in \text{int}(A \cup B)$ , then there exists  $B(x, r) \subseteq A \cup B$  for some  $r > 0$ . In addition, since  $\text{int}A = \phi$ , we find that  $B(x, r) \not\subseteq A$ . Hence,  $B(x, r) \cap (B - A) \neq \phi$ . It implies  $B(x, r) \cap (M - A) \neq \phi$ . Choose  $y \in B(x, r) \cap (M - A)$ , then we have

$$y \in B(x, r) \Rightarrow B(y, \varepsilon_1) \subseteq B(x, r), \text{ where } 0 < \varepsilon_1 < r$$

and

$$y \in M - A \Rightarrow B(y, \varepsilon_2) \subseteq M - A, \text{ for some } \varepsilon_2 > 0.$$

Choose  $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ , then we have

$$\begin{aligned} B(y, \varepsilon) &\subseteq B(x, r) \cap (M - A) \\ &\subseteq (A \cup B) \cap A^c \\ &\subseteq B. \end{aligned}$$

That is,  $\text{int}B \neq \phi$  which is absurd. Hence, we have  $\text{int}(A \cup B) = \phi$ .

3.50 Give an example in which  $\text{int}A = \text{int}B = \phi$  but  $\text{int}(A \cup B) = M$ .

**Solution:** Consider the Euclidean space  $(R^1, |\cdot|)$ . Let  $A = Q$ , and  $B = Q^c$ , then  $\text{int}A = \text{int}B = \phi$  but  $\text{int}(A \cup B) = R^1$ .

3.51  $\partial A = \text{cl}(A) \cap \text{cl}(M - A)$  and  $\partial A = \partial(M - A)$ .

**Proof:** By the definition of the boundary of a set, it is clear that  $\partial A = \text{cl}(A) \cap \text{cl}(M - A)$ . In addition,  $\partial A = \text{cl}(A) \cap \text{cl}(M - A)$ , and  $\partial(M - A) = \text{cl}(M - A) \cap \text{cl}(M - (M - A)) = \text{cl}(M - A) \cap \text{cl}(A)$ . Hence, we have  $\partial A = \partial(M - A)$ .

**Remark:** It had better regard the exercise as a formula.

3.52 If  $\text{cl}(A) \cap \text{cl}(B) = \phi$ , then  $\partial(A \cup B) = \partial A \cup \partial B$ .

**Proof:** We prove it by two steps.

(1) ( $\subseteq$ ) Let  $x \in \partial(A \cup B)$ , then for all  $r > 0$ ,

$$B(x, r) \cap (A \cup B) \neq \phi \Rightarrow [B(x, r) \cap A] \cup [B(x, r) \cap B] \neq \phi$$

and

$$B(x, r) \cap [(A \cup B)^c] \neq \phi \Rightarrow B(x, r) \cap A^c \cap B^c \neq \phi \quad *$$

Note that at least one of  $[B(x, r) \cap A]$  and  $[B(x, r) \cap B]$  is not empty. Without loss of generality, we say  $[B(x, r) \cap A] \neq \phi$ . Then by (\*), we have for all  $r > 0$ ,

$$B(x, r) \cap A \neq \phi, \text{ and } B(x, r) \cap A^c \neq \phi.$$

That is,  $x \in \partial A$ . Hence, we have proved  $\partial(A \cup B) \subseteq \partial A \cup \partial B$ .

(2) ( $\supseteq$ ) Let  $x \in \partial A \cup \partial B$ . Without loss of generality, we let  $x \in \partial A$ . Then

$$B(x, r) \cap A \neq \phi, \text{ and } B(x, r) \cap A^c \neq \phi.$$

Since  $B(x, r) \cap A \neq \phi$ , we have

$$B(x, r) \cap (A \cup B) = (B(x, r) \cap A) \cup (B(x, r) \cap B) \neq \phi. \quad **$$

Claim that  $B(x, r) \cap [(A \cup B)^c] = B(x, r) \cap A^c \cap B^c \neq \phi$ . Suppose **NOT**, it means that  $B(x, r) \cap A^c \cap B^c = \phi$ . Then we have

$$B(x, r) \subseteq A \Rightarrow B(x, r) \subseteq \text{cl}(A)$$

and

$$B(x, r) \subseteq B \Rightarrow B(x, r) \subseteq \text{cl}(B).$$

It implies that by hypothesis,  $B(x, r) \subseteq \text{cl}(A) \cap \text{cl}(B) = \phi$  which is absurd. Hence, we have proved the claim. We have proved that

$$B(x, r) \cap (A \cup B) \neq \phi \text{ by (**).}$$

and

$$B(x, r) \cap [(A \cup B)^c] \neq \phi.$$

That is,  $x \in \partial(A \cup B)$ . Hence, we have proved  $\partial(A \cup B) \supseteq \partial A \cup \partial B$ .

From (1) and (2), we have proved that  $\partial(A \cup B) = \partial A \cup \partial B$ .

### Supplement on a separable metric space

**Definition (Base)** A collection  $\{V_\alpha\}$  of open subsets of  $X$  is said to be a base for  $X$  if the following is true: For every  $x \in X$  and every open set  $G \subseteq X$  such that  $x \in G$ , we have

$$x \in V_\alpha \subseteq G \text{ for some } \alpha.$$

In other words, every open set in  $X$  is the union of a subcollection of  $\{V_\alpha\}$ .

**Theorem** Every separable metric space has a countable base.

**Proof:** Let  $(M, d)$  be a separable metric space with  $S = \{x_1, \dots, x_n, \dots\}$  satisfying  $cl(S) = M$ . Consider a collection  $\{B(x_i, \frac{1}{k}) : i, k \in N\}$ , then given any  $x \in M$  and  $x \in G$ , where  $G$  is open in  $X$ , we have  $B(x, \delta) \subseteq G$  for some  $\delta > 0$ . Since  $S$  is dense in  $M$ , we know that there is a set  $B(x_i, \frac{1}{k})$  for some  $i, k$ , such that  $x \in B(x_i, \frac{1}{k}) \subseteq B(x, \delta) \subseteq G$ . So, we know that  $M$  has a countable base.

**Corollary**  $R^k$ , where  $k \in N$ , has a countable base.

**Proof:** Since  $R^k$  is separable, by Theorem 1, we know that  $R^k$  has a countable base.

**Theorem** Every compact metric space is separable.

**Proof:** Let  $(K, d)$  be a compact metric space, and given a radius  $1/n$ , we have

$$K \subseteq \bigcup_{i=1}^p B(x_i^{(n)}, 1/n).$$

Let  $S = \{x_i^{(n)} : i, n \in N\}$ , then it is clear  $S$  is countable. In order to show that  $S$  is dense in  $K$ , given  $x \in K$ , we want to show that  $x$  is an adherent point of  $S$ .

Consider  $B(x, \delta)$  for any  $\delta > 0$ , there is a point  $x_i^{(n)}$  in  $S$  such that  $B(x_i^{(n)}, 1/n) \subseteq B(x, \delta)$  since  $1/n \rightarrow 0$ . Hence, we have shown that  $B(x, \delta) \cap S \neq \phi$ .

That is,  $x \in cl(S)$  which implies that  $K = cl(S)$ . So, we finally have  $K$  is separable.

**Corollary** Every compact metric space has a countable base.

**Proof:** It is immediately from Theorem 1.

**Remark** This corollary can be used to show that **Arzela-Ascoli Theorem**.