Some Basic Notations Of Set Theory

References

There are some good books about set theory; we write them down. We wish the reader can get more.

1. Set Theory and Related Topics by Seymour Lipschutz.
2. Set Theory by Charles C. Pinter.
3. Theory of sets by Kamke.
4. Naive set by Halmos.

2.1 Prove Theorem 2.2. Hint. $(a, b) = (c, d)$ means $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Now appeal to the definition of set equality.

Proof: ($\Leftarrow$) It is trivial.

($\Rightarrow$) Suppose that $(a, b) = (c, d)$, it means that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. It implies that

$$\{a\} \in \{\{c\}, \{c, d\}\} \text{ and } \{a, b\} \in \{\{c\}, \{c, d\}\}.$$

So, if $a \neq c$, then $\{a\} = \{c, d\}$. It implies that $c \in \{a\}$ which is impossible. Hence, $a = c$. Similarly, we have $b = d$.

2.2 Let S be a relation and let $D(S)$ be its domain. The relation S is said to be

(i) reflexive if $a \in D(S)$ implies $(a, a) \in S$,
(ii) symmetric if $(a, b) \in S$ implies $(b, a) \in S$,
(iii) transitive if $(a, b) \in S$ and $(b, c) \in S$ implies $(a, c) \in S$.

A relation which is symmetric, reflexive, and transitive is called an equivalence relation. Determine which of these properties is possessed by S, if S is the set of all pairs of real numbers (x, y) such that

(a) $x \leq y$

Proof: Write $S = \{(x, y) : x \leq y\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = R$.

(i) Since $x \leq x$, $(x, x) \in S$. That is, S is reflexive.

(ii) If $(x, y) \in S$, i.e., $x \leq y$, then $y \leq x$. So, $(y, x) \in S$. That is, S is symmetric.

(iii) If $(x, y) \in S$ and $(y, z) \in S$, i.e., $x \leq y$ and $y \leq z$, then $x \leq z$. So, $(x, z) \in S$. That is, S is transitive.

(b) $x < y$

Proof: Write $S = \{(x, y) : x < y\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = R$.

(i) It is clear that for any real x, we cannot have $x < x$. So, S is not reflexive.

(ii) It is clear that for any real x, and y, we cannot have $x < y$ and $y < x$ at the same time. So, S is not symmetric.

(iii) If $(x, y) \in S$ and $(y, z) \in S$, then $x < y$ and $y < z$. So, $x < z$ wich implies $(x, z) \in S$. That is, S is transitive.

(c) $x < |y|$

Proof: Write $S = \{(x, y) : x < |y|\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = R$.

(i) Since it is impossible for $0 < |0|$, S is not reflexive.

(ii) Since $(-1, 2) \in S$ but $(2, -1) \notin S$, S is not symmetric.

(iii) Since $(0, -1) \in S$ and $(-1, 0) \in S$, but $(0, 0) \notin S$, S is not transitive.

(d) $x^2 + y^2 = 1$

Proof: Write $S = \{(x, y) : x^2 + y^2 = 1\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = [-1, 1]$, an closed interval with endpoints, -1 and 1.

(i) Since $1 \in D(S)$, and it is impossible for $(1, 1) \in S$ by $1^2 + 1^2 \neq 1$, S is not reflexive.

(ii) If $(x, y) \in S$, then $x^2 + y^2 = 1$. So, $(y, x) \in S$. That is, S is symmetric.

(iii) Since $(1, 0) \in S$ and $(0, 1) \in S$, but $(1, 1) \notin S$, S is not transitive.

(e) $x^2 + y^2 < 0$

Proof: Write $S = \{(x, y) : x^2 + y^2 < 1\} = \phi$, then S automatically satisfies (i) reflexive, (ii) symmetric, and (iii) transitive.

(f) $x^2 + x = y^2 + y$

Proof: Write $S = \{(x, y) : x^2 + x = y^2 + y\} = \{(x, y) : (x - y)(x + y - 1) = 0\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = R$.

(i) If $x \in R$, it is clear that $(x, x) \in S$. So, S is reflexive.

(ii) If $(x, y) \in S$, it is clear that $(y, x) \in S$. So, S is symmetric.

(iii) If $(x, y) \in S$ and $(y, z) \in S$, it is clear that $(x, z) \in S$. So, S is transitive.

2.3 The following functions F and G are defined for all real x by the equations given. In each case where the composite function $G \circ F$ can be formed, give the domain of $G \circ F$ and a formula (or formulas) for $(G \circ F)(x)$.

(a) $F(x) = 1 - x$, $G(x) = x^2 + 2x$

Proof: Write

$$G \circ F(x) = G[F(x)] = G[1 - x] = (1 - x)^2 + 2(1 - x) = x^2 - 4x + 3.$$

It is clear that the domain of $G \circ F(x)$ is R.

(b) $F(x) = x + 5$, $G(x) = |x|/x$ if $x \neq 0$, $G(0) = 0$.

Proof: Write

$$G \circ F(x) = G[F(x)] = \begin{cases} G(x+5) = \frac{|x+5|}{x+5} \text{ if } x \neq -5. \\ 0 \text{ if } x = -5. \end{cases}$$

It is clear that the domain of $G \circ F(x)$ is R.

(c) $F(x) = \begin{cases} 2x, \text{ if } 0 \leq x \leq 1 \\ 1, \text{ otherwise,} \end{cases}$ $G(x) = \begin{cases} x^2, \text{ if } 0 \leq x \leq 1 \\ 0, \text{ otherwise.} \end{cases}$

Proof: Write

$$G \circ F(x) = G[F(x)] = \begin{cases} 4x^2 \text{ if } x \in [0, 1/2] \\ 0 \text{ if } x \in (1/2, 1] \\ 1 \text{ if } x \in R - [0, 1] \end{cases}.$$

It is clear that the domain of $G \circ F(x)$ is R.

Find $F(x)$ if $G(x)$ and $G[F(x)]$ are given as follows:

(d) $G(x) = x^3$, $G[F(x)] = x^3 - 3x^2 + 3x - 1$.

Proof: With help of $(x - 1)^3 = x^3 - 3x^2 + 3x - 1$, it is easy to know that $F(x) = 1 - x$. In addition, there is not other function $H(x)$ such that $G[H(x)] = x^3 - 3x^2 + 3x - 1$ since $G(x) = x^3$ is 1-1.

(e) $G(x) = 3 + x + x^2$, $G[F(x)] = x^2 - 3x + 5$.

Proof: Write $G(x) = \left(x + \frac{1}{2}\right)^2 + \frac{11}{4}$, then

$$G[F(x)] = \left(F(x) + \frac{1}{2}\right)^2 + \frac{11}{4} = x^2 - 3x + 5$$

which implies that

$$(2F(x) + 1)^2 = (2x - 3)^2$$

which implies that

$$F(x) = x - 2 \text{ or } -x + 1.$$

2.4 Given three functions F, G, H, what restrictions must be placed on their domains so that the following four composite functions can be defined?

$$G \circ F,\ H \circ G,\ H \circ (G \circ F),\ (H \circ G) \circ F.$$

Proof: It is clear for answers,

$$R(F) \subseteq D(G) \text{ and } R(G) \subseteq D(H).$$

Assuming that $H \circ (G \circ F)$ and $(H \circ G) \circ F$ can be defined, prove that associative law:

$$H \circ (G \circ F) = (H \circ G) \circ F.$$

Proof: Given any $x \in D(F)$, then

$$\begin{aligned} ((H \circ G) \circ F)(x) &= (H \circ G)(F(x)) \\ &= H(G(F(x))) \\ &= H((G \circ F)(x)) \\ &= (H \circ (G \circ F))(x). \end{aligned}$$

So, $H \circ (G \circ F) = (H \circ G) \circ F$.

2.5 Prove the following set-theoretic identities for union and intersection:

(a) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof: For the part, $A \cup (B \cup C) = (A \cup B) \cup C$: Given $x \in A \cup (B \cup C)$, we have $x \in A$ or $x \in B \cup C$. That is, $x \in A$ or $x \in B$ or $x \in C$. Hence, $x \in A \cup B$ or $x \in C$. It implies $x \in (A \cup B) \cup C$. Similarly, if $x \in (A \cup B) \cup C$, then $x \in A \cup (B \cup C)$. Therefore, $A \cup (B \cup C) = (A \cup B) \cup C$.

For the part, $A \cap (B \cap C) = (A \cap B) \cap C$: Given $x \in A \cap (B \cap C)$, we have $x \in A$ and $x \in B \cap C$. That is, $x \in A$ and $x \in B$ and $x \in C$. Hence, $x \in A \cap B$ and $x \in C$. It implies $x \in (A \cap B) \cap C$. Similarly, if $x \in (A \cap B) \cap C$, then $x \in A \cap (B \cap C)$. Therefore, $A \cap (B \cap C) = (A \cap B) \cap C$.

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: Given $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. We consider two cases as follows.

If $x \in B$, then $x \in A \cap B$. So, $x \in (A \cap B) \cup (A \cap C)$.

If $x \in C$, then $x \in A \cap C$. So, $x \in (A \cap B) \cup (A \cap C)$.

So, we have shown that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C). \qquad (*)$$

Conversely, given $x \in (A \cap B) \cup (A \cap C)$, then $x \in A \cap B$ or $x \in A \cap C$. We consider two cases as follows.

If $x \in A \cap B$, then $x \in A \cap (B \cup C)$.

If $x \in A \cap C$, then $x \in A \cap (B \cup C)$.

So, we have shown that

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C). \qquad (**)$$

By (*) and (**), we have proved it.

(c) $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$

Proof: Given $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup B$ and $x \in A \cup C$. We consider two cases as follows.

If $x \in A$, then $x \in A \cup (B \cap C)$.

If $x \notin A$, then $x \in B$ and $x \in C$. So, $x \in B \cap C$. It implies that $x \in A \cup (B \cap C)$.

Therefore, we have shown that

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C). \qquad (*)$$

Conversely, if $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. We consider two cases as follows.

If $x \in A$, then $x \in (A \cup B) \cap (A \cup C)$.

If $x \in B \cap C$, then $x \in A \cup B$ and $x \in A \cup C$. So, $x \in (A \cup B) \cap (A \cup C)$.

Therefore, we have shown that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C). \tag{*}$$

By (*) and (**), we have proved it.

(d) $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (A \cap C) \cup (B \cap C)$

Proof: Given $x \in (A \cup B) \cap (B \cup C) \cap (C \cup A)$, then

$$x \in A \cup B \text{ and } x \in B \cup C \text{ and } x \in C \cup A. \tag{*}$$

We consider the cases to show $x \in (A \cap B) \cup (A \cap C) \cup (B \cap C)$ as follows.

For the case ($x \in A$):

If $x \in B$, then $x \in A \cap B$.

If $x \notin B$, then by (*), $x \in C$. So, $x \in A \cap C$.

Hence, in this case, we have proved that $x \in (A \cap B) \cup (A \cap C) \cup (B \cap C)$.

For the case ($x \notin A$):

If $x \in B$, then by (*), $x \in C$. So, $x \in B \cap C$.

If $x \notin B$, then by (*), it is impossible.

Hence, in this case, we have proved that $x \in (A \cap B) \cup (A \cap C) \cup (B \cap C)$.

From above,

$$(A \cup B) \cap (B \cup C) \cap (C \cup A) \subseteq (A \cap B) \cup (A \cap C) \cup (B \cap C)$$

Similarly, we also have

$$(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap (B \cup C) \cap (C \cup A).$$

So, we have proved it.

Remark: There is another proof, we write it as a reference.

Proof: Consider

$$\begin{aligned}
&(A \cup B) \cap (B \cup C) \cap (C \cup A) \\
&= [(A \cup B) \cap (B \cup C)] \cap (C \cup A) \\
&= [B \cup (A \cap C)] \cap (C \cup A) \\
&= [B \cap (C \cup A)] \cup [(A \cap C) \cap (C \cup A)] \\
&= [(B \cap C) \cup (B \cap A)] \cup (A \cap C) \\
&= (A \cap B) \cup (A \cap C) \cup (B \cap C).
\end{aligned}$$

(e) $A \cap (B - C) = (A \cap B) - (A \cap C)$

Proof: Given $x \in A \cap (B - C)$, then $x \in A$ and $x \in B - C$. So, $x \in A$ and $x \in B$ and $x \notin C$. So, $x \in A \cap B$ and $x \notin C$. Hence,

$$x \in (A \cap B) - C \subseteq (A \cap B) - (A \cap C). \tag{*}$$

Conversely, given $x \in (A \cap B) - (A \cap C)$, then $x \in A \cap B$ and $x \notin A \cap C$. So, $x \in A$ and $x \in B$ and $x \notin C$. So, $x \in A$ and $x \in B - C$. Hence,

$$x \in A \cap (B - C) \tag{**}$$

By (*) and (**), we have proved it.

(f) $(A - C) \cap (B - C) = (A \cap B) - C$

Proof: Given $x \in (A - C) \cap (B - C)$, then $x \in A - C$ and $x \in B - C$. So, $x \in A$ and $x \in B$ and $x \notin C$. So, $x \in (A \cap B) - C$. Hence,

$$(A - C) \cap (B - C) \subseteq (A \cap B) - C. \tag{*}$$

Conversely, given $x \in (A \cap B) - C$, then $x \in A$ and $x \in B$ and $x \notin C$. Hence, $x \in A - C$ and $x \in B - C$. Hence,

$$(A \cap B) - C \subseteq (A - C) \cap (B - C). \tag{**}$$

By (*) and (**), we have proved it.

(g) $(A - B) \cup B = A$ if, and only if, $B \subseteq A$

Proof: ($\Rightarrow$) Suppose that $(A - B) \cup B = A$, then it is clear that $B \subseteq A$.
($\Leftarrow$) Suppose that $B \subseteq A$, then given $x \in A$, we consider two cases.
If $x \in B$, then $x \in (A - B) \cup B$.
If $x \notin B$, then $x \in A - B$. Hence, $x \in (A - B) \cup B$.
From above, we have

$$A \subseteq (A - B) \cup B.$$

In addition, it is obviously $(A - B) \cup B \subseteq A$ since $A - B \subseteq A$ and $B \subseteq A$.

2.6

Let $f : S \to T$ be a function. If A and B are arbitrary subsets of S, prove that

$$f(A \cup B) = f(A) \cup f(B) \text{ and } f(A \cap B) \subseteq f(A) \cap f(B).$$

Generalize to arbitrary unions and intersections.

Proof: First, we prove $f(A \cup B) = f(A) \cup f(B)$ as follows. Let $y \in f(A \cup B)$, then $y = f(a)$ or $y = f(b)$, where $a \in A$ and $b \in B$. Hence, $y \in f(A) \cup f(B)$. That is,

$$f(A \cup B) \subseteq f(A) \cup f(B).$$

Conversely, if $y \in f(A) \cup f(B)$, then $y = f(a)$ or $y = f(b)$, where $a \in A$ and $b \in B$. Hence, $y \in f(A \cup B)$. That is,

$$f(A) \cup f(B) \subseteq f(A \cup B).$$

So, we have proved that $f(A \cup B) = f(A) \cup f(B)$.

For the part $f(A \cap B) \subseteq f(A) \cap f(B)$: Let $y \in f(A \cap B)$, then $y = f(x)$, where $x \in A \cap B$. Hence, $y \in f(A)$ and $y \in f(B)$. That is, $f(A \cap B) \subseteq f(A) \cap f(B)$.

For arbitrary unions and intersections, we have the following facts, and the proof is easy from above. So, we omit the detail.

$$f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i), \text{ where } I \text{ is an index set.}$$

And

$$f(\cap_{i \in I} A_i) \subseteq \cap_{i \in I} f(A_i), \text{ where } I \text{ is an index set.}$$

Remark: We should note why the equality does **NOT** hold for the case of intersection. for example, consider $A = \{1, 2\}$ and $B = \{1, 3\}$, where $f(1) = 1$ and $f(2) = 2$ and $f(3) = 2$.

$$f(A \cap B) = f(\{1\}) = \{1\} \subseteq \{1, 2\} \subseteq f(\{1, 2\}) \cap f(\{1, 3\}) = f(A) \cap f(B).$$

2.7 Let $f : S \to T$ be a function. If $Y \subseteq T$, we denote by $f^{-1}(Y)$ the largest subset of S which f maps into Y. That is,

$$f^{-1}(Y) = \{x : x \in S \text{ and } f(x) \in Y\}.$$

The set $f^{-1}(Y)$ is called the inverse image of Y under f. Prove that the following for arbitrary subsets X of S and Y of T.

(a) $X \subseteq f^{-1}[f(X)]$

Proof: Given $x \in X$, then $f(x) \in f(X)$. Hence, $x \in f^{-1}[f(X)]$ by definition of the inverse image of $f(X)$ under f. So, $X \subseteq f^{-1}[f(X)]$.

Remark: The equality may not hold, for example, let $f(x) = x^2$ on R, and let $X = [0, \infty)$, we have

$$f^{-1}[f(X)] = f^{-1}[[0, \infty)] = R.$$

(b) $f(f^{-1}(Y)) \subseteq Y$

Proof: Given $y \in f(f^{-1}(Y))$, then there exists a point $x \in f^{-1}(Y)$ such that $f(x) = y$. Since $x \in f^{-1}(Y)$, we know that $f(x) \in Y$. Hence, $y \in Y$. So, $f(f^{-1}(Y)) \subseteq Y$

Remark: The equality may not hold, for example, let $f(x) = x^2$ on R, and let $Y = R$, we have

$$f\left(f^{-1}(Y)\right) = f(R) = [0, \infty) \subseteq R.$$

(c) $f^{-1}[Y_1 \cup Y_2] = f^{-1}(Y_1) \cup f^{-1}(Y_2)$

Proof: Given $x \in f^{-1}[Y_1 \cup Y_2]$, then $f(x) \in Y_1 \cup Y_2$. We consider two cases as follows.

If $f(x) \in Y_1$, then $x \in f^{-1}(Y_1)$. So, $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$.

If $f(x) \notin Y_1$, i.e., $f(x) \in Y_2$, then $x \in f^{-1}(Y_2)$. So, $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$.

From above, we have proved that

$$f^{-1}[Y_1 \cup Y_2] \subseteq f^{-1}(Y_1) \cup f^{-1}(Y_2). \qquad (*)$$

Conversely, since $f^{-1}(Y_1) \subseteq f^{-1}[Y_1 \cup Y_2]$ and $f^{-1}(Y_2) \subseteq f^{-1}[Y_1 \cup Y_2]$, we have

$$f^{-1}(Y_1) \cup f^{-1}(Y_2) \subseteq f^{-1}[Y_1 \cup Y_2]. \qquad (**)$$

From (*) and (**), we have proved it.

(d) $f^{-1}[Y_1 \cap Y_2] = f^{-1}(Y_1) \cap f^{-1}(Y_2)$

Proof: Given $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$, then $f(x) \in Y_1$ and $f(x) \in Y_2$. So, $f(x) \in Y_1 \cap Y_2$. Hence, $x \in f^{-1}[Y_1 \cap Y_2]$. That is, we have proved that

$$f^{-1}(Y_1) \cap f^{-1}(Y_2) \subseteq f^{-1}[Y_1 \cap Y_2]. \qquad (*)$$

Conversely, since $f^{-1}[Y_1 \cap Y_2] \subseteq f^{-1}(Y_1)$ and $f^{-1}[Y_1 \cap Y_2] \subseteq f^{-1}(Y_2)$, we have

$$f^{-1}[Y_1 \cap Y_2] \subseteq f^{-1}(Y_1) \cap f^{-1}(Y_2). \tag{**}$$

From (*) and (**), we have proved it.

(e) $f^{-1}(T - Y) = S - f^{-1}(Y)$

Proof: Given $x \in f^{-1}(T - Y)$, then $f(x) \in T - Y$. So, $f(x) \notin Y$. We want to show that $x \in S - f^{-1}(Y)$. Suppose **NOT**, then $x \in f^{-1}(Y)$ which implies that $f(x) \in Y$. That is impossible. Hence, $x \in S - f^{-1}(Y)$. So, we have

$$f^{-1}(T - Y) \subseteq S - f^{-1}(Y). \tag{*}$$

Conversely, given $x \in S - f^{-1}(Y)$, then $x \notin f^{-1}(Y)$. So, $f(x) \notin Y$. That is, $f(x) \in T - Y$. Hence, $x \in f^{-1}(T - Y)$. So, we have

$$S - f^{-1}(Y) \subseteq f^{-1}(T - Y). \tag{**}$$

From (*) and (**), we have proved it.

(f) Generalize (c) and (d) to arbitrary unions and intersections.

Proof: We give the statement without proof since it is the same as (c) and (d). In general, we have

$$f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i).$$

and

$$f^{-1}(\cap_{i \in I} A_i) = \cap_{i \in I} f^{-1}(A_i).$$

Remark: From above sayings and **Exercise 2.6**, we found that the inverse image f^{-1} and the operations of sets, such as intersection and union, can be exchanged. However, for a function, we only have the exchange of f and the operation of union. The reader also see the **Exercise 2.9** to get more.

2.8 Refer to Exercise 2.7. Prove that $f[f^{-1}(Y)] = Y$ for every subset Y of T if, and only if, $T = f(S)$.

Proof: ($\Rightarrow$) It is clear that $f(S) \subseteq T$. In order to show the equality, it suffices to show that $T \subseteq f(S)$. Consider $f^{-1}(T) \subseteq S$, then we have

$$f\left(f^{-1}(T)\right) \subseteq f(S).$$

By hyppothesis, we get $T \subseteq f(S)$.

($\Leftarrow$) Suppose **NOT**, i.e., $f[f^{-1}(Y)]$ is a proper subset of Y for some $Y \subseteq T$ by **Exercise 2.7 (b)**. Hence, there is a $y \in Y$ such that $y \notin f[f^{-1}(Y)]$. Since $Y \subseteq f(S) = T$, $f(x) = y$ for some $x \in S$. It implies that $x \in f^{-1}(Y)$. So, $f(x) \in f[f^{-1}(Y)]$ which is impossible by the choice of y. Hence, $f[f^{-1}(Y)] = Y$ for every subset Y of T.

2.9 Let $f : S \to T$ be a function. Prove that the following statements are equivalent.

(a) f is one-to-one on S.
(b) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B of S.
(c) $f^{-1}[f(A)] = A$ for every subset A of S.
(d) For all disjoint subsets A and B of S, the image $f(A)$ and $f(B)$ are disjoint.
(e) For all subsets A and B of S with $B \subseteq A$, we have

$$f(A - B) = f(A) - f(B).$$

Proof: $(a) \Rightarrow (b)$: Suppose that f is 1-1 on S. By **Exercise 2.6**, we have proved that $f(A \cap B) \subseteq f(A) \cap f(B)$ for all A, B of S. In order to show the equality, it suffices to show that $f(A) \cap f(B) \subseteq f(A \cap B)$.

Given $y \in f(A) \cap f(B)$, then $y = f(a)$ and $y = f(b)$ where $a \in A$ and $b \in B$. Since f is 1-1, we have $a = b$. That is, $y \in f(A \cap B)$. So, $f(A) \cap f(B) \subseteq f(A \cap B)$.

$(b) \Rightarrow (c)$: Suppose that $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B of S. If $A \neq f^{-1}[f(A)]$ for some A of S, then by **Exercise 2.7 (a),** there is an element $a \notin A$ and $a \in f^{-1}[f(A)]$. Consider

$$\phi = f(A \cap \{a\}) = f(A) \cap f(\{a\}) \text{ by (b)} \qquad (*)$$

Since $a \in f^{-1}[f(A)]$, we have $f(a) \in f(A)$ which contradicts to (*). Hence, no such a exists. That is, $f^{-1}[f(A)] = A$ for every subset A of S.

$(c) \Rightarrow (d)$: Suppose that $f^{-1}[f(A)] = A$ for every subset A of S. If $A \cap B = \phi$, then Consider

$$\begin{aligned}\phi &= A \cap B \\ &= f^{-1}[f(A)] \cap f^{-1}[f(B)] \\ &= f^{-1}(f(A) \cap f(B)) \text{ by } \textbf{Exercise 2.7 (d)}\end{aligned}$$

which implies that $f(A) \cap f(B) = \phi$.

$(d) \Rightarrow (e)$: Suppose that for all disjoint subsets A and B of S, the image $f(A)$ and $f(B)$ are disjoint. If $B \subseteq A$, then since $(A - B) \cap B = \phi$, we have

$$f(A - B) \cap f(B) = \phi$$

which implies that

$$f(A - B) \subseteq f(A) - f(B). \tag{**}$$

Conversely, we consider if $y \in f(A) - f(B)$, then $y = f(x)$, where $x \in A$ and $x \notin B$. It implies that $x \in A - B$. So, $y = f(x) \in f(A - B)$. That is,

$$f(A) - f(B) \subseteq f(A - B). \tag{***}$$

By (**) and (***), we have proved it.

$(d) \Rightarrow (a)$: Suppose that $f(A - B) = f(A) - f(B)$ for all subsets A and B of S with $B \subseteq A$. If $f(a) = f(b)$, consider $A = \{a, b\}$ and $B = \{b\}$, we have

$$f(A - B) = \phi$$

which implies that $A = B$. That is, $a = b$. So, f is 1-1.

2.10 Prove that if $A\tilde{}B$ and $B\tilde{}C$, then $A\tilde{}C$.

Proof: Since $A\tilde{}B$ and $B\tilde{}C$, then there exists bijection f and g such that

$$f : A \rightarrow B \text{ and } g : B \rightarrow C.$$

So, if we consider $g \circ f : A \rightarrow C$, then $A\tilde{}C$ since $g \circ f$ is bijection.

2.11 If $\{1, 2, ..., n\} \tilde{} \{1, 2, ..., m\}$, prove that $m = n$.

Proof: Since $\{1, 2, ..., n\} \tilde{} \{1, 2, ..., m\}$, there exists a bijection function

$$f : \{1, 2, ..., n\} \rightarrow \{1, 2, ..., m\}.$$

Since f is 1-1, then $n \leq m$. Conversely, consider f^{-1} is 1-1 to get $m \leq n$. So, $m = n$.

2.12 If S is an infinite set, prove that S contains a countably infinite subset. Hint. Choose an element a_1 in S and consider $S - \{a_1\}$.

Proof: Since S is an infinite set, then choose a_1 in S and thus $S - \{a_1\}$ is still infinite. From this, we have $S - \{a_1, .., a_n\}$ is infinite. So, we finally have

$$\{a_1, ..., a_n, ...\} \, (\subseteq S)$$

which is countably infinite subset.

2.13 Prove that every infinite set S contains a proper subset similar to S.

Proof: By Exercise 2.12, we write $S = \tilde{S} \cup \{x_1, ..., x_n, ...\}$, where $\tilde{S} \cap \{x_1, ..., x_n, ...\} = \phi$ and try to show

$$\tilde{S} \cup \{x_2, ..., x_n, ...\} \,\tilde{}\, S$$

as follows. Define

$$f : \tilde{S} \cup \{x_2, ..., x_n, ...\} \to S = \tilde{S} \cup \{x_1, ..., x_n, ...\}$$

by

$$f(x) = \begin{cases} x \text{ if } x \in \tilde{S} \\ x_i \text{ if } x = x_{i+1} \end{cases}.$$

Then it is clear that f is 1-1 and onto. So, we have proved that every infinite set S contains a proper subset similar to S.

Remark: In the proof, we may choose the map

$$f : \tilde{S} \cup \{x_{N+1}, ..., x_n, ...\} \to S = \tilde{S} \cup \{x_1, ..., x_n, ...\}$$

by

$$f(x) = \begin{cases} x \text{ if } x \in \tilde{S} \\ x_i \text{ if } x = x_{i+N} \end{cases}.$$

2.14 If A is a countable set and B an uncountable set, prove that $B - A$ is similar to B.

Proof: In order to show it, we consider some cases as follows. (i) $B \cap A = \phi$ (ii) $B \cap A$ is a finite set, and (iii) $B \cap A$ is an infinite set.

For case (i), $B - A = B$. So, $B - A$ is similar to B.

For case (ii), it follows from the **Remark in Exercise 2.13**.

For case (iii), note that $B \cap A$ is countable, and let $C = B - A$, we have $B = C \cup (B \cap A)$. We want to show that

$$(B - A) \,\tilde{}\, B \Leftrightarrow C \,\tilde{}\, C \cup (B \cap A).$$

By **Exercise 2.12**, we write $C = \tilde{C} \cup D$, where D is countably infinite and $\tilde{C} \cap D = \phi$. Hence,

$$C\tilde{\ }C \cup (B \cap A) \Leftrightarrow \left(\tilde{C} \cup D\right) \tilde{\ } \left[\tilde{C} \cup (D \cup (B \cap A))\right]$$
$$\Leftrightarrow \left(\tilde{C} \cup D\right) \tilde{\ } \left(\tilde{C} \cup D'\right)$$

where $D' = D \cup (B \cap A)$ which is countably infinite. Since $\left(\tilde{C} \cup D\right) \tilde{\ } \left(\tilde{C} \cup D'\right)$ is clear, we have proved it.

2.15 A real number is called **algebraic** if it is a root of an algebraic equation $f(x) = 0$, where $f(x) = a_0 + a_1 x + ... + a_n x^n$ is a polynomial with integer coefficients. Prove that the set of all polynomials with integer coefficients is countable and deduce that the set of algebraic numbers is also countable.

Proof: Given a positive integer $N (\geq 2)$, there are only finitely many eqautions with

$$n + \sum_{k=1}^{n} |a_k| = N, \text{ where } a_k \in Z. \tag{*}$$

Let $S_N = \{f : f(x) = a_0 + a_1 x + ... + a_n x^n \text{ satisfies } (*)\}$, then S_N is a finite set. Hence, $\cup_{n=2}^{\infty} S_n$ is countable. Clearly, the set of all polynomials with integer coefficients is a subset of $\cup_{n=2}^{\infty} S_n$. So, the set of all polynomials with integer coefficients is countable. In addition, a polynomial of degree k has at most k roots. Hence, the set of algebraic numbers is also countable.

2.16 Let S be a finite set consisting of n elements and let T be the collection of all subsets of S. Show that T is a finite set and find the number of elements in T.

Proof: Write $S = \{x_1, ..., x_n\}$, then T =the collection of all subsets of S. Clearly, T is a finite set with 2^n elements.

2.17 Let R denote the set of real numbers and let S denote the set of all real-valued functions whose domain in R. Show that S and R are not **equinumrous**. Hint. Assume $S\tilde{\ }R$ and let f be a one-to-one function such that $f(R) = S$. If $a \in R$, let $g_a = f(a)$ be the real-valued function in S which correspouds to real number a. Now define h by the equation $h(x) = 1 + g_x(x)$ if $x \in R$, and show that $h \notin S$.

Proof: Assume $S\tilde{\ }R$ and let f be a one-to-one function such that $f(R) = S$. If $a \in R$, let $g_a = f(a)$ be the real-valued function in S which correspouds to real number a. Define h by the equation $h(x) = 1 + g_x(x)$ if $x \in R$, then

$$h = f(b) = g_b$$

which implies that

$$h(b) := 1 + g_b(b) = g_b(b)$$

which is impossible. So, S and R are not **equinumrous.**

Remark: There is a similar exercise, we write it as a reference. The cardinal number of $C[a,b]$ is $2^{\aleph_0}$, where $\aleph_0 = \#(N)$.

Proof: First, $\#(R) = 2^{\aleph_0} \leq \#(C[a,b])$ by considering constant function. Second, we consider the map

$$f : C[a,b] \to P(Q \times Q), \text{ the power set of } Q \times Q$$

by

$$f(\varphi) = \{(x,y) \in Q \times Q : x \in [a,b] \text{ and } y \leq \varphi(x)\}.$$

Clearly, f is 1-1. It implies that $\#(C[a,b]) \leq \#(P(Q \times Q)) = 2^{\aleph_0}$.

So, we have proved that $\#(C[a,b]) = 2^{\aleph_0}$.

Note: For notations, the reader can see the textbook, in **Chapter 4, pp 102.** Also, see the book, **Set Theory and Related Topics by Seymour Lipschutz, Chapter 9, pp 157-174. (Chinese Version)**

2.18 Let S be the collection of all sequences whose terms are the integers 0 and 1. Show that S is uncountable.

Proof: Let E be a countable subet of S, and let E consists of the sequences $s_1, .., s_n, ...$ We construct a sequence s as follows. If the nth digit in s_n is 1, we let the nth digit of s be 0, and vice versa. Then the sequence s differes from every member of E in at least one place; hence $s \notin E$. But clearly $s \in S$, so that E is a proper subset of S.

We have shown that every countable subset of S is a proper subset of S. It follows that S is uncountable (for otherwise S would be a proper subset of S, which is absurb).

Remark: In this exercise, we have proved that R, the set of real numbers, is uncountable. It can be regarded as the **Exercise 1.22** for $k = 2$. (**Binary System**).

2.19 Show that the following sets are countable:

(a) the set of circles in the complex plane having the ratiional radii and centers with rational coordinates.

Proof: Write the set of circles in the complex plane having the ratiional radii and centers with rational coordinates, $\{C(x_n; q_n) : x_n \in Q \times Q \text{ and } q_n \in Q\} :=$ S. Clearly, S is countable.

(b) any collection of disjoint intervals of positive length.

Proof: Write the collection of disjoint intervals of positive length, $\{I : I$ is an interval of positive
S. Use the reason in **Exercise 2.21**, we have proved that S is countable.

2.20 Let f be a real-valued function defined for every x in the interval $0 \leq x \leq 1$. Suppose there is a positive number M having the following property: for every choice of a finite number of points $x_1, x_2, ..., x_n$ in the interval $0 \leq x \leq 1$, the sum

$$|f(x_1) + ... + f(x_n)| \leq M.$$

Let S be the set of those x in $0 \leq x \leq 1$ for which $f(x) \neq 0$. Prove that S is countable.

Proof: Let $S_n = \{x \in [0,1] : |f(x)| \geq 1/n\}$, then S_n is a finite set by hypothesis. In addition, $S = \cup_{n=1}^{\infty} S_n$. So, S is countable.

2.21 Find the fallacy in the following "proof" that the set of all intervals of positive length is countable.

Let $\{x_1, x_2, ...\}$ denote the countable set of rational numbers and let I be any interval of positive length. Then I contains infinitely many rational points x_n, but among these there will be one with **smallest index** n. Define a function F by means of the eqaution $F(I) = n$ if x_n is the rational number with smallest index in the interval I. This function establishes a one-to-one correspondence between the set of all intervals and a subset of the positive integers. Hence, the set of all intervals is countable.

Proof: Note that F is not a one-to-one correspondence between the set of all intervals and a subset of the positive integers. So, this is not a proof. In fact, the set of all intervals of positive length is uncountable.

Remark: Compare with **Exercise 2.19,** and the set of all intervals of positive length is uncountable is clear by considering $\{(0, x) : 0 < x < 1\}$.

2.22 Let S denote the collection of all subsets of a given set T. Let $f : S \to R$ be a real-valued function defined on S. The function f is called **additive** if $f(A \cup B) = f(A) + f(B)$ whenever A and B are disjoint subsets of T. If f is additive, prove that for any two subsets A and B we have

$$f(A \cup B) = f(A) + f(B - A)$$

and

$$f(A \cup B) = f(A) + f(B) - f(A \cap B).$$

Proof: Since $A \cap (B - A) = \phi$ and $A \cup B = A \cup (B - A)$, we have

$$f(A \cup B) = f(A \cup (B - A)) = f(A) + f(B - A). \qquad (*)$$

In addition, since$(B - A) \cap (A \cap B) = \phi$ and $B = (B - A) \cup (A \cap B)$, we have

$$f(B) = f((B - A) \cup (A \cap B)) = f(B - A) + f(A \cap B)$$

which implies that

$$f(B - A) = f(B) - f(A \cap B) \qquad (**)$$

By (*) and (**), we have proved that

$$f(A \cup B) = f(A) + f(B) - f(A \cap B).$$

2.23 Refer to Exercise 2.22. Assume f is additive and assume also that the following relations hold for two particular subsets A and B of T :

$$f(A \cup B) = f(A') + f(B') - f(A') f(B')$$

and

$$f(A \cap B) = f(A) f(B)$$

and

$$f(A) + f(B) \neq f(T),$$

where $A' = T - A$, $B' = T - B$. Prove that these relations determine $f(T)$, and compute the value of $f(T)$.

Proof: Write

$$f(T) = f(A) + f(A') = f(B) + f(B'),$$

then

$$\begin{aligned}
[f(T)]^2 &= [f(A) + f(A')]\,[f(B) + f(B')] \\
&= f(A)\,f(B) + f(A)\,f(B') + f(A')\,f(B) + f(A')\,f(B') \\
&= f(A)\,f(B) + f(A)\,[f(T) - f(B)] + [f(T) - f(A)]\,f(B) + f(A')\,f(B') \\
&= [f(A) + f(B)]\,f(T) - f(A)\,f(B) + f(A')\,f(B') \\
&= [f(A) + f(B)]\,f(T) - f(A)\,f(B) + f(A') + f(B') - f(A \cup B) \\
&= [f(A) + f(B)]\,f(T) - f(A)\,f(B) + [f(T) - f(A)] + [f(T) - f(B)] \\
&\quad - [f(A) + f(B) - f(A \cap B)] \\
&= [f(A) + f(B) + 2]\,f(T) - f(A)\,f(B) - 2\,[f(A) + f(B)] + f(A \cap B) \\
&= [f(A) + f(B) + 2]\,f(T) - 2\,[f(A) + f(B)]
\end{aligned}$$

which implies that

$$[f(T)]^2 - [f(A) + f(B) + 2]\,f(T) + 2\,[f(A) + f(B)] = 0$$

which implies that

$$x^2 - (a + 2)\,x + 2a = 0 \Rightarrow (x - a)\,(x - 2) = 0$$

where $a = f(A) + f(B)$. So, $x = 2$ since $x \neq a$ by $f(A) + f(B) \neq f(T)$.