The Real And Complex Number Systems

Integers

1.1 Prove that there is no largest prime.

Proof: Suppose $p$ is the largest prime. Then $p! + 1$ is NOT a prime. So, there exists a prime $q$ such that

$$q | p! + 1 \Rightarrow q | 1$$

which is impossible. So, there is no largest prime.

Remark: There are many and many proofs about it. The proof that we give comes from Archimedes 287-212 B.C. In addition, Euler Leonhard (1707-1783) find another method to show it. The method is important since it develops to study the theory of numbers by analytic method. The reader can see the book, An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 91-93. (Chinese Version)

1.2 If $n$ is a positive integer, prove the algebraic identity

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k}$$

Proof: It suffices to show that

$$x^n - 1 = (x - 1) \sum_{k=0}^{n-1} x^k.$$
Consider the right hand side, we have
\[
(x - 1) \sum_{k=0}^{n-1} x^k = \sum_{k=0}^{n-1} x^{k+1} - \sum_{k=0}^{n-1} x^k = \sum_{k=1}^{n} x^k - \sum_{k=0}^{n-1} x^k = x^n - 1.
\]

1.3 If \(2^n - 1\) is a prime, prove that \(n\) is prime. A prime of the form \(2^p - 1\), where \(p\) is prime, is called a Mersenne prime.

**Proof:** If \(n\) is not a prime, then say \(n = ab\), where \(a > 1\) and \(b > 1\). So, we have
\[
2^{ab} - 1 = (2^a - 1) \sum_{k=0}^{b-1} (2^a)^k
\]
which is not a prime by Exercise 1.2. So, \(n\) must be a prime.

**Remark:** The study of Mersenne prime is important; it is related with so called Perfect number. In addition, there are some OPEN problem about it. For example, is there infinitely many Mersenne numbers? The reader can see the book, An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 13-15. (Chinese Version)

1.4 If \(2^n + 1\) is a prime, prove that \(n\) is a power of 2. A prime of the form \(2^{2^n} + 1\) is called a Fermat prime. Hint. Use exercise 1.2.

**Proof:** If \(n\) is a not a power of 2, say \(n = ab\), where \(b\) is an odd integer. So,
\[
2^a + 1 \mid 2^{ab} + 1
\]
and \(2^a + 1 < 2^{ab} + 1\). It implies that \(2^n + 1\) is not a prime. So, \(n\) must be a power of 2.

**Remark:** (1) In the proof, we use the identity
\[
x^{2n-1} + 1 = (x + 1) \sum_{k=0}^{2n-2} (-1)^k x^k.
\]

2
**Proof:** Consider

\[
(x + 1) \sum_{k=0}^{2n-2} (-1)^k x^k = \sum_{k=0}^{2n-2} (-1)^k x^{k+1} + \sum_{k=0}^{2n-2} (-1)^k x^k
\]

\[
= \sum_{k=1}^{2n-1} (-1)^{k+1} x^k + \sum_{k=0}^{2n-2} (-1)^k x^k
\]

\[
= x^{2n+1} + 1.
\]

(2) The study of **Fermat number** is important; for the details the reader can see the book, *An Introduction To The Theory Of Numbers* by Loo-Keng Hua, pp 15. (Chinese Version)

1.5 The Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ... are defined by the recursion formula \( x_{n+1} = x_n + x_{n-1} \), with \( x_1 = x_2 = 1 \). Prove that \((x_n, x_{n+1}) = 1\) and that \( x_n = (a^n - b^n) / (a - b) \), where \( a \) and \( b \) are the roots of the quadratic equation \( x^2 - x - 1 = 0 \).

**Proof:** Let \( d = \text{g.c.d.} (x_n, x_{n+1}) \), then

\[
d | x_n \text{ and } d | x_{n+1} = x_n + x_{n-1}.
\]

So,

\[
d | x_{n-1}.
\]

Continue the process, we finally have

\[
d | 1.
\]

So, \( d = 1 \) since \( d \) is positive.

Observe that

\[
x_{n+1} = x_n + x_{n-1},
\]

and thus we consider

\[
x^{n+1} = x^n + x^{n-1},
\]

i.e., consider

\[
x^2 = x + 1 \text{ with two roots, } a \text{ and } b.
\]

If we let

\[
F_n = (a^n - b^n) / (a - b),
\]

3
then it is clear that

$$F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for} \quad n > 1.$$  

So, $F_n = x_n$ for all $n$.

**Remark:** The study of the Fibonacci numbers is important; the reader can see the book, *Fibonacci and Lucas Numbers with Applications* by Koshy and Thomas.

1.6 Prove that every nonempty set of positive integers contains a smallest member. This is called the **well-ordering Principle**.

**Proof:** Given $(\phi \neq) S (\subseteq N)$, we prove that if $S$ contains an integer $k$, then $S$ contains the smallest member. We prove it by **Mathematical Induction of second form** as follows.

As $k = 1$, it trivially holds. Assume that as $k = 1, 2, \ldots, m$ holds, consider as $k = m + 1$ as follows. In order to show it, we consider two cases.

(1) If there is a member $s \in S$ such that $s < m + 1$, then by Induction hypothesis, we have proved it.

(2) If every $s \in S$, $s \geq m + 1$, then $m + 1$ is the smallest member.

Hence, by **Mathematical Induction**, we complete it.

**Remark:** We give a fundamental result to help the reader get more. We will prove the followings are equivalent:

(A. **Well-ordering Principle**) every nonempty set of positive integers contains a smallest member.

(B. **Mathematical Induction of first form**) Suppose that $S (\subseteq N)$, if $S$ satisfies that

(1). 1 in $S$

(2). As $k \in S$, then $k + 1 \in S$.

Then $S = N$.

(C. **Mathematical Induction of second form**) Suppose that $S (\subseteq N)$, if $S$ satisfies that

(1). 1 in $S$

(2). As $1, \ldots, k \in S$, then $k + 1 \in S$. 

4
Then $S = N$.

**Proof**: ($A \Rightarrow B$): If $S \neq N$, then $N - S \neq \emptyset$. So, by (A), there exists the smallest integer $w$ such that $w \in N - S$. Note that $w > 1$ by (1), so we consider $w - 1$ as follows.

Since $w - 1 \notin N - S$, we know that $w - 1 \in S$. By (2), we know that $w \in S$ which contradicts to $w \in N - S$. Hence, $S = N$.

($B \Rightarrow C$): It is obvious.

($C \Rightarrow A$): We have proved it by this exercise.

**Rational and irrational numbers**

1.7 Find the rational number whose decimal expansion is 0.3344444444....

**Proof**: Let $x = 0.3344444444....$, then

$$x = \frac{3}{10} + \frac{3}{10^2} + \frac{4}{10^3} + \ldots + \frac{4}{10^n} + \ldots,$$

where $n \geq 3$

$$= \frac{33}{10^2} + \frac{4}{10^3} \left( 1 + \frac{1}{10} + \ldots + \frac{1}{10^n} + \ldots \right)$$

$$= \frac{33}{10^2} + \frac{4}{10^3} \left( \frac{1}{1} - \frac{1}{10} \right)$$

$$= \frac{33}{10^2} + \frac{4}{900}$$

$$= \frac{301}{900}.$$  

1.8 Prove that the decimal expansion of $x$ will end in zeros (or in nines) if, and only if, $x$ is a rational number whose denominator is of the form $2^n5^m$, where $m$ and $n$ are nonnegative integers.

**Proof**: ($\Leftarrow$) Suppose that $x = \frac{k}{2^n5^m}$, if $n \geq m$, we have

$$\frac{k5^{n-m}}{2^n5^n} = \frac{5^{n-m}k}{10^n}.$$  

So, the decimal expansion of $x$ will end in zeros. Similarly for $m \geq n$.

($\Rightarrow$) Suppose that the decimal expansion of $x$ will end in zeros (or in nines).
For case \( x = a_0.a_1a_2 \cdots a_n \). Then

\[
x = \sum_{k=0}^{n} \frac{10^{n-k}a_k}{10^n} = \sum_{k=0}^{n} \frac{10^{n-k}a_k}{2^n5^n}.
\]

For case \( x = a_0.a_1a_2 \cdots a_n999999 \cdots \). Then

\[
x = \sum_{k=0}^{n} \frac{10^{n-k}a_k}{2^n5^n} + \frac{9}{10^{n+1}} + \frac{9}{10^{n+m}} + \ldots
\]

\[
= \sum_{k=0}^{n} \frac{10^{n-k}a_k}{2^n5^n} + \frac{9}{10^{n+1}} \sum_{j=0}^{\infty} 10^{-j}
\]

\[
= \sum_{k=0}^{n} \frac{10^{n-k}a_k}{2^n5^n} + \frac{1}{10^n}
\]

\[
= 1 + \sum_{k=0}^{n} \frac{10^{n-k}a_k}{2^n5^n}.
\]

So, in both case, we prove that \( x \) is a rational number whose denominator is of the form \( 2^n5^n \), where \( m \) and \( n \) are nonnegative integers.

1.9 Prove that \( \sqrt{2} + \sqrt{3} \) is irrational.

**Proof:** If \( \sqrt{2} + \sqrt{3} \) is rational, then consider

\[
\left( \sqrt{3} + \sqrt{2} \right) \left( \sqrt{3} - \sqrt{2} \right) = 1
\]

which implies that \( \sqrt{3} - \sqrt{2} \) is rational. Hence, \( \sqrt{3} \) would be rational. It is impossible. So, \( \sqrt{2} + \sqrt{3} \) is irrational.

**Remark:** (1) \( \sqrt{p} \) is an irrational if \( p \) is a prime.

**Proof:** If \( \sqrt{p} \in \mathbb{Q} \), write \( \sqrt{p} = \frac{a}{b} \), where \( \text{g.c.d.} (a, b) = 1 \). Then

\[
b^2p = a^2 \Rightarrow p \mid a^2 \Rightarrow p \mid a
\]

Write \( a = pq \). So,

\[
b^2p = p^2q^2 \Rightarrow b^2 = pq^2 \Rightarrow p \mid b^2 \Rightarrow p \mid b.
\]

By (*) and (**), we get

\[
p \mid \text{g.c.d.} (a, b) = 1
\]

which implies that \( p = 1 \), a contradiction. So, \( \sqrt{p} \) is an irrational if \( p \) is a prime.
Note: There are many and many methods to prove it. For example, the reader can see the book, *An Introduction To The Theory Of Numbers* by Loo-Keng Hua, pp 19-21. (Chinese Version)

(2) Suppose $a, b \in \mathbb{N}$. Prove that $\sqrt{a} + \sqrt{b}$ is rational if and only if, $a = k^2$ and $b = h^2$ for some $h, k \in \mathbb{N}$.

**Proof**: ($\Leftarrow$) It is clear.

($\Rightarrow$) Consider

$$\left( \sqrt{a} + \sqrt{b} \right) \left( \sqrt{a} - \sqrt{b} \right) = a^2 - b^2,$$

then $\sqrt{a} \in \mathbb{Q}$ and $\sqrt{b} \in \mathbb{Q}$. Then it is clear that $a = h^2$ and $b = h^2$ for some $h, k \in \mathbb{N}$.

1.10 If $a, b, c, d$ are rational and if $x$ is irrational, prove that $(ax + b) / (cx + d)$ is usually irrational. When do exceptions occur?

**Proof**: We claim that $(ax + b) / (cx + d)$ is rational if and only if $ad = bc$.

($\Rightarrow$) If $(ax + b) / (cx + d)$ is rational, say $(ax + b) / (cx + d) = q/p$. We consider two cases as follows.

(i) If $q = 0$, then $ax + b = 0$. If $a \neq 0$, then $x$ would be rational. So, $a = 0$ and $b = 0$. Hence, we have $ad = 0 = bc$.

(ii) If $q \neq 0$, then $(pa -qc)x + (pb - qd) = 0$. If $pa -qc \neq 0$, then $x$ would be rational. So, $pa -qc = 0$ and $pb - qd = 0$. It implies that $qcb = qad \Rightarrow ad = bc$.

($\Leftarrow$) Suppose $ad = bc$. If $a = 0$, then $b = 0$ or $c = 0$. So,

$$\frac{ax + b}{cx + d} = \begin{cases} 0 & \text{if } a = 0 \text{ and } b = 0 \\ \frac{b}{d} & \text{if } a = 0 \text{ and } c = 0 \end{cases}.$$

If $a \neq 0$, then $d = bc/a$. So,

$$\frac{ax + b}{cx + d} = \frac{ax + b}{cx + bc/a} = \frac{a(\frac{ax + b}{c})}{c} = a.$$

Hence, we proved that if $ad = bc$, then $(ax + b) / (cx + d)$ is rational.
1.11 Given any real $x > 0$, prove that there is an irrational number between 0 and $x$.

**Proof:** If $x \in Q^c$, we choose $y = x/2 \in Q^c$. Then $0 < y < x$. If $x \in Q$, we choose $y = x/\sqrt{2} \in Q$, then $0 < y < x$.

**Remark:** (1) There are many and many proofs about it. We may prove it by the concept of **Perfect set**. The reader can see the book, *Principles of Mathematical Analysis* written by Walter Rudin, Theorem 2.43, pp 41. Also see the textbook, Exercise 3.25.

(2) Given $a$ and $b \in R$ with $a < b$, there exists $r \in Q^c$, and $q \in Q$ such that $a < r < b$ and $a < q < b$.

**Proof:** We show it by considering four cases. (i) $a \in Q$, $b \in Q$. (ii) $a \in Q$, $b \in Q^c$, (iii) $a \in Q^c$, $b \in Q$. (iv) $a \in Q^c$, $b \in Q^c$.

(i) $(a \in Q, b \in Q)$ Choose $q = a + b$ and $r = \frac{1}{\sqrt{2}}a + \left(1 - \frac{1}{\sqrt{2}}\right)b$.

(ii) $(a \in Q, b \in Q^c)$ Choose $r = \frac{a + b}{2}$ and let $c = \frac{1}{2} < b - a$, then $a + c := q$.

(iii) $(a \in Q^c, b \in Q)$ Similarly for (iii).

(iv) $(a \in Q^c, b \in Q^c)$ It suffices to show that there exists a rational number $q \in (a, b)$ by (ii). Write

$$b = b_0.b_1b_2\ldots b_n\ldots$$

Choose $n$ large enough so that

$$a < q = b_0.b_1b_2\ldots b_n < b.$$  

(It works since $b - q = 0.000 \ldots b_{n+1} \ldots \leq \frac{1}{10^n}$)

1.12 If $a/b < c/d$ with $b > 0$, $d > 0$, prove that $(a+c)/(b+d)$ lies between the two fractions $a/b$ and $c/d$.

**Proof:** It only needs to consider the substraction. So, we omit it.

**Remark:** The result of this exercise is often used, so we suggest the reader keep it in mind.

1.13 Let $a$ and $b$ be positive integers. Prove that $\sqrt{2}$ always lies between the two fractions $a/b$ and $(a + 2b) / (a + b)$. Which fraction is closer to $\sqrt{2}$?

**Proof:** Suppose $a/b \leq \sqrt{2}$, then $a \leq \sqrt{2}b$. So,

$$\frac{a + 2b}{a + b} - \sqrt{2} = \frac{(\sqrt{2} - 1)(\sqrt{2}b - a)}{a + b} \geq 0.$$
In addition,

\[
(\sqrt{2} - \frac{a}{b}) - \left( \frac{a + 2b}{a + b} - \sqrt{2} \right) = 2\sqrt{2} - \left( \frac{a}{b} + \frac{a + 2b}{a + b} \right)
= 2\sqrt{2} - \frac{a^2 + 2ab + 2b^2}{ab + b^2}
= \frac{1}{ab + b^2} \left[ (2\sqrt{2} - 2) ab + \left( 2\sqrt{2} - 2 \right) b^2 - a^2 \right]
\geq \frac{1}{ab + b^2} \left[ (2\sqrt{2} - 2) a \frac{a}{\sqrt{2}} + \left( 2\sqrt{2} - 2 \right) \left( \frac{a}{\sqrt{2}} \right)^2 - a^2 \right]
= 0.
\]

So, \(\frac{a + 2b}{a + b}\) is closer to \(\sqrt{2}\).

Similarly, we also have if \(a/b > \sqrt{2}\), then \(\frac{a + 2b}{a + b} < \sqrt{2}\). Also, \(\frac{a + 2b}{a + b}\) is closer to \(\sqrt{2}\) in this case.

**Remark**: Note that

\[
\frac{a}{b} < \sqrt{2} < \frac{a + 2b}{a + b} < \frac{2b}{a} \text{ by Exercise 12 and 13.}
\]

And we know that \(\frac{a + 2b}{a + b}\) is closer to \(\sqrt{2}\). We can use it to approximate \(\sqrt{2}\). Similarly for the case

\[
\frac{2b}{a} < \frac{a + 2b}{a + b} < \sqrt{2} < \frac{a}{b}.
\]

1.14 Prove that \(\sqrt{n - 1} + \sqrt{n + 1}\) is irrational for every integer \(n \geq 1\).

**Proof**: Suppose that \(\sqrt{n - 1} + \sqrt{n + 1}\) is rational, and thus consider

\[
\left( \sqrt{n + 1} + \sqrt{n - 1} \right) \left( \sqrt{n + 1} - \sqrt{n - 1} \right) = 2
\]

which implies that \(\sqrt{n + 1} - \sqrt{n - 1}\) is rational. Hence, \(\sqrt{n + 1}\) and \(\sqrt{n - 1}\) are rational. So, \(n - 1 = k^2\) and \(n + 1 = h^2\), where \(k\) and \(h\) are positive integer. It implies that

\[
h = \frac{3}{2} \text{ and } k = \frac{1}{2}
\]

which is absurd. So, \(\sqrt{n - 1} + \sqrt{n + 1}\) is irrational for every integer \(n \geq 1\).
1.15 Given a real $x$ and an integer $N > 1$, prove that there exist integers $h$ and $k$ with $0 < k \leq N$ such that $|kx - h| < 1/N$. Hint. Consider the $N + 1$ numbers $tx - [tx]$ for $t = 0, 1, 2, ..., N$ and show that some pair differs by at most $1/N$.

Proof: Given $N > 1$, and thus consider $tx - [tx]$ for $t = 0, 1, 2, ..., N$ as follows. Since 

$$0 \leq tx - [tx] := a_t < 1,$$

so there exists two numbers $a_i$ and $a_j$ where $i \neq j$ such that

$$|a_i - a_j| < \frac{1}{N} \Rightarrow |(i - j)x - p| < \frac{1}{N},$$


Hence, there exist integers $h$ and $k$ with $0 < k \leq N$ such that $|kx - h| < 1/N$.

1.16 If $x$ is irrational prove that there are infinitely many rational numbers $h/k$ with $k > 0$ such that $|x - h/k| < 1/k^2$. Hint. Assume there are only a finite number $h_1/k_1, ..., h_r/k_r$ and obtain a contradiction by applying Exercise 1.15 with $N > 1/\delta$, where $\delta$ is the smallest of the numbers $|x - h_i/k_i|$.

Proof: Assume there are only a finite number $h_1/k_1, ..., h_r/k_r$, and let $\delta = \min_{i=1}^r |x - h_i/k_i| > 0$ since $x$ is irrational. Choose $N > 1/\delta$, then by Exercise 1.15, we have

$$\frac{1}{N} < \delta \leq \left| x - \frac{h}{k} \right| < \frac{1}{kN},$$

which implies that

$$\frac{1}{N} < \frac{1}{kN}$$

which is impossible. So, there are infinitely many rational numbers $h/k$ with $k > 0$ such that $|x - h/k| < 1/k^2$.

Remark: (1) There is another proof by continued fractions. The reader can see the book, An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 270. (Chinese Version)

(2) The exercise is useful to help us show the following lemma. \{ar + b : a \in \mathbb{Z}, b \in \mathbb{Z}\}, where $r \in Q^c$ is dense in $R$. It is equivalent to \{ar : a \in \mathbb{Z}\}, where $r \in Q^c$ is dense in $[0, 1]$ modulus 1.
Proof: Say \( \{ ar + b : a \in Z, b \in Z \} = S \), and since \( r \in Q^c \), then by Exercise 1.16, there are infinitely many rational numbers \( h/k \) with \( k > 0 \) such that \( |kr - h| < \frac{1}{k} \). Consider \((x - \delta, x + \delta) := I\), where \( \delta > 0 \), and thus choosing \( k_0 \) large enough so that \( 1/k_0 < \delta \). Define \( L = |k_0 r - h_0| \), then we have \( sL \in I \) for some \( s \in Z \). So, \( sL = (\pm)((sk_0)r - (sh_0)) \in S \). That is, we have proved that \( S \) is dense in \( R \).

1.17 Let \( x \) be a positive rational number of the form

\[
x = \sum_{k=1}^{n} \frac{a_k}{k!},
\]

where each \( a_k \) is nonnegative integer with \( a_k \leq k - 1 \) for \( k \geq 2 \) and \( a_n > 0 \).

Let \( [x] \) denote the largest integer in \( x \). Prove that \( a_1 = [x] \), that \( a_k = [k!x] - k [(k - 1)!x] \) for \( k = 2, ..., n \), and that \( n \) is the smallest integer such that \( n!x \) is an integer. Conversely, show that every positive rational number \( x \) can be expressed in this form in one and only one way.

Proof: \((\Rightarrow)\) First,

\[
[x] = \left[ a_1 + \sum_{k=2}^{n} \frac{a_k}{k!} \right]
\]

\[
= a_1 + \left[ \sum_{k=2}^{n} \frac{a_k}{k!} \right] \text{ since } a_1 \in N
\]

\[
= a_1 \text{ since } \sum_{k=2}^{n} \frac{a_k}{k!} \leq \sum_{k=2}^{n} \frac{k - 1}{k!} = \sum_{k=2}^{n} \frac{1}{(k - 1)!} - \frac{1}{k!} = 1 - \frac{1}{n!} < 1.
\]

Second, fixed \( k \) and consider

\[
k!x = k! \sum_{j=1}^{n} \frac{a_j}{j!} = k! \sum_{j=1}^{k-1} \frac{a_j}{j!} + a_k + k! \sum_{j=k+1}^{n} \frac{a_j}{j!}
\]

and

\[
(k - 1)!x = (k - 1)! \sum_{j=1}^{n} \frac{a_j}{j!} = (k - 1)! \sum_{j=1}^{k-1} \frac{a_j}{j!} + (k - 1)! \sum_{j=k}^{n} \frac{a_j}{j!}.
\]
So, 

\[ [k!x] = \left[ k! \sum_{j=1}^{k-1} \frac{a_j}{j!} + a_k + k! \sum_{j=k+1}^{n} \frac{a_j}{j!} \right] \]

\[ = k! \sum_{j=1}^{k-1} \frac{a_j}{j!} + a_k \text{ since } k! \sum_{j=k+1}^{n} \frac{a_j}{j!} < 1 \]

and

\[ k [(k-1)!x] = k \left[ (k-1)! \sum_{j=1}^{k-1} \frac{a_j}{j!} + (k-1)! \sum_{j=k}^{n} \frac{a_j}{j!} \right] \]

\[ = k (k-1)! \sum_{j=1}^{k-1} \frac{a_j}{j!} \text{ since } (k-1)! \sum_{j=k}^{n} \frac{a_j}{j!} < 1 \]

\[ = k! \sum_{j=1}^{k-1} \frac{a_j}{j!} \]

which implies that

\[ a_k = [k!x] - k [(k-1)!x] \text{ for } k = 2, \ldots, n. \]

Last, in order to show that \( n \) is the smallest integer such that \( n!x \) is an integer. It is clear that

\[ n!x = n! \sum_{k=1}^{n} \frac{a_k}{k!} \in Z. \]

In addition,

\[ (n-1)!x = (n-1)! \sum_{k=1}^{n} \frac{a_k}{k!} \]

\[ = (n-1)! \sum_{k=1}^{n-1} \frac{a_k}{k!} + \frac{a_n}{n} \]

\[ \notin Z \text{ since } \frac{a_n}{n} \notin Z. \]

So, we have proved it.
It is clear since every $a_n$ is uniquely determined.

**Upper bounds**

1.18 Show that the sup and the inf of a set are uniquely determined whenever they exist.

**Proof:** Given a nonempty set $S \subseteq \mathbb{R}$, and assume $\sup S = a$ and $\sup S = b$, we show $a = b$ as follows. Suppose that $a > b$, and thus choose $\varepsilon = \frac{a-b}{2}$, then there exists a $x \in S$ such that

$$b < \frac{a+b}{2} = a - \varepsilon < x < a$$

which implies that

$$b < x$$

which contradicts to $b = \sup S$. Similarly for $a < b$. Hence, $a = b$.

1.19 Find the sup and inf of each of the following sets of real numbers:

(a) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$, where $p, q,$ and $r$ take on all positive integer values.

**Proof:** Define $S = \{2^{-p} + 3^{-q} + 5^{-r} : p, q, r \in \mathbb{N}\}$. Then it is clear that $\sup S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$, and $\inf S = 0$.

(b) $S = \{x : 3x^2 - 10x + 3 < 0\}$

**Proof:** Since $3x^2 - 10x + 3 = (x - 3) (3x - 1)$, we know that $S = \left(\frac{1}{3}, 3\right)$. Hence, $\sup S = 3$ and $\inf S = \frac{1}{3}$.

(c) $S = \{x : (x - a) (x - b) (x - c) (x - d) < 0\}$, where $a < b < c < d$.

**Proof:** It is clear that $S = (a, b) \cup (c, d)$. Hence, $\sup S = d$ and $\inf S = a$.

1.20 Prove the comparison property for suprema (**Theorem 1.16**)

**Proof:** Since $s \leq t$ for every $s \in S$ and $t \in T$, fixed $t_0 \in T$, then $s \leq t_0$ for all $s \in S$. Hence, by **Axiom 10**, we know that $\sup S$ exists. In addition, it is clear $\sup S \leq \sup T$.

**Remark:** There is a useful result, we write it as a reference. Let $S$ and $T$ be two nonempty subsets of $\mathbb{R}$. If $S \subseteq T$ and $\sup T$ exists, then $\sup S$ exists and $\sup S \leq \sup T$. 

13
Proof: Since sup $T$ exists and $S \subseteq T$, we know that for every $s \in S$, we have

$$s \leq \sup T.$$ 

Hence, by Axiom 10, we have proved the existence of sup $S$. In addition, sup $S \leq$ sup $T$ is trivial.

1.21 Let $A$ and $B$ be two sets of positive numbers bounded above, and let $a = \sup A$, $b = \sup B$. Let $C$ be the set of all products of the form $xy$, where $x \in A$ and $y \in B$. Prove that $ab = \sup C$.

Proof: Given $\varepsilon > 0$, we want to find an element $c \in C$ such that $ab - \varepsilon < c$. If we can show this, we have proved that sup $C$ exists and equals $ab$.

Since sup $A = a > 0$ and sup $B = b > 0$, we can choose $n$ large enough such that $a - \varepsilon/n > 0$, $b - \varepsilon/n > 0$, and $n > a + b$. So, for this $\varepsilon' = \varepsilon/n$, there exists $a' \in A$ and $b' \in B$ such that

$$a - \varepsilon' < a' \text{ and } b - \varepsilon' < b'$$

which implies that

$$ab - \varepsilon'(a + b - \varepsilon') < ab' \text{ since } a - \varepsilon' > 0 \text{ and } b - \varepsilon' > 0$$

which implies that

$$ab - \frac{\varepsilon}{n} (a + b) < ab' := c$$

which implies that

$$ab - \varepsilon < c.$$ 

1.22 Given $x > 0$, and an integer $k \geq 2$. Let $a_0$ denote the largest integer $\leq x$ and, assuming that $a_0, a_1, ..., a_{n-1}$ have been defined, let $a_n$ denote the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} \leq x.$$ 

Note: When $k = 10$ the integers $a_0, a_1, ...$ are the digits in a decimal representation of $x$. For general $k$ they provide a representation in the scale of $k$.

(a) Prove that $0 \leq a_i \leq k - 1$ for each $i = 1, 2, ...$
**Proof:** Choose \( a_0 = [x] \), and thus consider

\[
[kx - ka_0] := a_1
\]

then

\[
0 \leq k(x - a_0) < k \Rightarrow 0 \leq a_1 \leq k - 1
\]

and

\[
a_0 + \frac{a_1}{k} \leq x \leq a_0 + \frac{a_1}{k} + \frac{1}{k}.
\]

Continue the process, we then have

\[
0 \leq a_i \leq k - 1 \text{ for each } i = 1, 2, ...
\]

and

\[
a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \ldots + \frac{a_n}{k^n} \leq x < a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \ldots + \frac{a_n}{k^n} + \frac{1}{k^n}.
\]

\[(*)\]

(b) Let \( r_n = a_0 + a_1k^{-1} + a_2k^{-2} + \ldots + a_nk^{-n} \) and show that \( x \) is the sup of the set of rational numbers \( r_1, r_2, \ldots \)

**Proof:** It is clear by (a)-(*)

**Inequality**

1.23 Prove Lagrange’s identity for real numbers:

\[
\left( \sum_{k=1}^{n} a_k b_k \right)^2 = \left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 \right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2.
\]

Note that this identity implies that **Cauchy-Schwarz inequality**.

**Proof:** Consider

\[
\left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 \right) = \sum_{1 \leq k, j \leq n} a_k^2 b_j^2 = \sum_{k=j} a_k^2 b_j^2 + \sum_{k \neq j} a_k^2 b_j^2 = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k \neq j} a_k^2 b_j^2
\]
and
\[
\left( \sum_{k=1}^{n} a_k b_k \right) \left( \sum_{k=1}^{n} a_k b_k \right) = \sum_{1 \leq k,j \leq n} a_k b_k a_j b_j = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k \neq j} a_k b_k a_j b_j
\]

So,
\[
\left( \sum_{k=1}^{n} a_k b_k \right)^2 = \left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 \right) + \sum_{k \neq j} a_k b_k a_j b_j - \sum_{1 \leq k,j \leq n} a_k^2 b_j^2 + a_j^2 b_k^2
\]
\[
= \left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 \right) - \sum_{1 \leq k<j \leq n} (a_k b_j - a_j b_k)^2.
\]

**Remark:** (1) The reader may recall the relation with Cross Product and Inner Product, we then have a fancy formula:
\[
\|x \times y\|^2 + |< x, y >|^2 = \|x\|^2 \|y\|^2,
\]
where \( x, y \in \mathbb{R}^3 \).

(2) We often write
\[
< a, b > := \sum_{k=1}^{n} a_k b_k,
\]
and the Cauchy-Schwarz inequality becomes
\[
|< x, y >| \leq \|x\| \|y\| \text{ by Remark (1)}.
\]

1.24 Prove that for arbitrary real \( a_k, b_k, c_k \) we have
\[
\left( \sum_{k=1}^{n} a_k b_k c_k \right)^4 \leq \left( \sum_{k=1}^{n} a_k^4 \right) \left( \sum_{k=1}^{n} b_k^2 \right) \left( \sum_{k=1}^{n} c_k^4 \right).
\]
Proof: Use Cauchy-Schwarz inequality twice, we then have
\[
\left( \sum_{k=1}^{n} a_k b_k c_k \right)^4 \leq \left( \sum_{k=1}^{n} a_k^2 b_k c_k \right)^2 \leq \left( \sum_{k=1}^{n} a_k^2 \right)^2 \left( \sum_{k=1}^{n} b_k^2 \right)^2 \leq \left( \sum_{k=1}^{n} a_k^4 \right)^{1/2} \left( \sum_{k=1}^{n} b_k^2 \right)^{1/2} \left( \sum_{k=1}^{n} c_k^4 \right)^{1/2}.
\]

1.25 Prove that Minkowski’s inequality:
\[
\left( \sum_{k=1}^{n} (a_k + b_k)^2 \right)^{1/2} \leq \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2} + \left( \sum_{k=1}^{n} b_k^2 \right)^{1/2}.
\]
This is the triangle inequality \(\|a + b\| \leq \|a\| + \|b\|\) for \(n\)-dimensional vectors, where \(a = (a_1, \ldots, a_n)\), \(b = (b_1, \ldots, b_n)\) and
\[
\|a\| = \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2}.
\]

Proof: Consider
\[
\sum_{k=1}^{n} (a_k + b_k)^2 = \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2 \sum_{k=1}^{n} a_k b_k \leq \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2 \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2} \left( \sum_{k=1}^{n} b_k^2 \right)^{1/2} \text{ by Cauchy-Schwarz inequality}
\]
\[
= \left[ \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2} + \left( \sum_{k=1}^{n} b_k^2 \right)^{1/2} \right]^2.
\]
So,
\[(\sum_{k=1}^{n} (a_k + b_k)^2)^{1/2} \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}.
\]

1.26 If \(a_1 \geq \ldots \geq a_n\) and \(b_1 \geq \ldots \geq b_n\), prove that
\[
\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leq n\left(\sum_{k=1}^{n} a_k b_k\right).
\]

Hint. \(\sum_{1 \leq j \leq k \leq n} (a_k - a_j)(b_k - b_j) \geq 0\).

Proof: Consider
\[
0 \leq \sum_{1 \leq j \leq k \leq n} (a_k - a_j)(b_k - b_j) = \sum_{1 \leq j \leq k \leq n} a_k b_k + a_j b_j - \sum_{1 \leq j \leq k \leq n} a_k b_j + a_j b_k
\]

which implies that
\[
\sum_{1 \leq j \leq k \leq n} a_k b_j + a_j b_k \leq \sum_{1 \leq j \leq k \leq n} a_k b_k + a_j b_j. 
\]

Since
\[
\sum_{1 \leq j \leq k \leq n} a_k b_j + a_j b_k = \sum_{1 \leq j < k \leq n} a_k b_j + a_j b_k + 2\sum_{k=1}^{n} a_k b_k
\]
\[
= \left(\sum_{1 \leq j < k \leq n} a_k b_j + a_j b_k + \sum_{k=1}^{n} a_k b_k\right) + \sum_{k=1}^{n} a_k b_k
\]
\[
= \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) + \sum_{k=1}^{n} a_k b_k,
\]

we then have, by (*)
\[
\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) + \sum_{k=1}^{n} a_k b_k \leq \sum_{1 \leq j \leq k \leq n} a_k b_k + a_j b_j. 
\]

(**)
In addition,

\[
\sum_{1 \leq j \leq k \leq n} a_k b_k + a_j b_j
\]

\[
= \sum_{k=1}^{n} a_k b_k + na_1 b_1 + \sum_{k=2}^{n} a_k b_k + (n-1) a_2 b_2 + \ldots + \sum_{k=n-1}^{n} a_k b_k + 2a_{n-1} b_{n-1} + \sum_{k=n}^{n} a_k b_k
\]

\[
= n \sum_{k=1}^{n} a_k b_k + a_1 b_1 + a_2 b_2 + \ldots + a_n b_n
\]

\[
= (n + 1) \sum_{k=1}^{n} a_k b_k
\]

which implies that, by (**),

\[
\left( \sum_{k=1}^{n} a_k \right) \left( \sum_{k=1}^{n} b_k \right) \leq n \left( \sum_{k=1}^{n} a_k b_k \right).
\]

**Complex numbers**

1.27 Express the following complex numbers in the form \(a + bi\).

(a) \((1 + i)^3\)

**Solution:** \((1 + i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i\).

(b) \((2 + 3i) / (3 - 4i)\)

**Solution:** \(\frac{2+3i}{3-4i} = \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)} = \frac{-6+17i}{25} = \frac{-6}{25} + \frac{17}{25}i\).

(c) \(i^5 + i^{16}\)

**Solution:** \(i^5 + i^{16} = i + 1\).

(d) \(\frac{1}{2} (1 + i) (1 + i^{-8})\)

**Solution:** \(\frac{1}{2} (1 + i) (1 + i^{-8}) = 1 + i\).

1.28 In each case, determine all real \(x\) and \(y\) which satisfy the given relation.
(a) \( x + iy = |x - iy| \)
\[ \text{Proof:} \text{ Since } |x - iy| \geq 0, \text{ we have} \]
\[ x \geq 0 \text{ and } y = 0. \]

(b) \( x + iy = (x - iy)^2 \)
\[ \text{Proof:} \text{ Since } (x - iy)^2 = x^2 - (2xy)i - y^2, \text{ we have} \]
\[ x = x^2 - y^2 \text{ and } y = -2xy. \]

We consider two cases: (i) \( y = 0 \) and (ii) \( y \neq 0 \).

(i) As \( y = 0 \) : \( x = 0 \text{ or } 1 \).

(ii) As \( y \neq 0 \) : \( x = -1/2, \) and \( y = \pm \sqrt{3}/2 \).

(c) \( \sum_{k=0}^{100} i^k = x + iy \)
\[ \text{Proof:} \text{ Since } \sum_{k=0}^{100} i^k = \frac{1-i^{101}}{1-i} = \frac{1-i}{1-i} = 1, \text{ we have} x = 1 \text{ and } y = 0. \]

1.29 If \( z = x + iy, \) \( x \) and \( y \) real, the complex conjugate of \( z \) is the complex number \( \bar{z} = x - iy. \) Prove that:

(a) Conjugate of \((z_1 + z_2) = \bar{z}_1 + \bar{z}_2\)
\[ \text{Proof:} \text{ Write } z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2, \text{ then} \]
\[ \overline{z_1 + z_2} = (x_1 + x_2) + i(y_1 + y_2) \]
\[ = (x_1 + x_2) - i(y_1 + y_2) \]
\[ = (x_1 - iy_1) + (x_2 - iy_2) \]
\[ = \bar{z}_1 + \bar{z}_2. \]

(b) \( \overline{z_1z_2} = \bar{z}_1 \bar{z}_2 \)
\[ \text{Proof:} \text{ Write } z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2, \text{ then} \]
\[ \overline{z_1z_2} = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \]
\[ = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \]
and
\[ \bar{z}_1 \bar{z}_2 = (x_1 - iy_1)(x_2 - iy_2) \]
\[ = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1). \]
So, $z_1 \bar{z}_2 = \bar{z}_1 z_2$

(c) $z \bar{z} = |z|^2$

**Proof:** Write $z = x + iy$ and thus

$$z \bar{z} = x^2 + y^2 = |z|^2.$$  

(d) $z + \bar{z} = $twice the real part of $z$

**Proof:** Write $z = x + iy$, then

$$z + \bar{z} = 2x,$$

twice the real part of $z$.

(e) $(z - \bar{z})/i = $twice the imaginary part of $z$

**Proof:** Write $z = x + iy$, then

$$\frac{z - \bar{z}}{i} = 2y,$$

twice the imaginary part of $z$.

1.30 Describe geometrically the set of complex numbers $z$ which satisfies each of the following conditions:

(a) $|z| = 1$

**Solution:** The unit circle centered at zero.

(b) $|z| < 1$

**Solution:** The open unit disk centered at zero.

(c) $|z| \leq 1$

**Solution:** The closed unit disk centered at zero.

(d) $z + \bar{z} = 1$

**Solution:** Write $z = x + iy$, then $z + \bar{z} = 1$ means that $x = 1/2$. So, the set is the line $x = 1/2$.

(e) $z - \bar{z} = i$
Proof: Write \( z = x + iy \), then \( z - \bar{z} = i \) means that \( y = 1/2 \). So, the set is the line \( y = 1/2 \).

(f) \( z + \bar{z} = |z|^2 \)

Proof: Write \( z = x + iy \), then \( 2x = x^2 + y^2 \Leftrightarrow (x - 1)^2 + y^2 = 1 \). So, the set is the unit circle centered at \((1,0)\).

1.31 Given three complex numbers \( z_1, z_2, z_3 \) such that \( |z_1| = |z_2| = |z_3| = 1 \) and \( z_1 + z_2 + z_3 = 0 \). Show that these numbers are vertices of an equilateral triangle inscribed in the unit circle with center at the origin.

Proof: It is clear that three numbers are vertices of triangle inscribed in the unit circle with center at the origin. It remains to show that \( |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \). In addition, it suffices to show that

\[ |z_1 - z_2| = |z_2 - z_3| . \]

Note that

\[ |2z_1 + z_3| = |2z_3 + z_1| \quad \text{by} \quad z_1 + z_2 + z_3 = 0 \]

which is equivalent to

\[ |2z_1 + z_3|^2 = |2z_3 + z_1|^2 \]

which is equivalent to

\[ (2z_1 + z_3) (2\bar{z}_1 + \bar{z}_3) = (2z_3 + z_1) (2\bar{z}_3 + \bar{z}_1) \]

which is equivalent to

\[ |z_1| = |z_3| . \]

1.32 If \( a \) and \( b \) are complex numbers, prove that:

(a) \( |a - b|^2 \leq (1 + |a|^2) (1 + |b|^2) \)

Proof: Consider

\[ (1 + |a|^2) (1 + |b|^2) - |a - b|^2 = (1 + \bar{a}a) (1 + \bar{b}b) - (a - b) (\bar{a} - \bar{b}) \]

\[ = (1 + \bar{a}b) (1 + a\bar{b}) \]

\[ = |1 + \bar{a}b|^2 \geq 0 , \]
so, \(|a - b|^2 \leq (1 + |a|^2) (1 + |b|^2)\)

(b) If \(a \neq 0\), then \(|a + b| = |a| + |b|\) if, and only if, \(b/a\) is real and nonnegative.

**Proof:** \((\Rightarrow)\) Since \(|a + b| = |a| + |b|\), we have

\[|a + b|^2 = (|a| + |b|)^2\]

which implies that

\[\text{Re}(\bar{a}b) = |a| |b| = |\bar{a}| |b|\]

which implies that

\[\bar{a}b = |\bar{a}| |b|\]

which implies that

\[\frac{b}{a} = \frac{\bar{a}b}{\bar{a}a} = \frac{|\bar{a}| |b|}{|a|^2} \geq 0.\]

\((\Leftarrow)\) Suppose that

\[\frac{b}{a} = k, \text{ where } k \geq 0.\]

Then

\[|a + b| = |a + ka| = (1 + k) |a| = |a| + k |a| = |a| + |b|.\]

1.33 If \(a\) and \(b\) are complex numbers, prove that

\[|a - b| = |1 - \bar{a}b|\]

if, and only if, \(|a| = 1\) or \(|b| = 1\). For which \(a\) and \(b\) is the inequality \(|a - b| < |1 - \bar{a}b|\) valid?

**Proof:** \((\Leftrightarrow)\) Since

\[|a - b| = |1 - \bar{a}b|\]

\[\iff (\bar{a} - \bar{b}) (a - b) = (1 - \bar{a}b) (1 - a\bar{b})\]

\[\iff |a|^2 + |b|^2 = 1 + |a|^2 |b|^2\]

\[\iff (|a|^2 - 1) (|b|^2 - 1) = 0\]

\[\iff |a|^2 = 1 \text{ or } |b|^2 = 1.\]
By the preceding, it is easy to know that

\[ |a - b| < |1 - \bar{a}b| \Leftrightarrow 0 < (|a|^2 - 1) (|b|^2 - 1). \]

So, \(|a - b| < |1 - \bar{a}b|\) if, and only if, \(|a| > 1\) and \(|b| > 1\). (Or \(|a| < 1\) and \(|b| < 1\)).

1.34 If \(a\) and \(c\) are real constant, \(b\) complex, show that the equation

\[ azi + b\bar{z} + \bar{b}z + c = 0 \quad (a \neq 0, z = x + iy) \]

represents a circle in the \(x - y\) plane.

**Proof:** Consider

\[ z\bar{z} - \frac{b}{-a} \bar{z} - \frac{\bar{b}}{-a} z + \frac{b}{-a} \left( \frac{b}{-a} \right) = \frac{-ac + |b|^2}{a^2}, \]

so, we have

\[ \left| z - \left( \frac{b}{-a} \right) \right|^2 = \frac{-ac + |b|^2}{a^2}. \]

Hence, as \(|b|^2 - ac > 0\), it is a circle. As \(\frac{-ac + |b|^2}{a^2} = 0\), it is a point. As \(\frac{-ac + |b|^2}{a^2} < 0\), it is not a circle.

**Remark:** The idea is easy from the fact

\[ |z - q| = r. \]

We square both sides and thus

\[ z\bar{z} - q\bar{z} - \bar{q}z + \bar{q}q = r^2. \]

1.35 Recall the definition of the inverse tangent: given a real number \(t\), \(\tan^{-1}(t)\) is the unique real number \(\theta\) which satisfies the two conditions

\[ -\frac{\pi}{2} < \theta < +\frac{\pi}{2}, \quad \tan \theta = t. \]

If \(z = x + iy\), show that

(a) \(\arg(z) = \tan^{-1} \left( \frac{y}{x} \right)\), if \(x > 0\)
Proof: Note that in this textbook, we say \( \arg(z) \) is the principal argument of \( z \), denoted by \( \theta = \arg z \), where \( -\pi < \theta \leq \pi \).

So, as \( x > 0 \), \( \arg z = \tan^{-1} \left( \frac{y}{x} \right) \).

(b) \( \arg(z) = \tan^{-1} \left( \frac{y}{x} \right) + \pi \), if \( x < 0, \ y \geq 0 \)

Proof: As \( x < 0 \), and \( y \geq 0 \). The point \((x, y)\) is lying on \( S = \{(x, y) : x < 0, \ y \geq 0 \} \).

Note that \( -\pi < \arg z \leq \pi \), so we have \( \arg(z) = \tan^{-1} \left( \frac{y}{x} \right) + \pi \).

(c) \( \arg(z) = \tan^{-1} \left( \frac{y}{x} \right) - \pi \), if \( x < 0, \ y < 0 \)

Proof: Similarly for (b). So, we omit it.

(d) \( \arg(z) = \frac{\pi}{2} \) if \( x = 0, \ y > 0 \); \( \arg(z) = -\frac{\pi}{2} \) if \( x = 0, \ y < 0 \).

Proof: It is obvious.

1.36 Define the following "pseudo-ordering" of the complex numbers: we say \( z_1 < z_2 \) if we have either

(i) \( |z_1| < |z_2| \) or (ii) \( |z_1| = |z_2| \) and \( \arg(z_1) < \arg(z_2) \).

Which of Axioms 6, 7, 8, 9 are satisfied by this relation?

Proof: (1) For axiom 6, we prove that it holds as follows. Given \( z_1 = r_1 e^{i \arg(z_1)} \) and \( r_2 e^{i \arg(z_2)} \), then if \( z_1 = z_2 \), there is nothing to prove it. If \( z_1 \neq z_2 \), there are two possibilities: (a) \( r_1 \neq r_2 \), or (b) \( r_1 = r_2 \) and \( \arg(z_1) \neq \arg(z_2) \). So, it is clear that axiom 6 holds.

(2) For axiom 7, we prove that it does not hold as follows. Given \( z_1 = 1 \) and \( z_2 = -1 \), then it is clear that \( z_1 < z_2 \) since \( |z_1| = |z_2| = 1 \) and \( \arg(z_1) = 0 < \arg(z_2) = \pi \). However, let \( z_3 = -i \), we have

\[
z_1 + z_3 = 1 - i > z_2 + z_3 = -1 + i
\]

since

\[
|z_1 + z_3| = |z_2 + z_3| = \sqrt{2}
\]

and

\[
\arg(z_1 + z_3) = -\frac{\pi}{4} > -\frac{3\pi}{4} = \arg(z_2 + z_3).
\]

(3) For axiom 8, we prove that it holds as follows. If \( z_1 > 0 \) and \( z_2 > 0 \), then \( |z_1| > 0 \) and \( |z_2| > 0 \). Hence, \( z_1 z_2 > 0 \) by \( |z_1 z_2| = |z_1| |z_2| > 0 \).

(4) For axiom 9, we prove that it holds as follows. If \( z_1 > z_2 \) and \( z_2 > z_3 \), we consider the following cases. Since \( z_1 > z_2 \), we may have (a) \( |z_1| > |z_2| \) or (b) \( |z_1| = |z_2| \) and \( \arg(z_1) < \arg(z_2) \).

As \( |z_1| > |z_2| \), it is clear that \( |z_1| > |z_3| \). So, \( z_1 > z_3 \).
As $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$, we have $\arg(z_1) > \arg(z_3)$. So, $z_1 > z_3$.

1.37 Which of Axioms 6, 7, 8, 9 are satisfied if the pseudo-ordering is defined as follows? We say $(x_1, y_1) < (x_2, y_2)$ if we have either (i) $x_1 < x_2$ or (ii) $x_1 = x_2$ and $y_1 < y_2$.

Proof: (1) For axiom 6, we prove that it holds as follows. Given $x = (x_1, y_1)$ and $y = (x_2, y_2)$. If $x = y$, there is nothing to prove it. We consider $x \neq y$: As $x \neq y$, we have $x_1 \neq x_2$ or $y_1 \neq y_2$. Both cases imply $x < y$ or $y < x$.

(2) For axiom 7, we prove that it holds as follows. Given $x = (x_1, y_1)$, $y = (x_2, y_2)$ and $z = (z_1, z_3)$. If $x < y$, then there are two possibilities: (a) $x_1 < x_2$ or (b) $x_1 = x_2$ and $y_1 < y_2$.

For case (a), it is clear that $x_1 + z_1 < y_1 + z_1$. So, $x + z < y + z$.

For case (b), it is clear that $x_1 + z_1 = y_1 + z_1$ and $x_2 + z_2 < y_2 + z_2$. So, $x + z < y + z$.

(3) For axiom 8, we prove that it does not hold as follows. Consider $x = (1, 0)$ and $y = (0, 1)$, then it is clear that $x > 0$ and $y > 0$. However, $xy = (0, 0) = 0$.

(4) For axiom 9, we prove that it holds as follows. Given $x = (x_1, y_1)$, $y = (x_2, y_2)$ and $z = (z_1, z_3)$. If $x > y$ and $y > z$, then we consider the following cases. (a) $x_1 > y_1$, or (b) $x_1 = y_1$.

For case (a), it is clear that $x_1 > z_1$. So, $x > z$.

For case (b), it is clear that $x_2 > y_2$. So, $x > z$.

1.38 State and prove a theorem analogous to Theorem 1.48, expressing $\arg(z_1/z_2)$ in terms of $\arg(z_1)$ and $\arg(z_2)$.

Proof: Write $z_1 = r_1e^{i\arg(z_1)}$ and $z_2 = r_2e^{i\arg(z_2)}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i[\arg(z_1) - \arg(z_2)]}.$$

Hence,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2),$$

where

$$n(z_1, z_2) = \begin{cases} 
0 & \text{if } -\pi < \arg(z_1) - \arg(z_2) \leq \pi \\
1 & \text{if } -2\pi < \arg(z_1) - \arg(z_2) \leq -\pi \\
-1 & \text{if } \pi < \arg(z_1) - \arg(z_2) < 2\pi 
\end{cases}.$$
1.39 State and prove a theorem analogous to Theorem 1.54, expressing $\log\left(\frac{z_1}{z_2}\right)$ in terms of $\log(z_1)$ and $\log(z_2)$.

**Proof:** Write $z_1 = r_1 e^{i \arg(z_1)}$ and $z_2 = r_2 e^{i \arg(z_2)}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i [\arg(z_1) - \arg(z_2)]}.$$ 

Hence,

$$\log\left(\frac{z_1}{z_2}\right) = \log \left| \frac{z_1}{z_2} \right| + i \arg \left( \frac{z_1}{z_2} \right)$$

$$= \log |z_1| - \log |z_2| + i \left[ \arg (z_1) - \arg (z_2) + 2\pi n (z_1, z_2) \right] \text{ by exercise 1.38}$$

$$= \log (z_1) - \log (z_2) + i2\pi n (z_1, z_2).$$

1.40 Prove that the $n$th roots of 1 (also called the $n$th roots of unity) are given by $\alpha, \alpha^2, ..., \alpha^n$, where $\alpha = e^{2\pi i/n}$, and show that the roots $\neq 1$ satisfy the equation

$$1 + x + x^2 + ... + x^{n-1} = 0.$$

**Proof:** By Theorem 1.51, we know that the roots of 1 are given by $\alpha, \alpha^2, ..., \alpha^n$, where $\alpha = e^{2\pi i/n}$. In addition, since

$$x^n = 1 \Rightarrow (x - 1) \left( 1 + x + x^2 + ... + x^{n-1} \right) = 0$$

which implies that

$$1 + x + x^2 + ... + x^{n-1} = 0 \text{ if } x \neq 1.$$ 

So, all roots except 1 satisfy the equation

$$1 + x + x^2 + ... + x^{n-1} = 0.$$ 

1.41 (a) Prove that $|z^i| < e^n$ for all complex $z \neq 0$.

**Proof:** Since

$$z^i = e^{i \log(z)} = e^{-\arg(z) + i \log |z|},$$

$$|z^i| = e^{-\arg(z)},$$

which is less than $e^n$ for all $z \neq 0$. 

27
we have
\[ |z^i| = e^{-\arg(z)} < e^\pi \]
by \(-\pi < \arg(z) \leq \pi\).

(b) Prove that there is no constant \(M > 0\) such that \(|\cos z| < M\) for all complex \(z\).

**Proof:** Write \(z = x + iy\) and thus,
\[
\cos z = \cos x \cosh y - i \sin x \sinh y
\]
which implies that
\[
|\cos x \cosh y| \leq |\cos z|.
\]
Let \(x = 0\) and \(y\) be real, then
\[
\frac{e^y}{2} \leq \frac{1}{2} |e^y + e^{-y}| \leq |\cos z|.
\]
So, there is no constant \(M > 0\) such that \(|\cos z| < M\) for all complex \(z\).

**Remark:** There is an important theorem related with this exercise. We state it as a reference. (Liouville’s Theorem) A bounded entire function is constant. The reader can see the book, *Complex Analysis by Joseph Bak, and Donald J. Newman*, pp 62-63. Liouville’s Theorem can be used to prove the much important theorem, *Fundamental Theorem of Algebra.*

1.42 If \(w = u + iv\) (\(u, v\) real), show that
\[
z^w = e^{u \log |z| - v \arg(z)} e^{i[v \log |z| + u \arg(z)]}.
\]

**Proof:** Write \(z^w = e^{w \log(z)}\), and thus
\[
w \log(z) = (u + iv) (\log |z| + i \arg(z))
= [u \log |z| - v \arg(z)] + i [v \log |z| + u \arg(z)].
\]
So,
\[
z^w = e^{u \log |z| - v \arg(z)} e^{i[v \log |z| + u \arg(z)]}.
\]
1.43 (a) Prove that \( \log (z^w) = w \log z + 2\pi i n \).

**Proof:** Write \( w = u + iv \), where \( u \) and \( v \) are real. Then

\[
\log (z^w) = \log |z^w| + i \arg (z^w)
\]
\[
= \log \left[ e^{u \log |z| - v \arg (z)} \right] + i \left[ v \log |z| + u \arg (z) \right] + 2\pi i n \\
= u \log |z| - v \arg (z) + i \left[ v \log |z| + u \arg (z) \right] + 2\pi i n.
\]

On the other hand,

\[
w \log z + 2\pi i n = (u + iv) (\log |z| + i \arg (z)) + 2\pi i n
\]
\[
= u \log |z| - v \arg (z) + i \left[ v \log |z| + u \arg (z) \right] + 2\pi i n.
\]

Hence, \( \log (z^w) = w \log z + 2\pi i n \).

**Remark:** There is another proof by considering

\[
e^\log (z^w) = z^w = e^{w \log (z)}
\]

which implies that

\[
\log (z^w) = w \log z + 2\pi i n
\]

for some \( n \in \mathbb{Z} \).

(b) Prove that \( (z^w)^\alpha = z^{w\alpha} e^{2\pi i n \alpha} \), where \( n \) is an integer.

**Proof:** By (a), we have

\[
(z^w)^\alpha = e^{\alpha \log (z^w)} = e^{\alpha (w \log z + 2\pi i n)} = e^{\alpha w \log z} e^{2\pi i n \alpha} = z^{\alpha w} e^{2\pi i n \alpha},
\]

where \( n \) is an integer.

1.44 (i) If \( \theta \) and \( a \) are real numbers, \(-\pi < \theta \leq \pi\), prove that

\[
(\cos \theta + i \sin \theta)^a = \cos (a \theta) + i \sin (a \theta).
\]

**Proof:** Write \( \cos \theta + i \sin \theta = z \), we then have

\[
(\cos \theta + i \sin \theta)^a = z^a = e^{a \log z} = e^{a \left[ \log |z| + i \arg (z) \right]} = e^{ia \theta}
\]
\[
= \cos (a \theta) + i \sin (a \theta).
\]
Remark: Compare with the Exercise 1.43-(b).

(ii) Show that, in general, the restriction $-\pi < \theta \leq \pi$ is necessary in (i) by taking $\theta = -\pi$, $a = \frac{1}{2}$.

Proof: As $\theta = -\pi$, and $a = \frac{1}{2}$, we have

$(-1)^{\frac{1}{2}} = e^{\frac{1}{2} \text{Log}(-1)} = e^{\frac{\pi}{2}i} = i \neq -i = \cos \left(\frac{-\pi}{2}\right) + i \sin \left(\frac{-\pi}{2}\right)$.

(iii) If $a$ is an integer, show that the formula in (i) holds without any restriction on $\theta$. In this case it is known as DeMorvre’s theorem.

Proof: By Exercise 1.43, as $a$ is an integer we have

$$(z^w)^a = z^{wa},$$

where $z^w = e^{i\theta}$. Then

$$(e^{i\theta})^a = e^{i\theta a} = \cos (a\theta) + i \sin (a\theta).$$

1.45 Use DeMorvre’s theorem (Exercise 1.44) to derive the trigonometric identities

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$$
$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta,$$

valid for real $\theta$. Are these valid when $\theta$ is complex?

Proof: By Exercise 1.44-(iii), we have for any real $\theta$,

$$(\cos \theta + i \sin \theta)^3 = \cos (3\theta) + i \sin (3\theta).$$

By Binomial Theorem, we have

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$$

and

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta.$$
For complex $\theta$, we show that it holds as follows. Note that $\sin z = e^{iz} - e^{-iz}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, we have

$$3 \cos^2 z \sin z - \sin^3 z = 3 \left( \frac{e^{iz} + e^{-iz}}{2} \right)^2 \left( \frac{e^{iz} - e^{-iz}}{2i} \right) - \left( \frac{e^{iz} - e^{-iz}}{2i} \right)^3$$

$$= 3 \left( \frac{e^{2iz} + e^{-2iz} + 2}{4} \right) \left( \frac{e^{iz} - e^{-iz}}{2i} \right) + \frac{e^{3iz} - 3e^{iz} + 3e^{-iz} - e^{-3iz}}{8i}$$

$$= \frac{1}{8i} \left[ 3 \left( e^{2zi} + e^{-2zi} + 2 \right) \left( e^{zi} - e^{-zi} \right) + \left( e^{3zi} - 3e^{iz} + 3e^{-iz} - e^{-3zi} \right) \right]$$

$$= \frac{1}{8i} \left[ \left( 3e^{3zi} + 3e^{iz} - 3e^{-iz} - 3e^{-3zi} \right) + \left( e^{3zi} - 3e^{iz} + 3e^{-iz} - e^{-3zi} \right) \right]$$

$$= \frac{4}{8i} \left( e^{3zi} - e^{-3zi} \right)$$

$$= \frac{1}{2i} \left( e^{3zi} - e^{-3zi} \right)$$

$$= \sin 3z.$$

Similarly, we also have

$$\cos^3 z - 3 \cos z \sin^2 z = \cos 3z.$$

1.46 Define $\tan z = \frac{\sin z}{\cos z}$ and show that for $z = x + iy$, we have

$$\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$
Proof: Since
\[ \tan z = \frac{\sin z}{\cos z} = \frac{\sin (x + iy)}{\cos (x + iy)} = \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y} \]
\[ = \frac{(\sin x \cosh y + i \cos x \sinh y) (\cos x \cosh y + i \sin x \sinh y)}{(\cos x \cosh y - i \sin x \sinh y) (\cos x \cosh y + i \sin x \sinh y)} = \frac{\sin x \cosh y + i \cos x \sinh y}{(\cos x \cosh y)^2 - (i \sin x \sinh y)^2} \]
\[ = \frac{\sin x \cos x (\cosh^2 y - \sinh^2 y) + i (\cosh y \sinh y)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \text{ since } \sin^2 x + \cos^2 x = 1 \]
\[ = \frac{\sin x \cos x + i (\cosh y \sinh y)}{\cos^2 x + \sinh^2 y} \text{ since } \cosh^2 y = 1 + \sinh^2 y \]
\[ = \frac{1}{2} \sin 2x + \frac{i}{2} \sinh 2y \text{ since } 2 \cos y \sinh y = \sinh 2y \text{ and } 2 \sin x \cos x = \sin 2x \]
\[ = \frac{2 \cos^2 x + 2 \sinh^2 y}{\cos 2x + \sinh 2y} \text{ since } \cos 2x = 2 \cos^2 x - 1 \text{ and } 2 \sinh^2 y + 1 = \cosh 2y. \]

1.47 Let \( w \) be a given complex number. If \( w \neq \pm 1 \), show that there exists two values of \( z = x + iy \) satisfying the conditions \( \cos z = w \) and \( -\pi < x \leq \pi \). Find these values when \( w = i \) and when \( w = 2 \).

Proof: Since \( \cos z = \frac{e^{iz} + e^{-iz}}{2} \), if we let \( e^{iz} = u \), then \( \cos z = w \) implies that
\[ w = \frac{u^2 + 1}{2u} \Rightarrow u^2 - 2wu + 1 = 0 \]
which implies that
\[ (u - w)^2 = w^2 - 1 \neq 0 \text{ since } w \neq \pm 1. \]

So, by Theorem 1.51,
\[ e^{iz} = u = w + |w^2 - 1|^{1/2} e^{i\phi_k}, \text{ where } \phi_k = \frac{\arg (w^2 - 1)}{2} + \frac{2\pi k}{2}, \text{ } k = 0, 1. \]
\[ = w \pm |w^2 - 1|^{1/2} e^{i \left( \frac{\arg (w^2 - 1)}{2} \right)} \]

32
So,

\[ ix - y = i(x + iy) = iz = \log \left| w \pm |w^2 - 1|^{1/2} e^{i \text{arg} \left( \frac{w^2 - 1}{2} \right)} \right| + i \arg \left( w \pm |w^2 - 1|^{1/2} e^{i \left( \frac{\text{arg} \left( \frac{w^2 - 1}{2} \right)}{2} \right)} \right) \]

Hence, there exists two values of \( z = x + iy \) satisfying the conditions \( \cos z = w \) and

\[-\pi < x = \arg \left( w \pm |w^2 - 1|^{1/2} e^{i \left( \frac{\text{arg} \left( \frac{w^2 - 1}{2} \right)}{2} \right)} \right) \leq \pi.\]

For \( w = i \), we have

\[ iz = \log \left| (1 \pm \sqrt{2}) i \right| + i \arg \left( (1 \pm \sqrt{2}) i \right) \]

which implies that

\[ z = \arg \left( (1 \pm \sqrt{2}) i \right) - i \log \left| (1 \pm \sqrt{2}) i \right|. \]

For \( w = 2 \), we have

\[ iz = \log \left| 2 \pm \sqrt{3} \right| + i \arg \left( 2 \pm \sqrt{3} \right) \]

which implies that

\[ z = \arg \left( 2 \pm \sqrt{3} \right) - i \log \left| 2 \pm \sqrt{3} \right|. \]

1.48 Prove Lagrange’s identity for complex numbers:

\[ \left| \sum_{k=1}^{n} a_k b_k \right|^2 = \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \sum_{1 \leq k < j \leq n} (a_k \overline{b_j} - \overline{a_k} b_j)^2. \]

Use this to deduce a Cauchy-Schwarz inequality for complex numbers.

**Proof:** It is the same as the Exercise 1.23; we omit the details.

1.49 (a) By equating imaginary parts in DeMoivre’s formula prove that

\[ \sin n\theta = \sin^n \theta \left\{ \binom{n}{1} \cot^{n-1} \theta - \binom{n}{3} \cot^{n-3} \theta + \binom{n}{5} \cot^{n-5} \theta - +... \right\} \]
Proof: By Exercise 1.44 (i), we have
\[
\sin n\theta = \sum_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n}{2k-1} \sin^{2k-1} \theta \cos^{n-(2k-1)} \theta
\]
\[
= \sin^n \theta \left\{ \sum_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n}{2k-1} \cot^{n-(2k-1)} \theta \right\}
\]
\[
= \sin^n \theta \left\{ \left( \frac{1}{1} \right) \cot^{n-1} \theta - \left( \frac{1}{3} \right) \cot^{n-3} \theta + \left( \frac{1}{5} \right) \cot^{n-5} \theta - +... \right\}.
\]

(b) If $0 < \theta < \pi/2$, prove that
\[
\sin (2m+1) \theta = \sin^{2m+1} \theta P_m (\cot^2 \theta)
\]
where $P_m$ is the polynomial of degree $m$ given by
\[
P_m (x) = \left( \frac{2m+1}{1} \right) x^m - \left( \frac{2m+1}{3} \right) x^{m-1} + \left( \frac{2m+1}{5} \right) x^{m-2} - +...
\]
Use this to show that $P_m$ has zeros at the $m$ distinct points $x_k = \cot^2 \left\{ \pi k / (2m+1) \right\}$ for $k = 1, 2, ..., m$.

Proof: By (a),
\[
\sin (2m+1) \theta
\]
\[
= \sin^{2m+1} \theta \left\{ \left( \frac{2m+1}{1} \right) (\cot^2 \theta)^m - \left( \frac{2m+1}{3} \right) (\cot^2 \theta)^{m-1} + \left( \frac{2m+1}{5} \right) (\cot^2 \theta)^{m-2} - +... \right\}
\]
\[
= \sin^{2m+1} \theta P_m (\cot^2 \theta), \text{ where } P_m (x) = \sum_{k=1}^{m+1} \binom{2m+1}{2k-1} x^{m+1-k}. \quad (*)
\]
In addition, by (*), \( \sin (2m+1) \theta = 0 \) if, and only if, \( P_m (\cot^2 \theta) = 0 \). Hence, \( P_m \) has zeros at the $m$ distinct points $x_k = \cot^2 \left\{ \pi k / (2m+1) \right\}$ for $k = 1, 2, ..., m$.

(c) Show that the sum of the zeros of $P_m$ is given by
\[
\sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m+1} = \frac{m (2m-1)}{3},
\]
34
and the sum of their squares is given by
\[
\sum_{k=1}^{m} \cot^4 \frac{\pi k}{2m + 1} = \frac{m (2m - 1) (4m^2 + 10m - 9)}{45}.
\]

Note. There identities can be used to prove that \( \sum_{n=1}^{\infty} n^{-2} = \pi^2/6 \) and \( \sum_{n=1}^{\infty} n^{-4} = \pi^4/90 \). (See Exercises 8.46 and 8.47.)

**Proof:** By (b), we know that sum of the zeros of \( P_m \) is given by
\[
\sum_{k=1}^{m} x_k = \sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m + 1} = \left( -\frac{(2m+1)}{(3m+1)} \right) = \frac{m (2m - 1)}{3}.
\]

And the sum of their squares is given by
\[
\sum_{k=1}^{m} x_k^2 = \sum_{k=1}^{m} \cot^4 \frac{\pi k}{2m + 1}
\]
\[
= \left( \sum_{k=1}^{m} x_k \right)^2 - 2 \left( \sum_{1 \leq i < j \leq n} x_i x_j \right)
\]
\[
= \left( \frac{m (2m - 1)}{3} \right)^2 - 2 \left( \frac{(2m+1)}{(3m+1)} \right)
\]
\[
= \frac{m (2m - 1) (4m^2 + 10m - 9)}{45}.
\]

1.50 Prove that \( z^n - 1 = \prod_{k=1}^{n} (z - e^{2\pi ik/n}) \) for all complex \( z \). Use this to derive the formula
\[
\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}.
\]

**Proof:** Since \( z^n = 1 \) has exactly \( n \) distinct roots \( e^{2\pi ik/n} \), where \( k = 0, ..., n - 1 \) by Theorem 1.51. Hence, \( z^n - 1 = \prod_{k=1}^{n} (z - e^{2\pi ik/n}) \). It implies that
\[
z^{n-1} + ... + 1 = \prod_{k=1}^{n-1} (z - e^{2\pi ik/n}).
\]
So, let $z = 1$, we obtain that

$$n = \prod_{k=1}^{n-1} \left( 1 - e^{2\pi i k/n} \right) = \prod_{k=1}^{n-1} \left[ \left( 1 - \cos \frac{2\pi k}{n} \right) - i \left( \sin \frac{2\pi k}{n} \right) \right]$$

$$= \prod_{k=1}^{n-1} \left( 2 \sin^2 \frac{\pi k}{n} \right) - i \left( 2 \sin \frac{\pi k}{n} \cos \frac{\pi k}{n} \right)$$

$$= \prod_{k=1}^{n-1} 2 \left( \sin \frac{\pi k}{n} \right) \left( \sin \frac{\pi k}{n} - i \cos \frac{\pi k}{n} \right)$$

$$= 2^{n-1} \prod_{k=1}^{n-1} \left( \sin \frac{\pi k}{n} \right) \left( \cos \left( \frac{3\pi}{2} + \frac{\pi k}{n} \right) + i \sin \left( \frac{3\pi}{2} + \frac{\pi k}{n} \right) \right)$$

$$= 2^{n-1} \prod_{k=1}^{n-1} \left( \sin \frac{\pi k}{n} \right) e^{i \left( \frac{3\pi}{2} + \frac{\pi k}{n} \right)}$$

$$= \left[ 2^{n-1} \prod_{k=1}^{n-1} \left( \sin \frac{\pi k}{n} \right) \right] e^{\sum_{k=1}^{n-1} \frac{3\pi}{2} + \frac{\pi k}{n}}$$

$$= 2^{n-1} \prod_{k=1}^{n-1} \left( \sin \frac{\pi k}{n} \right).$$