The Real And Complex Number Systems

Integers

1.1 Prove that there is no largest prime.

Proof: Suppose p is the largest prime. Then $p!+1$ is **NOT** a prime. So, there exists a prime q such that

$$q\,|p!+1 \Rightarrow q\,|1$$

which is impossible. So, there is no largest prime.

Remark: There are many and many proofs about it. The proof that we give comes from A**rchimedes 287-212 B. C.** In addition, **Euler Leonhard (1707-1783)** find another method to show it. The method is important since it develops to study the theory of numbers by analytic method. The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 91-93. (Chinese Version)**

1.2 If n is a positive integer, prove the algebraic identity

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$$

Proof: It suffices to show that

$$x^n - 1 = (x-1)\sum_{k=0}^{n-1} x^k.$$

Consider the right hand side, we have

$$\begin{aligned}(x-1)\sum_{k=0}^{n-1} x^k &= \sum_{k=0}^{n-1} x^{k+1} - \sum_{k=0}^{n-1} x^k \\ &= \sum_{k=1}^{n} x^k - \sum_{k=0}^{n-1} x^k \\ &= x^n - 1.\end{aligned}$$

1.3 If $2^n - 1$ is a prime, prove that n is prime. A prime of the form $2^p - 1$, where p is prime, is called a Mersenne prime.

Proof: If n is not a prime, then say $n = ab$, where $a > 1$ and $b > 1$. So, we have

$$2^{ab} - 1 = (2^a - 1)\sum_{k=0}^{b-1} (2^a)^k$$

which is not a prime by **Exercise 1.2**. So, n must be a prime.

Remark: The study of **Mersenne prime** is important; it is related with so called **Perfect number**. In addition, there are some **OPEN** problem about it. For example, **is there infinitely many Mersenne nembers?** The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 13-15. (Chinese Version)**

1.4 If $2^n + 1$ is a prime, prove that n is a power of 2. A prime of the form $2^{2^m} + 1$ is called a **Fermat prime.** Hint. Use exercise 1.2.

Proof: If n is a not a power of 2, say $n = ab$, where b is an odd integer. So,

$$2^a + 1 \,\big|\, 2^{ab} + 1$$

and $2^a + 1 < 2^{ab} + 1$. It implies that $2^n + 1$ is not a prime. So, n must be a power of 2.

Remark: (1) In the proof, we use the identity

$$x^{2n-1} + 1 = (x+1)\sum_{k=0}^{2n-2} (-1)^k x^k.$$

Proof: Consider

$$\begin{aligned}(x+1)\sum_{k=0}^{2n-2}(-1)^k x^k &= \sum_{k=0}^{2n-2}(-1)^k x^{k+1} + \sum_{k=0}^{2n-2}(-1)^k x^k \\ &= \sum_{k=1}^{2n-1}(-1)^{k+1} x^k + \sum_{k=0}^{2n-2}(-1)^k x^k \\ &= x^{2n+1}+1.\end{aligned}$$

(2) The study of **Fermat number** is important; for the details the reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 15. (Chinese Version)**

1.5 The Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, ...$ are defined by the recursion formula $x_{n+1} = x_n + x_{n-1}$, with $x_1 = x_2 = 1$. Prove that $(x_n, x_{n+1}) = 1$ and that $x_n = (a^n - b^n) / (a - b)$, where a and b are the roots of the quadratic equation $x^2 - x - 1 = 0$.

Proof: Let $d = g.c.d.\,(x_n, x_{n+1})$, then

$$d\,|x_n \text{ and } d\,|x_{n+1} = x_n + x_{n-1}\ .$$

So,

$$d\,|x_{n-1}\ .$$

Continue the process, we finally have

$$d\,|1\ .$$

So, $d = 1$ since d is positive.

Observe that

$$x_{n+1} = x_n + x_{n-1},$$

and thus we consider

$$x^{n+1} = x^n + x^{n-1},$$

i.e., consider

$$x^2 = x + 1 \text{ with two roots, } a \text{ and } b.$$

If we let

$$F_n = (a^n - b^n) / (a - b)\,,$$

then it is clear that

$$F_1 = 1,\ F_2 = 1,\ \text{and}\ F_{n+1} = F_n + F_{n-1}\ \text{for}\ n > 1.$$

So, $F_n = x_n$ for all n.

Remark: The study of the **Fibonacci numbers** is important; the reader can see the book, **Fibonacci and Lucas Numbers with Applications by Koshy and Thomas.**

1.6 Prove that every nonempty set of positive integers contains a smallest member. This is called the **well–ordering Principle.**

Proof: Given $(\phi \neq) S (\subseteq N)$, we prove that if S contains an integer k, then S contains the smallest member. We prove it by **Mathematical Induction of second form** as follows.

As $k = 1$, it trivially holds. Assume that as $k = 1, 2, ..., m$ holds, consider as $k = m + 1$ as follows. In order to show it, we consider two cases.

(1) If there is a member $s \in S$ such that $s < m + 1$, then by Induction hypothesis, we have proved it.

(2) If every $s \in S$, $s \geq m + 1$, then $m + 1$ is the smallest member.

Hence, by **Mathematical Induction**, we complete it.

Remark: We give a fundamental result to help the reader get more. We will prove the followings are equivalent:

(A. Well–ordering Principle) every nonempty set of positive integers contains a smallest member.

(B. Mathematical Induction of first form) Suppose that $S (\subseteq N)$, if S satisfies that

$$\begin{gathered}(1).\ 1 \text{ in } S \\ (2).\ \text{As } k \in S,\ \text{then } k + 1 \in S.\end{gathered}$$

Then $S = N$.

(C. Mathematical Induction of second form) Suppose that $S (\subseteq N)$, if S satisfies that

$$\begin{gathered}(1).\ 1 \text{ in } S \\ (2).\ \text{As } 1, ..., k \in S,\ \text{then } k + 1 \in S.\end{gathered}$$

Then $S = N$.

Proof: $(A \Rightarrow B)$: If $S \neq N$, then $N - S \neq \phi$. So, by (**A**), there exists the smallest integer w such that $w \in N - S$. Note that $w > 1$ by (1), so we consider $w - 1$ as follows.

Since $w - 1 \notin N - S$, we know that $w - 1 \in S$. By (2), we know that $w \in S$ which contadicts to $w \in N - S$. Hence, $S = N$.

$(B \Rightarrow C)$: It is obvious.

$(C \Rightarrow A)$: We have proved it by this exercise.

Rational and irrational numbers

1.7 Find the rational number whose decimal expansion is 0.3344444444....

Proof: Let $x = 0.3344444444...$, then

$$\begin{aligned} x &= \frac{3}{10} + \frac{3}{10^2} + \frac{4}{10^3} + ... + \frac{4}{10^n} + .., \text{ where } n \geq 3 \\ &= \frac{33}{10^2} + \frac{4}{10^3}\left(1 + \frac{1}{10} + ... + \frac{1}{10^n} + ..\right) \\ &= \frac{33}{10^2} + \frac{4}{10^3}\left(\frac{1}{1 - \frac{1}{10}}\right) \\ &= \frac{33}{10^2} + \frac{4}{900} \\ &= \frac{301}{900}. \end{aligned}$$

1.8 Prove that the decimal expansion of x will end in zeros (or in nines) if, and only if, x is a rational number whose denominator is of the form $2^n 5^m$, where m and n are nonnegative integers.

Proof: $(\Leftarrow)$Suppose that $x = \frac{k}{2^n 5^m}$, if $n \geq m$, we have

$$\frac{k 5^{n-m}}{2^n 5^n} = \frac{5^{n-m} k}{10^n}.$$

So, the decimal expansion of x will end in zeros. Similarly for $m \geq n$.

$(\Rightarrow)$Suppose that the decimal expansion of x will end in zeros (or in nines).

For case $x = a_0.a_1a_2 \cdots a_n$. Then

$$x = \frac{\sum_{k=0}^{n} 10^{n-k} a_k}{10^n} = \frac{\sum_{k=0}^{n} 10^{n-k} a_k}{2^n 5^n}.$$

For case $x = a_0.a_1a_2 \cdots a_n 999999 \cdots$. Then

$$\begin{aligned} x &= \frac{\sum_{k=0}^{n} 10^{n-k} a_k}{2^n 5^n} + \frac{9}{10^{n+1}} + ... + \frac{9}{10^{n+m}} + ... \\ &= \frac{\sum_{k=0}^{n} 10^{n-k} a_k}{2^n 5^n} + \frac{9}{10^{n+1}} \sum_{j=0}^{\infty} 10^{-j} \\ &= \frac{\sum_{k=0}^{n} 10^{n-k} a_k}{2^n 5^n} + \frac{1}{10^n} \\ &= \frac{1 + \sum_{k=0}^{n} 10^{n-k} a_k}{2^n 5^n}. \end{aligned}$$

So, in both case, we prove that x is a rational number whose denominator is of the form $2^n 5^m$, where m and n are nonnegative integers.

1.9 Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Proof: If $\sqrt{2} + \sqrt{3}$ is rational, then consider

$$\left(\sqrt{3} + \sqrt{2}\right)\left(\sqrt{3} - \sqrt{2}\right) = 1$$

which implies that $\sqrt{3} - \sqrt{2}$ is rational. Hence, $\sqrt{3}$ would be rational. It is impossible. So, $\sqrt{2} + \sqrt{3}$ is irrational.

Remark: (1)$\sqrt{p}$ is an irrational if p is a prime.

Proof: If $\sqrt{p} \in Q$, write $\sqrt{p} = \frac{a}{b}$, where $g.c.d.\,(a, b) = 1$. Then

$$b^2 p = a^2 \Rightarrow p \left| a^2 \right. \Rightarrow p \left| a \right. \qquad (*)$$

Write $a = pq$. So,

$$b^2 p = p^2 q^2 \Rightarrow b^2 = pq^2 \Rightarrow p \left| b^2 \right. \Rightarrow p \left| b \right. . \qquad (*')$$

By (*) and (*'), we get

$$p \left| g.c.d.\,(a, b) = 1 \right.$$

which implies that $p = 1$, a contradiction. So, $\sqrt{p}$ is an irrational if p is a prime.

Note: There are many and many methods to prove it. For example, the reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 19-21. (Chinese Version)**

(2) Suppose $a, b \in N$. Prove that $\sqrt{a}+\sqrt{b}$ is rational if and only if, $a = k^2$ and $b = h^2$ for some $h, k \in N$.

Proof: ($\Leftarrow$) It is clear.

($\Rightarrow$) Consider

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a^2 - b^2,$$

then $\sqrt{a} \in Q$ and $\sqrt{b} \in Q$. Then it is clear that $a = h^2$ and $b = h^2$ for some $h, k \in N$.

1.10 If a, b, c, d are rational and if x is irrational, prove that $(ax+b)/(cx+d)$ is usually irrational. When do exceptions occur?

Proof: We claim that $(ax+b)/(cx+d)$ is rational if and only if $ad = bc$.

($\Rightarrow$)If $(ax+b)/(cx+d)$ is rational, say $(ax+b)/(cx+d) = q/p$. We consider two cases as follows.

(i) If $q = 0$, then $ax + b = 0$. If $a \neq 0$, then x would be rational. So, $a = 0$ and $b = 0$. Hence, we have

$$ad = 0 = bc.$$

(ii) If $q \neq 0$, then $(pa - qc)x + (pb - qd) = 0$. If $pa - qc \neq 0$, then x would be rational. So, $pa - qc = 0$ and $pb - qd = 0$. It implies that

$$qcb = qad \Rightarrow ad = bc.$$

($\Leftarrow$)Suppose $ad = bc$. If $a = 0$, then $b = 0$ or $c = 0$. So,

$$\frac{ax+b}{cx+d} = \begin{cases} 0 \text{ if } a = 0 \text{ and } b = 0 \\ \frac{b}{d} \text{ if } a = 0 \text{ and } c = 0 \end{cases}.$$

If $a \neq 0$, then $d = bc/a$. So,

$$\frac{ax+b}{cx+d} = \frac{ax+b}{cx+bc/a} = \frac{a(ax+b)}{c(ax+b)} = \frac{a}{c}.$$

Hence, we proved that if $ad = bc$, then $(ax+b)/(cx+d)$ is rational.

1.11 Given any real $x > 0$, prove that there is an irrational number between 0 and x.

Proof: If $x \in Q^c$, we choose $y = x/2 \in Q^c$. Then $0 < y < x$. If $x \in Q$, we choose $y = x/\sqrt{2} \in Q$, then $0 < y < x$.

Remark: (1) There are many and many proofs about it. We may prove it by the concept of **Perfect set**. The reader can see the book, **Principles of Mathematical Analysis written by Walter Rudin, Theorem 2.43, pp 41.** Also see the textbook, **Exercise 3.25**.

(2) Given a and $b \in R$ with $a < b$, there exists $r \in Q^c$, and $q \in Q$ such that $a < r < b$ and $a < q < b$.

Proof: We show it by considering four cases. (i) $a \in Q$, $b \in Q$. (ii) $a \in Q$, $b \in Q^c$. (iii) $a \in Q^c$, $b \in Q$. (iv) $a \in Q^c$, $b \in Q^c$.

(i) ($a \in Q$, $b \in Q$) Choose $q = \frac{a+b}{2}$ and $r = \frac{1}{\sqrt{2}}a + \left(1 - \frac{1}{\sqrt{2}}\right) b$.

(ii) ($a \in Q$, $b \in Q^c$) Choose $r = \frac{a+b}{2}$ and let $c = \frac{1}{2^n} < b - a$, then $a + c := q$.

(iii) ($a \in Q^c$, $b \in Q$) Similarly for (iii).

(iv) ($a \in Q^c$, $b \in Q^c$) It suffices to show that there exists a rational number $q \in (a, b)$ by (ii). Write

$$b = b_0.b_1b_2 \cdots b_n \cdots$$

Choose n large enough so that

$$a < q = b_0.b_1b_2 \cdots b_n < b.$$

(It works since $b - q = 0.000..000b_{n+1}... \leq \frac{1}{10^n}$)

1.12 If $a/b < c/d$ with $b > 0$, $d > 0$, prove that $(a + c) / (b + d)$ lies bwtween the two fractions a/b and c/d

Proof: It only needs to conisder the substraction. So, we omit it.

Remark: The result of this exercise is often used, so we suggest the reader keep it in mind.

1.13 Let a and b be positive integers. Prove that $\sqrt{2}$ always lies between the two fractions a/b and $(a + 2b) / (a + b)$. Which fraction is closer to $\sqrt{2}$?

Proof: Suppose $a/b \leq \sqrt{2}$, then $a \leq \sqrt{2}b$. So,

$$\frac{a + 2b}{a + b} - \sqrt{2} = \frac{\left(\sqrt{2} - 1\right)\left(\sqrt{2}b - a\right)}{a + b} \geq 0.$$

In addition,

$$\begin{aligned}
\left(\sqrt{2}-\frac{a}{b}\right)-\left(\frac{a+2b}{a+b}-\sqrt{2}\right) &= 2\sqrt{2}-\left(\frac{a}{b}+\frac{a+2b}{a+b}\right) \\
&= 2\sqrt{2}-\frac{a^2+2ab+2b^2}{ab+b^2} \\
&= \frac{1}{ab+b^2}\left[\left(2\sqrt{2}-2\right)ab+\left(2\sqrt{2}-2\right)b^2-a^2\right] \\
&\geq \frac{1}{ab+b^2}\left[\left(2\sqrt{2}-2\right)a\frac{a}{\sqrt{2}}+\left(2\sqrt{2}-2\right)\left(\frac{a}{\sqrt{2}}\right)^2-a^2\right] \\
&= 0.
\end{aligned}$$

So, $\frac{a+2b}{a+b}$ is closer to $\sqrt{2}$.

Similarly, we also have if $a/b > \sqrt{2}$, then $\frac{a+2b}{a+b} < \sqrt{2}$. Also, $\frac{a+2b}{a+b}$ is closer to $\sqrt{2}$ in this case.

Remark: Note that

$$\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b} < \frac{2b}{a} \text{ by Exercise 12 and 13.}$$

And we know that $\frac{a+2b}{a+b}$ is closer to $\sqrt{2}$. We can use it to approximate $\sqrt{2}$. Similarly for the case

$$\frac{2b}{a} < \frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}.$$

1.14 Prove that $\sqrt{n-1}+\sqrt{n+1}$ is irrational for every integer $n \geq 1$.

Proof: Suppose that $\sqrt{n-1}+\sqrt{n+1}$ is rational, and thus consider

$$\left(\sqrt{n+1}+\sqrt{n-1}\right)\left(\sqrt{n+1}-\sqrt{n-1}\right) = 2$$

which implies that $\sqrt{n+1}-\sqrt{n-1}$ is rational. Hence, $\sqrt{n+1}$ and $\sqrt{n-1}$ are rational. So, $n-1 = k^2$ and $n+1 = h^2$, where k and h are positive integer. It implies that

$$h = \frac{3}{2} \text{ and } k = \frac{1}{2}$$

which is absurb. So, $\sqrt{n-1}+\sqrt{n+1}$ is irrational for every integer $n \geq 1$.

1.15 Given a real x and an integer $N > 1$, prove that there exist integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$. Hint. Consider the $N+1$ numbers $tx - [tx]$ for $t = 0, 1, 2, ..., N$ and show that some pair differs by at most $1/N$.

Proof: Given $N > 1$, and thus consider $tx - [tx]$ for $t = 0, 1, 2, ..., N$ as follows. Since

$$0 \leq tx - [tx] := a_t < 1,$$

so there exists two numbers a_i and a_j where $i \neq j$ such that

$$|a_i - a_j| < \frac{1}{N} \Rightarrow |(i-j)\,x - p| < \frac{1}{N},\ \text{where } p = [jx] - [ix]\,.$$

Hence, there exist integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$.

1.16 If x is irrational prove that there are infinitely many rational numbers h/k with $k > 0$ such that $|x - h/k| < 1/k^2$. Hint. Assume there are only a finite number $h_1/k_1, ..., h_r/k_r$ and obtain a contradiction by applying Exercise 1.15 with $N > 1/\delta$, where δ is the smallest of the numbers $|x - h_i/k_i|$.

Proof: Assume there are only a finite number $h_1/k_1, ..., h_r/k_r$ and let $\delta = \min_{i=1}^{r} |x - h_i/k_i| > 0$ since x is irrational. Choose $N > 1/\delta$, then by **Exercise 1.15**, we have

$$\frac{1}{N} < \delta \leq \left| x - \frac{h}{k} \right| < \frac{1}{kN}$$

which implies that

$$\frac{1}{N} < \frac{1}{kN}$$

which is impossible. So, there are infinitely many rational numbers h/k with $k > 0$ such that $|x - h/k| < 1/k^2$.

Remark: (1) There is another proof by **continued fractions.** The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 270. (Chinese Version)**

(2) The exercise is useful to help us show the following lemma. $\{ar + b : a \in Z, b \in Z\}$, where $r \in Q^c$ is dense in R. It is equivalent to $\{ar : a \in Z\}$, where $r \in Q^c$ is dense in $[0, 1]$ modulus 1.

Proof: Say $\{ar+b : a \in Z, b \in Z\} = S$, and since $r \in Q^c$, then by **Exercise 1.16**, there are infinitely many rational numbers h/k with $k > 0$ such that $|kr - h| < \frac{1}{k}$. Consider $(x-\delta, x+\delta) := I$, where $\delta > 0$, and thus choosing k_0 large enough so that $1/k_0 < \delta$. Define $L = |k_0 r - h_0|$, then we have $sL \in I$ for some $s \in Z$. So, $sL = (\pm)\left[(sk_0)r - (sh_0)\right] \in S$. That is, we have proved that S is dense in R.

1.17 Let x be a positive rational number of the form

$$x = \sum_{k=1}^{n} \frac{a_k}{k!},$$

where each a_k is nonnegative integer with $a_k \leq k-1$ for $k \geq 2$ and $a_n > 0$. Let $[x]$ denote the largest integer in x. Prove that $a_1 = [x]$, that $a_k = [k!x] - k\left[(k-1)!x\right]$ for $k = 2, ..., n$, and that n is the smallest integer such that $n!x$ is an integer. Conversely, show that every positive rational number x can be expressed in this form in one and only one way.

Proof: ($\Rightarrow$)First,

$$\begin{aligned}
[x] &= \left[a_1 + \sum_{k=2}^{n} \frac{a_k}{k!}\right] \\
&= a_1 + \left[\sum_{k=2}^{n} \frac{a_k}{k!}\right] \text{ since } a_1 \in N \\
&= a_1 \text{ since } \sum_{k=2}^{n} \frac{a_k}{k!} \leq \sum_{k=2}^{n} \frac{k-1}{k!} = \sum_{k=2}^{n} \frac{1}{(k-1)!} - \frac{1}{k!} = 1 - \frac{1}{n!} < 1.
\end{aligned}$$

Second, fixed k and consider

$$k!x = k! \sum_{j=1}^{n} \frac{a_j}{j!} = k! \sum_{j=1}^{k-1} \frac{a_j}{j!} + a_k + k! \sum_{j=k+1}^{n} \frac{a_j}{j!}$$

and

$$(k-1)!x = (k-1)! \sum_{j=1}^{n} \frac{a_j}{j!} = (k-1)! \sum_{j=1}^{k-1} \frac{a_j}{j!} + (k-1)! \sum_{j=k}^{n} \frac{a_j}{j!}.$$

So,

$$
\begin{aligned}
[k!x] &= \left[k!\sum_{j=1}^{k-1}\frac{a_j}{j!} + a_k + k!\sum_{j=k+1}^{n}\frac{a_j}{j!}\right] \\
&= k!\sum_{j=1}^{k-1}\frac{a_j}{j!} + a_k \text{ since } k!\sum_{j=k+1}^{n}\frac{a_j}{j!} < 1
\end{aligned}
$$

and

$$
\begin{aligned}
k\left[(k-1)!x\right] &= k\left[(k-1)!\sum_{j=1}^{k-1}\frac{a_j}{j!} + (k-1)!\sum_{j=k}^{n}\frac{a_j}{j!}.\right] \\
&= k\,(k-1)!\sum_{j=1}^{k-1}\frac{a_j}{j!} \text{ since } \ (k-1)!\sum_{j=k}^{n}\frac{a_j}{j!} < 1 \\
&= k!\sum_{j=1}^{k-1}\frac{a_j}{j!}
\end{aligned}
$$

which implies that

$$
a_k = [k!x] - k\left[(k-1)!x\right] \text{ for } k = 2, ..., n.
$$

Last, in order to show that n is the smallest integer such that $n!x$ is an integer. It is clear that

$$
n!x = n!\sum_{k=1}^{n}\frac{a_k}{k!} \in Z.
$$

In addition,

$$
\begin{aligned}
(n-1)!x &= (n-1)!\sum_{k=1}^{n}\frac{a_k}{k!} \\
&= (n-1)!\sum_{k=1}^{n-1}\frac{a_k}{k!} + \frac{a_n}{n} \\
&\notin Z \text{ since } \frac{a_n}{n} \notin Z.
\end{aligned}
$$

So, we have proved it.

($\Leftarrow$)It is clear since every a_n is uniquely deermined.

Upper bounds

1.18 Show that the sup and the inf of a set are uniquely determined whenever they exists.

Proof: Given a nonempty set $S\,(\subseteq R)$, and assume $\sup S = a$ and $\sup S = b$, we show $a = b$ as follows. Suppose that $a > b$, and thus choose $\varepsilon = \frac{a-b}{2}$, then there exists a $x \in S$ such that

$$b < \frac{a+b}{2} = a - \varepsilon < x < a$$

which implies that

$$b < x$$

which contradicts to $b = \sup S$. Similarly for $a < b$. Hence, $a = b$.

1.19 Find the sup and inf of each of the following sets of real numbers:

(a) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$, where p, q, and r take on all positive integer values.

Proof: Define $S = \{2^{-p} + 3^{-q} + 5^{-r} : p, q, r \in N\}$. Then it is clear that $\sup S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$, and $\inf S = 0$.

(b) $S = \{x : 3x^2 - 10x + 3 < 0\}$

Proof: Since $3x^2 - 10x + 3 = (x-3)(3x-1)$, we know that $S = \left(\frac{1}{3}, 3\right)$. Hence, $\sup S = 3$ and $\inf S = \frac{1}{3}$.

(c) $S = \{x : (x-a)(x-b)(x-c)(x-d) < 0\}$, where $a < b < c < d$.

Proof: It is clear that $S = (a, b) \cup (c, d)$. Hence, $\sup S = d$ and $\inf S = a$.

1.20 Prove the comparison property for suprema (**Theorem 1.16**)

Proof: Since $s \le t$ for every $s \in S$ and $t \in T$, fixed $t_0 \in T$, then $s \le t_0$ for all $s \in S$. Hence, by **Axiom 10**, we know that $\sup S$ exists. In addition, it is clear $\sup S \le \sup T$.

Remark: There is a useful result, we write it as a reference. Let S and T be two nonempty subsets of R. If $S \subseteq T$ and $\sup T$ exists, then $\sup S$ exists and $\sup S \le \sup T$.

Proof: Since $\sup T$ exists and $S \subseteq T$, we know that for every $s \in S$, we have

$$s \leq \sup T.$$

Hence, by **Axiom 10**, we have proved the existence of $\sup S$. In addition, $\sup S \leq \sup T$ is trivial.

1.21 Let A and B be two sets of positive numbers bounded above, and let $a = \sup A$, $b = \sup B$. Let C be the set of all products of the form xy, where $x \in A$ and $y \in B$. Prove that $ab = \sup C$.

Proof: Given $\varepsilon > 0$, we want to find an element $c \in C$ such that $ab - \varepsilon < c$. If we can show this, we have proved that $\sup C$ exists and equals ab.

Since $\sup A = a > 0$ and $\sup B = b > 0$, we can choose n large enough such that $a - \varepsilon/n > 0$, $b - \varepsilon/n > 0$, and $n > a + b$. So, for this $\varepsilon' = \varepsilon/n$, there exists $a' \in A$ and $b' \in B$ such that

$$a - \varepsilon' < a' \text{ and } b - \varepsilon' < b'$$

which implies that

$$ab - \varepsilon' (a + b - \varepsilon') < a'b' \text{ since } a - \varepsilon' > 0 \text{ and } b - \varepsilon' > 0$$

which implies that

$$ab - \frac{\varepsilon}{n} (a + b) < a'b' := c$$

which implies that

$$ab - \varepsilon < c.$$

1.22 Given $x > 0$, and an integer $k \geq 2$. Let a_0 denote the largest integer $\leq x$ and, assumeing that $a_0, a_1, ..., a_{n-1}$ have been defined, let a_n denote the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} \leq x.$$

Note: When $k = 10$ the integers $a_0, a_1, ...$ are the digits in a decimal representation of x. For general k they provide a representation in the scale of k.

(a) Prove that $0 \leq a_i \leq k - 1$ for each $i = 1, 2, ...$

Proof: Choose $a_0 = [x]$, and thus consider

$$[kx - ka_0] := a_1$$

then

$$0 \leq k(x - a_0) < k \Rightarrow 0 \leq a_1 \leq k - 1$$

and

$$a_0 + \frac{a_1}{k} \leq x \leq a_0 + \frac{a_1}{k} + \frac{1}{k}.$$

Continue the process, we then have

$$0 \leq a_i \leq k - 1 \text{ for each } i = 1, 2, ...$$

and

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} \leq x < a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} + \frac{1}{k^n}. \qquad (*)$$

(b) Let $r_n = a_0 + a_1 k^{-1} + a_2 k^{-2} + ... + a_n k^{-n}$ and show that x is the sup of the set of rational numbers $r_1, r_2, ...$

Proof: It is clear by (a)-(*).

Inequality

1.23 Prove **Lagrange's identity** for real numbers:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2.$$

Note that this identity implies that **Cauchy-Schwarz inequality**.

Proof: Consider

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \sum_{1 \leq k, j \leq n} a_k^2 b_j^2 = \sum_{k=j} a_k^2 b_j^2 + \sum_{k \neq j} a_k^2 b_j^2 = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k \neq j} a_k^2 b_j^2$$

and

$$\left(\sum_{k=1}^{n} a_k b_k\right)\left(\sum_{k=1}^{n} a_k b_k\right) = \sum_{1\le k,j\le n} a_k b_k a_j b_j = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k\neq j} a_k b_k a_j b_j$$

So,

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k b_k\right)^2 &= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) + \sum_{k\neq j} a_k b_k a_j b_j - \sum_{k\neq j} a_k^2 b_j^2 \\ &= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) + 2\sum_{1\le k<j\le n} a_k b_k a_j b_j - \sum_{1\le k<j\le n} a_k^2 b_j^2 + a_j^2 b_k^2 \\ &= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\le k<j\le n} \left(a_k b_j - a_j b_k\right)^2 .\end{aligned}$$

Remark: (1) The reader may recall the relation with **Cross Product** and **Inner Product**, we then have a fancy formula:

$$\|x \times y\|^2 + |< x, y >|^2 = \|x\|^2 \|y\|^2 ,$$

where $x, y \in R^3$.

(2) We often write

$$< a, b > := \sum_{k=1}^{n} a_k b_k,$$

and the Cauchy-Schwarz inequality becomes

$$|< x, y >| \le \|x\| \|y\| \text{ by } \textbf{Remark} \text{ (1).}$$

1.24 Prove that for arbitrary real a_k, b_k, c_k we have

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \le \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^2\right)^2\left(\sum_{k=1}^{n} c_k^4\right).$$

Proof: Use **Cauchy-Schwarz inequality** twice, we then have

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 &= \left[\left(\sum_{k=1}^{n} a_k b_k c_k\right)^2\right]^2 \\ &\le \left(\sum_{k=1}^{n} a_k^2 c_k^2\right)^2 \left(\sum_{k=1}^{n} b_k^2\right)^2 \\ &\le \left(\sum_{k=1}^{n} a_k^4\right)^2 \left(\sum_{k=1}^{n} c_k^4\right) \left(\sum_{k=1}^{n} b_k^2\right)^2 \\ &= \left(\sum_{k=1}^{n} a_k^4\right) \left(\sum_{k=1}^{n} b_k^2\right)^2 \left(\sum_{k=1}^{n} c_k^4\right).\end{aligned}$$

1.25 Prove that **Minkowski's inequality**:

$$\left(\sum_{k=1}^{n} (a_k + b_k)^2\right)^{1/2} \le \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}.$$

This is the triangle inequality $\|a+b\| \le \|a\| + \|b\|$ for $n-$dimensional vectors, where $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and

$$\|a\| = \left(\sum_{k=1}^{n} a_k^2\right)^{1/2}.$$

Proof: Consider

$$\begin{aligned}\sum_{k=1}^{n} (a_k + b_k)^2 &= \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\sum_{k=1}^{n} a_k b_k \\ &\le \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2} \quad \text{by Cauchy-Schwarz inequality} \\ &= \left[\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}\right]^2.\end{aligned}$$

So,

$$\left(\sum_{k=1}^{n}(a_k+b_k)^2\right)^{1/2} \le \left(\sum_{k=1}^{n}a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n}b_k^2\right)^{1/2}.$$

1.26 If $a_1 \ge ... \ge a_n$ and $b_1 \ge ... \ge b_n$, prove that

$$\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right) \le n\left(\sum_{k=1}^{n}a_kb_k\right).$$

Hint. $\sum_{1\le j\le k\le n}(a_k-a_j)(b_k-b_j) \ge 0$.

Proof: Consider

$$0 \le \sum_{1\le j\le k\le n}(a_k-a_j)(b_k-b_j) = \sum_{1\le j\le k\le n}a_kb_k + a_jb_j - \sum_{1\le j\le k\le n}a_kb_j + a_jb_k$$

which implies that

$$\sum_{1\le j\le k\le n}a_kb_j + a_jb_k \le \sum_{1\le j\le k\le n}a_kb_k + a_jb_j. \qquad (*)$$

Since

$$\begin{aligned}\sum_{1\le j\le k\le n}a_kb_j + a_jb_k &= \sum_{1\le j<k\le n}a_kb_j + a_jb_k + 2\sum_{k=1}^{n}a_kb_k \\ &= \left(\sum_{1\le j<k\le n}a_kb_j + a_jb_k + \sum_{k=1}^{n}a_kb_k\right) + \sum_{k=1}^{n}a_kb_k \\ &= \left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right) + \sum_{k=1}^{n}a_kb_k,\end{aligned}$$

we then have, by (*)

$$\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right) + \sum_{k=1}^{n}a_kb_k \le \sum_{1\le j\le k\le n}a_kb_k + a_jb_j. \qquad (**)$$

In addition,

$$\sum_{1\leq j\leq k\leq n} a_k b_k + a_j b_j$$
$$= \sum_{k=1}^{n} a_k b_k + n a_1 b_1 + \sum_{k=2}^{n} a_k b_k + (n-1)\, a_2 b_2 + ... + \sum_{k=n-1}^{n} a_k b_k + 2a_{n-1} b_{n-1} + \sum_{k=n} a_k b_k$$
$$= n \sum_{k=1}^{n} a_k b_k + a_1 b_1 + a_2 b_2 + ... + a_n b_n$$
$$= (n+1) \sum_{k=1}^{n} a_k b_k$$

which implies that, by (**),

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leq n \left(\sum_{k=1}^{n} a_k b_k\right).$$

Complex numbers

1.27 Express the following complex numbers in the form $a+bi$.

(a) $(1+i)^3$

Solution: $(1+i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i$.

(b) $(2+3i)\,/\,(3-4i)$

Solution: $\frac{2+3i}{3-4i} = \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)} = \frac{-6+17i}{25} = \frac{-6}{25} + \frac{17}{25}i$.

(c) $i^5 + i^{16}$

Solution: $i^5 + i^{16} = i + 1$.

(d) $\frac{1}{2}(1+i)(1+i^{-8})$

Solution: $\frac{1}{2}(1+i)(1+i^{-8}) = 1 + i$.

1.28 In each case, determine all real x and y which satisfy the given relation.

(a) $x+iy=|x-iy|$

Proof: Since $|x-iy|\geq 0$, we have

$$x\geq 0 \text{ and } y=0.$$

(b) $x+iy=(x-iy)^2$

Proof: Since $(x-iy)^2=x^2-(2xy)\,i-y^2$, we have

$$x=x^2-y^2 \text{ and } y=-2xy.$$

We consider tow cases: (i) $y=0$ and (ii) $y\neq 0$.

(i) As $y=0: x=0$ or 1.

(ii) As $y\neq 0: x=-1/2$, and $y=\pm\frac{\sqrt{3}}{2}$.

(c) $\sum_{k=0}^{100} i^k=x+iy$

Proof: Since $\sum_{k=0}^{100} i^k=\frac{1-i^{101}}{1-i}=\frac{1-i}{1-i}=1$, we have $x=1$ and $y=0$.

1.29 If $z=x+iy$, x and y real, the complex conjugate of z is the complex number $\bar{z}=x-iy$. Prove that:

(a) Conjugate of $(z_1+z_2)=\bar{z}_1+\bar{z}_2$

Proof: Write $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, then

$$\begin{aligned}\overline{z_1+z_2}&=\overline{(x_1+x_2)+i\,(y_1+y_2)}\\&=(x_1+x_2)-i\,(y_1+y_2)\\&=(x_1-iy_1)+(x_2-iy_2)\\&=\bar{z}_1+\bar{z}_2.\end{aligned}$$

(b) $\overline{z_1z_2}=\bar{z}_1\bar{z}_2$

Proof: Write $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, then

$$\begin{aligned}\overline{z_1z_2}&=\overline{(x_1x_2-y_1y_2)+i\,(x_1y_2+x_2y_1)}\\&=(x_1x_2-y_1y_2)-i\,(x_1y_2+x_2y_1)\end{aligned}$$

and

$$\begin{aligned}\bar{z}_1\bar{z}_2&=(x_1-iy_1)\,(x_2-iy_2)\\&=(x_1x_2-y_1y_2)-i\,(x_1y_2+x_2y_1)\,.\end{aligned}$$

So, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(c) $z\bar{z} = |z|^2$

Proof: Write $z = x + iy$ and thus

$$z\bar{z} = x^2 + y^2 = |z|^2 .$$

(d) $z + \bar{z}$ =twice the real part of z

Proof: Write $z = x + iy$, then

$$z + \bar{z} = 2x,$$

twice the real part of z.

(e) $(z - \bar{z})/i$ =twice the imaginary part of z

Proof: Write $z = x + iy$, then

$$\frac{z - \bar{z}}{i} = 2y,$$

twice the imaginary part of z.

1.30 Describe geometrically the set of complex numbers z which satisfies each of the following conditions:

(a) $|z| = 1$

Solution: The unit circle centered at zero.

(b) $|z| < 1$

Solution: The open unit disk centered at zero.

(c) $|z| \leq 1$

Solution: The closed unit disk centered at zero.

(d) $z + \bar{z} = 1$

Solution: Write $z = x + iy$, then $z + \bar{z} = 1$ means that $x = 1/2$. So, the set is the line $x = 1/2$.

(e) $z - \bar{z} = i$

Proof: Write $z = x + iy$, then $z - \bar{z} = i$ means that $y = 1/2$. So, the set is the line $y = 1/2$.

(f) $z + \bar{z} = |z|^2$

Proof: Write $z = x + iy$, then $2x = x^2 + y^2 \Leftrightarrow (x-1)^2 + y^2 = 1$. So, the set is the unit circle centered at $(1, 0)$.

1.31 Given three complex numbers z_1, z_2, z_3 such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$. Show that these numbers are vertices of an equilateral triangle inscribed in the unit circle with center at the origin.

Proof: It is clear that three numbers are vertices of triangle inscribed in the unit circle with center at the origin. It remains to show that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$. In addition, it suffices to show that

$$|z_1 - z_2| = |z_2 - z_3|.$$

Note that

$$|2z_1 + z_3| = |2z_3 + z_1| \text{ by } z_1 + z_2 + z_3 = 0$$

which is equivalent to

$$|2z_1 + z_3|^2 = |2z_3 + z_1|^2$$

which is equivalent to

$$(2z_1 + z_3)(2\bar{z}_1 + \bar{z}_3) = (2z_3 + z_1)(2\bar{z}_3 + \bar{z}_1)$$

which is equivalent to

$$|z_1| = |z_3|.$$

1.32 If a and b are complex numbers, prove that:

(a) $|a - b|^2 \leq \left(1 + |a|^2\right)\left(1 + |b|^2\right)$

Proof: Consider

$$\begin{aligned}\left(1 + |a|^2\right)\left(1 + |b|^2\right) - |a - b|^2 &= (1 + \bar{a}a)\left(1 + \bar{b}b\right) - (a - b)\left(\bar{a} - \bar{b}\right) \\ &= (1 + \bar{a}b)\left(1 + a\bar{b}\right) \\ &= |1 + \bar{a}b|^2 \geq 0,\end{aligned}$$

so, $|a-b|^2 \leq \left(1+|a|^2\right)\left(1+|b|^2\right)$

(b) If $a \neq 0$, then $|a+b| = |a| + |b|$ if, and only if, b/a is real and nonnegative.

Proof: $(\Rightarrow)$Since $|a+b| = |a| + |b|$, we have

$$|a+b|^2 = (|a| + |b|)^2$$

which implies that

$$\operatorname{Re}(\bar{a}b) = |a||b| = |\bar{a}||b|$$

which implies that

$$\bar{a}b = |\bar{a}||b|$$

which implies that

$$\frac{b}{a} = \frac{\bar{a}b}{\bar{a}a} = \frac{|\bar{a}||b|}{|a|^2} \geq 0.$$

$(\Leftarrow)$ Suppose that

$$\frac{b}{a} = k, \text{ where } k \geq 0.$$

Then

$$|a+b| = |a+ka| = (1+k)|a| = |a| + k|a| = |a| + |b|.$$

1.33 If a and b are complex numbers, prove that

$$|a-b| = |1-\bar{a}b|$$

if, and only if, $|a| = 1$ or $|b| = 1$. For which a and b is the inequality $|a-b| < |1-\bar{a}b|$ valid?

Proof: $(\Leftrightarrow)$ Since

$$\begin{aligned}
|a-b| &= |1-\bar{a}b| \\
&\Leftrightarrow \left(\bar{a}-\bar{b}\right)(a-b) = (1-\bar{a}b)\left(1-a\bar{b}\right) \\
&\Leftrightarrow |a|^2 + |b|^2 = 1 + |a|^2|b|^2 \\
&\Leftrightarrow \left(|a|^2-1\right)\left(|b|^2-1\right) = 0 \\
&\Leftrightarrow |a|^2 = 1 \text{ or } |b|^2 = 1.
\end{aligned}$$

By the preceding, it is easy to know that

$$|a-b| < |1-\bar{a}b| \Leftrightarrow 0 < \left(|a|^2 - 1\right)\left(|b|^2 - 1\right).$$

So, $|a-b| < |1-\bar{a}b|$ if, and only if, $|a| > 1$ and $|b| > 1$. (Or $|a| < 1$ and $|b| < 1$).

1.34 If a and c are real constant, b complex, show that the equation

$$az\bar{z} + b\bar{z} + \bar{b}z + c = 0 \ (a \neq 0, z = x + iy)$$

represents a circle in the $x - y$ plane.

Proof: Consider

$$z\bar{z} - \frac{b}{-a}\bar{z} - \frac{\bar{b}}{-a}z + \frac{b}{-a}\left[\overline{\left(\frac{b}{-a}\right)}\right] = \frac{-ac + |b|^2}{a^2},$$

so, we have

$$\left|z - \left(\frac{b}{-a}\right)\right|^2 = \frac{-ac + |b|^2}{a^2}.$$

Hence, as $|b|^2 - ac > 0$, it is a circle. As $\frac{-ac+|b|^2}{a^2} = 0$, it is a point. As $\frac{-ac+|b|^2}{a^2} < 0$, it is not a circle.

Remark: The idea is easy from the fact

$$|z - q| = r.$$

We square both sides and thus

$$z\bar{z} - q\bar{z} - \bar{q}z + \bar{q}q = r^2.$$

1.35 Recall the definition of the inverse tangent: given a real number t, $\tan^{-1}(t)$ is the unique real number θ which satisfies the two conditions

$$-\frac{\pi}{2} < \theta < +\frac{\pi}{2}, \ \tan\theta = t.$$

If $z = x + iy$, show that

(a) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$, if $x > 0$

Proof: Note that in this text book, we say $\arg(z)$ is the principal argument of z, denoted by $\theta = \arg z$, where $-\pi < \theta \leq \pi$.

So, as $x > 0$, $\arg z = \tan^{-1}\left(\frac{y}{x}\right)$.

(b) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$, if $x < 0$, $y \geq 0$

Proof: As $x < 0$, and $y \geq 0$. The point (x, y) is lying on $S = \{(x, y) : x < 0,\ y \geq 0\}$. Note that $-\pi < \arg z \leq \pi$, so we have $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$.

(c) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) - \pi$, if $x < 0$, $y < 0$

Proof: Similarly for (b). So, we omit it.

(d) $\arg(z) = \frac{\pi}{2}$ if $x = 0$, $y > 0$; $\arg(z) = -\frac{\pi}{2}$ if $x = 0$, $y < 0$.

Proof: It is obvious.

1.36 Define the folowing "**pseudo-ordering**" of the complex numbers: we say $z_1 < z_2$ if we have either

(i) $|z_1| < |z_2|$ or (ii) $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$.

Which of Axioms 6,7,8,9 are satisfied by this relation?

Proof: (1) For axiom 6, we prove that it holds as follows. Given $z_1 = r_1 e^{i\arg(z_1)}$, and $r_2 e^{i\arg(z_2)}$, then if $z_1 = z_2$, there is nothing to prove it. If $z_1 \neq z_2$, there are two possibilities: (a) $r_1 \neq r_2$, or (b) $r_1 = r_2$ and $\arg(z_1) \neq \arg(z_2)$. So, it is clear that axiom 6 holds.

(2) For axiom 7, we prove that it does not hold as follows. Given $z_1 = 1$ and $z_2 = -1$, then it is clear that $z_1 < z_2$ since $|z_1| = |z_2| = 1$ and $\arg(z_1) = 0 < \arg(z_2) = \pi$. However, let $z_3 = -i$, we have

$$z_1 + z_3 = 1 - i > z_2 + z_3 = -1 - i$$

since

$$|z_1 + z_3| = |z_2 + z_3| = \sqrt{2}$$

and

$$\arg(z_1 + z_3) = -\frac{\pi}{4} > -\frac{3\pi}{4} = \arg(z_2 + z_3).$$

(3) For axiom 8, we prove that it holds as follows. If $z_1 > 0$ and $z_2 > 0$, then $|z_1| > 0$ and $|z_2| > 0$. Hence, $z_1 z_2 > 0$ by $|z_1 z_2| = |z_1| |z_2| > 0$.

(4) For axiom 9, we prove that it holds as follows. If $z_1 > z_2$ and $z_2 > z_3$, we consider the following cases. Since $z_1 > z_2$, we may have (a) $|z_1| > |z_2|$ or (b) $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$.

As $|z_1| > |z_2|$, it is clear that $|z_1| > |z_3|$. So, $z_1 > z_3$.

As $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$, we have $\arg(z_1) > \arg(z_3)$. So, $z_1 > z_3$.

1.37 Which of Axioms 6,7,8,9 are satisfied if the **pseudo-ordering** is defined as follows? We say $(x_1, y_1) < (x_2, y_2)$ if we have either (i) $x_1 < x_2$ or (ii) $x_1 = x_2$ and $y_1 < y_2$.

Proof: (1) For axiom 6, we prove that it holds as follows. Given $x = (x_1, y_1)$ and $y = (x_2, y_2)$. If $x = y$, there is nothing to prove it. We consider $x \neq y$: As $x \neq y$, we have $x_1 \neq x_2$ or $y_1 \neq y_2$. Both cases imply $x < y$ or $y < x$.

(2) For axiom 7, we prove that it holds as follows. Given $x = (x_1, y_1)$, $y = (x_2, y_2)$ and $z = (z_1, z_3)$. If $x < y$, then there are two possibilities: (a) $x_1 < x_2$ or (b) $x_1 = x_2$ and $y_1 < y_2$.

For case (a), it is clear that $x_1 + z_1 < y_1 + z_1$. So, $x + z < y + z$.

For case (b), it is clear that $x_1 + z_1 = y_1 + z_1$ and $x_2 + z_2 < y_2 + z_2$. So, $x + z < y + z$.

(3) For axiom 8, we prove that it does not hold as follows. Consider $x = (1, 0)$ and $y = (0, 1)$, then it is clear that $x > 0$ and $y > 0$. However, $xy = (0, 0) = 0$.

(4) For axiom 9, we prove that it holds as follows. Given $x = (x_1, y_1)$, $y = (x_2, y_2)$ and $z = (z_1, z_3)$. If $x > y$ and $y > z$, then we consider the following cases. (a) $x_1 > y_1$, or (b) $x_1 = y_1$.

For case (a), it is clear that $x_1 > z_1$. So, $x > z$.

For case (b), it is clear that $x_2 > y_2$. So, $x > z$.

1.38 State and prove a theorem analogous to Theorem 1.48, expressing $\arg(z_1/z_2)$ in terms of $\arg(z_1)$ and $\arg(z_2)$.

Proof: Write $z_1 = r_1 e^{i\arg(z_1)}$ and $z_2 = r_2 e^{i\arg(z_2)}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i[\arg(z_1) - \arg(z_2)]}.$$

Hence,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2),$$

where

$$n(z_1, z_2) = \begin{cases} 0 \text{ if } -\pi < \arg(z_1) - \arg(z_2) \leq \pi \\ 1 \text{ if } -2\pi < \arg(z_1) - \arg(z_2) \leq -\pi \\ -1 \text{ if } \pi < \arg(z_1) - \arg(z_2) < 2\pi \end{cases}.$$

1.39 State and prove a theorem analogous to Theorem 1.54, expressing $Log\,(z_1/z_2)$ in terms of $Log\,(z_1)$ and $Log\,(z_2)$.

Proof: Write $z_1 = r_1 e^{i\arg(z_1)}$ and $z_2 = r_2 e^{i\arg(z_2)}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i[\arg(z_1) - \arg(z_2)]}.$$

Hence,

$$\begin{aligned} Log\,(z_1/z_2) &= \log\left|\frac{z_1}{z_2}\right| + i\arg\left(\frac{z_1}{z_2}\right) \\ &= \log|z_1| - \log|z_2| + i\left[\arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2)\right] \text{ by xercise 1.38} \\ &= Log\,(z_1) - Log\,(z_2) + i2\pi n(z_1, z_2). \end{aligned}$$

1.40 Prove that the nth roots of 1 (also called the nth roots of unity) are given by $\alpha, \alpha^2, ..., \alpha^n$, where $\alpha = e^{2\pi i/n}$, and show that the roots $\neq 1$ satisfy the equation

$$1 + x + x^2 + ... + x^{n-1} = 0.$$

Proof: By **Theorem 1.51**, we know that the roots of 1 are given by $\alpha, \alpha^2, ..., \alpha^n$, where $\alpha = e^{2\pi i/n}$. In addition, since

$$x^n = 1 \Rightarrow (x-1)\left(1 + x + x^2 + ... + x^{n-1}\right) = 0$$

which implies that

$$1 + x + x^2 + ... + x^{n-1} = 0 \text{ if } x \neq 1.$$

So, all roots except 1 satisfy the equation

$$1 + x + x^2 + ... + x^{n-1} = 0.$$

1.41 (a) Prove that $|z^i| < e^\pi$ for all complex $z \neq 0$.

Proof: Since

$$z^i = e^{iLog(z)} = e^{-\arg(z) + i\log|z|},$$

we have

$$\left|z^i\right| = e^{-\arg(z)} < e^{\pi}$$

by $-\pi < \arg(z) \leq \pi$.

(b) Prove that there is no constant $M > 0$ such that $|\cos z| < M$ for all complex z.

Proof: Write $z = x + iy$ and thus,

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

which implies that

$$|\cos x \cosh y| \leq |\cos z|.$$

Let $x = 0$ and y be real, then

$$\frac{e^y}{2} \leq \frac{1}{2}\left|e^y + e^{-y}\right| \leq |\cos z|.$$

So, there is no constant $M > 0$ such that $|\cos z| < M$ for all complex z.

Remark: There is an important theorem related with this exercise. We state it as a reference. (**Liouville's Theorem**) A bounded entire function is constant. The reader can see the book, **Complex Analysis by Joseph Bak, and Donald J. Newman, pp 62-63. Liouville's Theorem** can be used to prove the much important theorem, **Fundamental Theorem of Algebra.**

1.42 If $w = u + iv$ (u, v real), show that

$$z^w = e^{u\log|z| - v\arg(z)} e^{i[v\log|z| + u\arg(z)]}.$$

Proof: Write $z^w = e^{wLog(z)}$, and thus

$$\begin{aligned} wLog(z) &= (u + iv)(\log|z| + i\arg(z)) \\ &= [u\log|z| - v\arg(z)] + i[v\log|z| + u\arg(z)]. \end{aligned}$$

So,

$$z^w = e^{u\log|z| - v\arg(z)} e^{i[v\log|z| + u\arg(z)]}.$$

1.43 (a) Prove that $Log\left(z^{w}\right) = wLog\ z + 2\pi in$.

Proof: Write $w = u + iv$, where u and v are real. Then

$$\begin{aligned} Log\left(z^{w}\right) &= \log\left|z^{w}\right| + i\arg\left(z^{w}\right) \\ &= \log\left[e^{u\log|z| - v\arg(z)}\right] + i\left[v\log|z| + u\arg(z)\right] + 2\pi in \text{ by Exercise1.42} \\ &= u\log|z| - v\arg(z) + i\left[v\log|z| + u\arg(z)\right] + 2\pi in. \end{aligned}$$

On the other hand,

$$\begin{aligned} wLogz + 2\pi in &= (u + iv)\left(\log|z| + i\arg(z)\right) + 2\pi in \\ &= u\log|z| - v\arg(z) + i\left[v\log|z| + u\arg(z)\right] + 2\pi in. \end{aligned}$$

Hence, $Log\left(z^{w}\right) = wLog\ z + 2\pi in$.

Remark: There is another proof by considering

$$e^{Log\left(z^{w}\right)} = z^{w} = e^{wLog(z)}$$

which implies that

$$Log\left(z^{w}\right) = wLogz + 2\pi in$$

for some $n \in Z$.

(b) Prove that $\left(z^{w}\right)^{\alpha} = z^{w\alpha}e^{2\pi in\alpha}$, where n is an integer.

Proof: By (a), we have

$$\left(z^{w}\right)^{\alpha} = e^{\alpha Log\left(z^{w}\right)} = e^{\alpha(wLogz + 2\pi in)} = e^{\alpha wLogz}e^{2\pi in\alpha} = z^{\alpha w}e^{2\pi in\alpha},$$

where n is an integer.

1.44 (i) If θ and a are real numbers, $-\pi < \theta \leq \pi$, prove that

$$(\cos\theta + i\sin\theta)^{a} = \cos(a\theta) + i\sin(a\theta).$$

Proof: Write $\cos\theta + i\sin\theta = z$, we then have

$$\begin{aligned} (\cos\theta + i\sin\theta)^{a} &= z^{a} = e^{aLogz} = e^{a\left[\log\left|e^{i\theta}\right| + i\arg\left(e^{i\theta}\right)\right]} = e^{ia\theta} \\ &= \cos(a\theta) + i\sin(a\theta). \end{aligned}$$

Remark: Compare with the **Exercise 1.43-(b)**.

(ii) Show that, in general, the restriction $-\pi < \theta \leq \pi$ is necessary in (i) by taking $\theta = -\pi$, $a = \frac{1}{2}$.

Proof: As $\theta = -\pi$, and $a = \frac{1}{2}$, we have

$$(-1)^{\frac{1}{2}} = e^{\frac{1}{2}Log(-1)} = e^{\frac{\pi}{2}i} = i \neq -i = \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right).$$

(iii) If a is an integer, show that the formula in (i) holds without any restriction on θ. In this case it is known as **DeMorvre's theorem**.

Proof: By **Exercise 1.43**, as a is an integer we have

$$(z^w)^a = z^{wa},$$

where $z^w = e^{i\theta}$. Then

$$\left(e^{i\theta}\right)^a = e^{i\theta a} = \cos(a\theta) + i\sin(a\theta).$$

1.45 Use **DeMorvre's theorem** (Exercise 1.44) to derive the trigonometric identities

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

valid for real θ. Are these valid when θ is complex?

Proof: By **Exercise 1.44-(iii)**, we have for any real θ,

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta).$$

By **Binomial Theorem**, we have

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

and

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

For complex θ, we show that it holds as follows. Note that $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, we have

$$\begin{aligned}
3\cos^2 z \sin z - \sin^3 z &= 3\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2\left(\frac{e^{iz}-e^{-iz}}{2i}\right) - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^3 \\
&= 3\left(\frac{e^{2zi}+e^{-2zi}+2}{4}\right)\left(\frac{e^{iz}-e^{-iz}}{2i}\right) + \frac{e^{3zi}-3e^{iz}+3e^{-iz}-e^{-3zi}}{8i} \\
&= \frac{1}{8i}\left[3\left(e^{2zi}+e^{-2zi}+2\right)\left(e^{zi}-e^{-zi}\right)+\left(e^{3zi}-3e^{iz}+3e^{-iz}-e^{-3zi}\right)\right] \\
&= \frac{1}{8i}\left[\left(3e^{3zi}+3e^{iz}-3e^{-iz}-3e^{-3zi}\right)+\left(e^{3zi}-3e^{iz}+3e^{-iz}-e^{-3zi}\right)\right] \\
&= \frac{4}{8i}\left(e^{3zi}-e^{-3zi}\right) \\
&= \frac{1}{2i}\left(e^{3zi}-e^{-3zi}\right) \\
&= \sin 3z.
\end{aligned}$$

Similarly, we also have

$$\cos^3 z - 3\cos z \sin^2 z = \cos 3z.$$

1.46 Define $\tan z = \sin z / \cos z$ and show that for $z = x + iy$, we have

$$\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

Proof: Since

$$\tan z = \frac{\sin z}{\cos z} = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{\sin x\cosh y + i\cos x\sinh y}{\cos x\cosh y - i\sin x\sinh y}$$
$$= \frac{(\sin x\cosh y + i\cos x\sinh y)(\cos x\cosh y + i\sin x\sinh y)}{(\cos x\cosh y - i\sin x\sinh y)(\cos x\cosh y + i\sin x\sinh y)}$$
$$= \frac{\left(\sin x\cos x\cosh^2 y - \sin x\cos x\sinh^2 y\right) + i\left(\sin^2 x\cosh y\sinh y + \cos^2 x\cosh y\sinh y\right)}{(\cos x\cosh y)^2 - (i\sin x\sinh y)^2}$$
$$= \frac{\sin x\cos x\left(\cosh^2 y - \sinh^2 y\right) + i(\cosh y\sinh y)}{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} \text{ since } \sin^2 x + \cos^2 x = 1$$
$$= \frac{(\sin x\cos x) + i(\cosh y\sinh y)}{\cos^2 x + \sinh^2 y} \text{ since } \cosh^2 y = 1 + \sinh^2 y$$
$$= \frac{\frac{1}{2}\sin 2x + \frac{i}{2}\sinh 2y}{\cos^2 x + \sinh^2 y} \text{ since } 2\cosh y\sinh y = \sinh 2y \text{ and } 2\sin x\cos x = \sin 2x$$
$$= \frac{\sin 2x + i\sinh 2y}{2\cos^2 x + 2\sinh^2 y}$$
$$= \frac{\sin 2x + i\sinh 2y}{2\cos^2 x - 1 + 2\sinh^2 y + 1}$$
$$= \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} \text{ since } \cos 2x = 2\cos^2 x - 1 \text{ and } 2\sinh^2 y + 1 = \cosh 2y.$$

1.47 Let w be a given complex number. If $w \neq \pm 1$, show that there exists two values of $z = x + iy$ satisfying the conditions $\cos z = w$ and $-\pi < x \leq \pi$. Find these values when $w = i$ and when $w = 2$.

Proof: Since $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, if we let $e^{iz} = u$, then $\cos z = w$ implies that

$$w = \frac{u^2+1}{2u} \Rightarrow u^2 - 2wu + 1 = 0$$

which implies that

$$(u-w)^2 = w^2 - 1 \neq 0 \text{ since } w \neq \pm 1.$$

So, by **Theorem 1.51**,

$$e^{iz} = u = w + \left|w^2-1\right|^{1/2} e^{i\phi_k}, \text{ where } \phi_k = \frac{\arg(w^2-1)}{2} + \frac{2\pi k}{2}, \; k = 0, 1.$$
$$= w \pm \left|w^2-1\right|^{1/2} e^{i\left(\frac{\arg\left(w^2-1\right)}{2}\right)}$$

So,

$$ix-y = i(x+iy) = iz = \log\left|w \pm \left|w^2-1\right|^{1/2} e^{i\frac{\arg\left(w^2-1\right)}{2}}\right| + i\arg\left(w \pm \left|w^2-1\right|^{1/2} e^{i\left(\frac{\arg\left(w^2-1\right)}{2}\right)}\right)$$

Hence, there exists two values of $z = x+iy$ satisfying the conditions $\cos z = w$ and

$$-\pi < x = \arg\left(w \pm \left|w^2-1\right|^{1/2} e^{i\left(\frac{\arg\left(w^2-1\right)}{2}\right)}\right) \leq \pi.$$

For $w = i$, we have

$$iz = \log\left|\left(1 \pm \sqrt{2}\right) i\right| + i\arg\left(\left(1 \pm \sqrt{2}\right) i\right)$$

which implies that

$$z = \arg\left(\left(1 \pm \sqrt{2}\right) i\right) - i\log\left|\left(1 \pm \sqrt{2}\right) i\right|.$$

For $w = 2$, we have

$$iz = \log\left|2 \pm \sqrt{3}\right| + i\arg\left(2 \pm \sqrt{3}\right)$$

which implies that

$$z = \arg\left(2 \pm \sqrt{3}\right) - i\log\left|2 \pm \sqrt{3}\right|.$$

1.48 Prove Lagrange's identity for complex numbers:

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 = \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \sum_{1\leq k<j\leq n} \left(a_k \bar{b}_j - \bar{a}_j b_k\right)^2.$$

Use this to deduce a Cauchy-Schwarz ineqality for complex numbers.

Proof: It is the same as the **Exercise 1.23**; we omit the details.

1.49 (a) By eqating imaginary parts in DeMoivre's formula prove that

$$\sin n\theta = \sin^n\theta \left\{\binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - +...\right\}$$

Proof: By **Exercise 1.44 (i)**, we have

$$\begin{aligned}
\sin n\theta &= \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} \sin^{2k-1}\theta \cos^{n-(2k-1)}\theta \\
&= \sin^n\theta \left\{ \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} \cot^{n-(2k-1)}\theta \right\} \\
&= \sin^n\theta \left\{ \binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - +... \right\}.
\end{aligned}$$

(b) If $0 < \theta < \pi/2$, prove that

$$\sin(2m+1)\theta = \sin^{2m+1}\theta P_m\left(\cot^2\theta\right)$$

where P_m is the polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - +...$$

Use this to show that P_m has zeros at the m distinct points $x_k = \cot^2\{\pi k/(2m+1)\}$ for $k = 1, 2, ..., m$.

Proof: By (a),

$$\begin{aligned}
&\sin(2m+1)\theta \\
&= \sin^{2m+1}\theta \left\{ \binom{2m+1}{1}\left(\cot^2\theta\right)^m - \binom{2m+1}{3}\left(\cot^2\theta\right)^{m-1} + \binom{2m+1}{5}\left(\cot^2\theta\right)^{m-2} - +... \right\} \\
&= \sin^{2m+1}\theta P_m\left(\cot^2\theta\right), \text{ where } P_m(x) = \sum_{k=1}^{m+1} \binom{2m+1}{2k-1} x^{m+1-k}. \qquad (*)
\end{aligned}$$

In addition, by (*), $\sin(2m+1)\theta = 0$ if, and only if, $P_m(\cot^2\theta) = 0$. Hence, P_m has zeros at the m distinct points $x_k = \cot^2\{\pi k/(2m+1)\}$ for $k = 1, 2, ..., m$.

(c) Show that the sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} \cot^2\frac{\pi k}{2m+1} = \frac{m(2m-1)}{3},$$

and the sum of their squares is given by

$$\sum_{k=1}^{m} \cot^4 \frac{\pi k}{2m+1} = \frac{m(2m-1)(4m^2+10m-9)}{45}.$$

Note. There identities can be used to prove that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ and $\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$. (See Exercises 8.46 and 8.47.)

Proof: By (b), we know that sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} x_k = \sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m+1} = -\left(\frac{-\binom{2m+1}{3}}{\binom{2m+1}{1}}\right) = \frac{m(2m-1)}{3}.$$

And the sum of their squares is given by

$$\begin{aligned}\sum_{k=1}^{m} x_k^2 &= \sum_{k=1}^{m} \cot^4 \frac{\pi k}{2m+1} \\ &= \left(\sum_{k=1}^{m} x_k\right)^2 - 2\left(\sum_{1\leq i<j\leq n} x_i x_j\right) \\ &= \left(\frac{m(2m-1)}{3}\right)^2 - 2\left(\frac{\binom{2m+1}{5}}{\binom{2m+1}{1}}\right) \\ &= \frac{m(2m-1)(4m^2+10m-9)}{45}.\end{aligned}$$

1.50 Prove that $z^n - 1 = \prod_{k=1}^{n} \left(z - e^{2\pi i k/n}\right)$ for all complex z. Use this to derive the formula

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

Proof: Since $z^n = 1$ has exactly n distinct roots $e^{2\pi i k/n}$, where $k = 0, ..., n-1$ by **Theorem 1.51.** Hence, $z^n - 1 = \prod_{k=1}^{n} \left(z - e^{2\pi i k/n}\right)$. It implies that

$$z^{n-1} + ... + 1 = \prod_{k=1}^{n-1} \left(z - e^{2\pi i k/n}\right).$$

So, let $z = 1$, we obtain that

$$\begin{aligned}
n &= \prod_{k=1}^{n-1} \left(1 - e^{2\pi ik/n}\right) = \prod_{k=1}^{n-1} \left[\left(1 - \cos\frac{2\pi k}{n}\right) - i\left(\sin\frac{2\pi k}{n}\right)\right] \\
&= \prod_{k=1}^{n-1} \left(2\sin^2\frac{\pi k}{n}\right) - i\left(2\sin\frac{\pi k}{n}\cos\frac{\pi k}{n}\right) \\
&= \prod_{k=1}^{n-1} 2\left(\sin\frac{\pi k}{n}\right)\left(\sin\frac{\pi k}{n} - i\cos\frac{\pi k}{n}\right) \\
&= 2^{n-1}\prod_{k=1}^{n-1}\left(\sin\frac{\pi k}{n}\right)\left(\cos\left(\frac{3\pi}{2} + \frac{\pi k}{n}\right) + i\sin\left(\frac{3\pi}{2} + \frac{\pi k}{n}\right)\right) \\
&= 2^{n-1}\prod_{k=1}^{n-1}\left(\sin\frac{\pi k}{n}\right) e^{i\left(\frac{3\pi}{2} + \frac{\pi k}{n}\right)} \\
&= \left[2^{n-1}\prod_{k=1}^{n-1}\left(\sin\frac{\pi k}{n}\right)\right] e^{\sum_{k=1}^{n-1}\frac{3\pi}{2} + \frac{\pi k}{n}} \\
&= 2^{n-1}\prod_{k=1}^{n-1}\left(\sin\frac{\pi k}{n}\right).
\end{aligned}$$