Something around the number $e$

1. Show that the sequence $\{(1 + \frac{1}{n})^n\}$ converges, and denote the limit by $e$.

**Proof:** Since

\[
(1 + \frac{1}{n})^n = \sum_{k=0}^{n} \left(\frac{n}{k}\right) \left(\frac{1}{n}\right)^k
\]

\[
= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \left(\frac{1}{n}\right)^2 + \ldots + \frac{n(n-1) \cdots 1}{n!} \left(\frac{1}{n}\right)^n
\]

\[
= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right)
\]

\[
\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2(n-1)} + \ldots
\]

\[
= 3,
\]

and by (1), we know that the sequence is increasing. Hence, the sequence is convergent. We denote its limit $e$. That is,

\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.
\]

**Remark:** 1. The sequence and $e$ first appear in the mail that Euler wrote to Goldbach. It is a beautiful formula involving

\[
e^{i\pi} + 1 = 0.
\]

2. Use the exercise, we can show that $\sum_{k=0}^{\infty} \frac{1}{k!} = e$ as follows.

**Proof:** Let $x_n = (1 + \frac{1}{n})^n$, and let $k > n$, we have

\[
1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{k}\right) + \ldots + \frac{1}{n!} \left(1 - \frac{1}{k}\right) \cdots \left(1 - \frac{n-1}{k}\right) \leq x_k
\]

which implies that (let $k \to \infty$)

\[
y_n := \sum_{i=0}^{n} \frac{1}{i!} \leq e.
\]

On the other hand,

\[
x_n \leq y_n
\]

So, by (2) and (3), we finally have

\[
\sum_{k=0}^{\infty} \frac{1}{k!} = e.
\]

3. $e$ is an irrational number.

**Proof:** Assume that $e$ is a rational number, say $e = p/q$, where g.c.d. $(p, q) = 1$. Note that $q > 1$. Consider

\[
(q!)e = (q!) \left(\sum_{k=0}^{\infty} \frac{1}{k!}\right)
\]

\[
= (q!) \left(\sum_{k=0}^{q} \frac{1}{k!}\right) + (q!) \left(\sum_{k=q+1}^{\infty} \frac{1}{k!}\right),
\]

and since $(q!) \left(\sum_{k=0}^{q} \frac{1}{k!}\right)$ and $(q!)e$ are integers, we have $(q!) \left(\sum_{k=q+1}^{\infty} \frac{1}{k!}\right)$ is also an integer. However,
\[(q!) \left( \sum_{k=q+1}^{\infty} \frac{1}{k!} \right) = \sum_{k=q+1}^{\infty} \frac{q!}{k!} \]
\[= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots \]
\[< \frac{1}{q+1} + \left( \frac{1}{q+1} \right)^2 + \ldots \]
\[= \frac{1}{q} \]
\[< 1, \]
a contradiction. So, we know that \(e\) is not a rational number.

4. Here is an estimate about \(e = \sum_{k=0}^{\infty} \frac{1}{k!} + \frac{\theta}{n(n!)}\), where \(0 < \theta < 1\). (In fact, we know that \(e = 2.71828 18284 59045 \ldots\)).

**Proof:** Since \(e = \sum_{k=0}^{\infty} \frac{1}{k!}\), we have
\[0 < e - x_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}\]
where \(x_n = \sum_{k=0}^{n} \frac{1}{k!}\)
\[\leq \frac{1}{(n+1)!} \left( 1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \ldots \right) \]
\[\leq \frac{1}{(n+1)!} \left( 1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \ldots \right) \]
\[\leq \frac{1}{(n+1)!} \cdot \frac{n+2}{n+1} \]
\[\leq \frac{1}{n(n!)} \text{ since } \frac{n+2}{(n+1)^2} < \frac{1}{n}. \]
So, we finally have
\[e = \sum_{k=0}^{n} \frac{1}{k!} + \frac{\theta}{n(n!)}\], where \(0 < \theta < 1\).

**Note:** We can use the estimate directly to show \(e\) is an irrational number.

2. For continuous variables, we have the same result as follows. That is,
\[\lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = e. \]

**Proof:** (1) Since \((1 + \frac{1}{n})^n \to e \text{ as } n \to \infty\), we know that for any sequence \(\{a_n\} \subseteq N\), with \(a_n \to \infty\), we have
\[\lim_{a \to \infty} \left( 1 + \frac{1}{a_n} \right)^{a_n} = e. \]
(2) Given a sequence \(\{x_n\}\) with \(x_n \to +\infty\), and define \(a_n = \lfloor x_n \rfloor\), then \(a_n \leq x_n < a_n + 1\), then we have
\[\left( 1 + \frac{1}{a_n + 1} \right)^{a_n} \leq \left( 1 + \frac{1}{x_n} \right)^{x_n} \leq \left( 1 + \frac{1}{a_n} \right)^{a_n+1}. \]
Since
\[\left( 1 + \frac{1}{a_n + 1} \right)^{a_n} \to e \text{ and } \left( 1 + \frac{1}{a_n} \right)^{a_n+1} \to e \text{ as } x \to +\infty \text{ by (5)} \]
we know that
\[
\lim_{n \to \infty} \left( 1 + \frac{1}{x_n} \right)^{x_n} = e.
\]

Since \( \{x_n\} \) is arbitrary chosen so that it goes infinity, we finally obtain that
\[
\lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^{x} = e.
\]

(3) In order to show \( (1 + \frac{1}{x})^{x} \to e \) as \( x \to -\infty \), we let \( x = -y \), then
\[
(1 + \frac{1}{x})^{x} = \left( 1 + \frac{1}{-y} \right)^{-y}
= \left( \frac{y}{y - 1} \right)^{y}
= \left( 1 + \frac{1}{y - 1} \right)^{y-1} \left( 1 + \frac{1}{y - 1} \right).
\]

Note that \( x \to -\infty (\Leftrightarrow y \to +\infty) \), by (6), we have shown that
\[
e = \lim_{y \to \infty} \left( 1 + \frac{1}{y - 1} \right)^{y-1} \left( 1 + \frac{1}{y - 1} \right)
= \lim_{x \to -\infty} \left( 1 + \frac{1}{x} \right)^{x}.
\]

3. Prove that as \( x > 0 \), we have \((1 + \frac{1}{x})^{x}\) is strictly increasing, and \((1 + \frac{1}{x})^{x+1}\) is strictly decreasing.

**Proof:** Since, by **Mean Value Theorem**
\[
\frac{1}{x + 1} < \log \left( 1 + \frac{1}{x} \right) = \log(x + 1) - \log(x) = \frac{1}{x} < \frac{1}{x}
\]
for all \( x > 0 \), we have
\[
\left[ x \log \left( 1 + \frac{1}{x} \right) \right]' = \log \left( 1 + \frac{1}{x} \right) - \frac{1}{x + 1} > 0
\]
for all \( x > 0 \)
and
\[
\left[ (x + 1) \log \left( 1 + \frac{1}{x} \right) \right]' = \log \left( 1 + \frac{1}{x} \right) - \frac{1}{x} < 0
\]
for all \( x > 0 \).

Hence, we know that
\[
x \log \left( 1 + \frac{1}{x} \right) \text{ is strictly increasing on } (0, \infty)
\]
and
\[
(x + 1) \log \left( 1 + \frac{1}{x} \right) \text{ is strictly decreasing on } (0, \infty).
\]

It implies that
\[
(1 + \frac{1}{x})^{x} \text{ is strictly increasing } (0, \infty), \text{ and } (1 + \frac{1}{x})^{x+1} \text{ is strictly decreasing on } (0, \infty).
\]

**Remark:** By exercise 2, we know that
\[
\lim_{x \to +\infty} \left( 1 + \frac{1}{x} \right)^{x} = e = \lim_{x \to +\infty} \left( 1 + \frac{1}{x} \right)^{x+1}.
\]

4. Follow the Exercise 3 to find the smallest \( a \) such that \((1 + \frac{1}{x})^{x+a} > e \) and strictly decreasing for all \( x \in (0, \infty) \).

**Proof:** Let \( f(x) = (1 + \frac{1}{x})^{x+a} \), and consider
\[
\log f(x) = (x + a) \log \left( 1 + \frac{1}{x} \right) := g(x),
\]
Let us consider
$g'(x) = \log\left(1 + \frac{1}{x}\right) - \frac{x + a}{x^2 + x}$

$= -\log(1 - y) + [-y + (1 - a)y^2] \frac{1}{1 - y}$, where $0 < y = \frac{1}{1 + x} < 1$

$= \sum_{k=1}^{\infty} \frac{y^k}{k} + [-y + (1 - a)y^2] \sum_{k=0}^{\infty} y^k$

$= \left(\frac{1}{2} - a\right)y^2 + \left(\frac{1}{3} - a\right)y^3 + \ldots + \left(\frac{1}{n} - a\right)y^n + \ldots$

It is clear that for $a \geq 1/2$, we have $g'(x) < 0$ for all $x \in (0, \infty)$. Note that for $a < 1/2$, if there exists such $a$ so that $f$ is strictly decreasing for all $x \in (0, \infty)$. Then $g'(x) \leq 0$ for all $x \in (0, \infty)$. However, it is impossible since

$g'(x) = \left(\frac{1}{2} - a\right)y^2 + \left(\frac{1}{3} - a\right)y^3 + \ldots + \left(\frac{1}{n} - a\right)y^n + \ldots$

$\to \frac{1}{2} - a > 0$ as $y \to 1^-$.

So, we have proved that the smallest value of $a$ is $1/2$.

**Remark:** There is another proof to show that $(1 + \frac{1}{x})^{x+1/2}$ is strictly decreasing on $(0, \infty)$.

**Proof:** Consider $h(t) = 1/t$, and two points $(1, 1)$ and $(1 + \frac{1}{x}, 1 + \frac{1}{x})$ lying on the graph

From three areas, the idea is that

The area of lower rectangle < The area of the curve < The area of trapezoid

So, we have

$\frac{1}{1+x} = \frac{1}{x}\left(\frac{1}{1 + \frac{1}{x}}\right) < \log\left(1 + \frac{1}{x}\right) < \frac{1}{2x}\left(1 + \frac{1}{1 + \frac{1}{x}}\right) = \left(x + \frac{1}{2}\right)\left(\frac{1}{x(x + 1)}\right)$.  

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Consider

$\left[(1 + \frac{1}{x})^{x+1/2}\right]' = \left[(1 + \frac{1}{x})^{x+1/2}\right]\left[\log\left(1 + \frac{1}{x}\right) - \left(x + \frac{1}{2}\right)\left(\frac{1}{x(x + 1)}\right)\right]$

$< 0$ by (7);

hence, we know that $(1 + \frac{1}{x})^{x+1/2}$ is strictly decreasing on $(0, \infty)$.

**Note:** Use the method of remark, we know that $(1 + \frac{1}{x})^x$ is strictly increasing on $(0, \infty)$. 