Exercises of Quotient Space

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This note is for some exercises related to quotient space from the book *Linear Algebra, 4th edition* by Stephen H. Friedberg, Arnold J. Insel and Lawrence E. Spence.
Given a linear transformation $T : V \to W$, we want to "rectify" T to be an isomorphsim so that we can identify V with W, given their bases. To make T onto, it's easy to restrict the co-domain to be $R(T)$, since elements not in $R(T)$ has nothing to do with the linear transformation. Let the kernel of T be Z. If T is not one to one, $Z \neq \{0\}$. To make T one-to-one, we regard the set Z as a single element. We want to define a new linear function $\overline{T}$ with $\overline{T}(Z) = 0$. The domain of $\overline{T}$ is actually the quotient space $V/Z = \{v + Z : \ v \in V\}$.

1 Problem 31 in Sec 1.3

Let W be a vector subspace of a vector space V over F. For any $v \in V$ the set $\{v\} + W = \{v + w : w \in W\}$ is called the coset of W containing v. It is customary to denote the coset by $v + W$ rather than $\{v\} + W$.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$.
Sol. We have

$$v + W \text{ is a subspace of } V \iff 0 \in v + W \iff (-v) \in W \iff v \in W \qquad \blacksquare$$

(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.
Sol. We have

$$v_1 + W = v_2 + W \iff v_1 = v_1 + 0 = v_2 + w \text{ for some } w \in W \iff v_1 - v_2 \in W \qquad \blacksquare$$

(c) Show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$$

and for all $a \in F$,

$$a(v_1 + W) = a(v_1') + W$$

Sol. We have $v_1 - v_1' \in W$ and $v_2 - v_2' \in W$. Hence

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ &= ((v_1' + v_2') + w) + W \qquad \text{for some } w \in W \\ &= (v_1' + v_2') + W \end{aligned} \qquad \blacksquare$$

The scalar part is trivial.

(d) Prove that the set S is a vector space with the operations defined in (c). This vector space is called the quotient space of V modulo W and is denoted by V/W.
Sol. The closedness of scalar and addition is ensured by (c). The identity is (W). The inverse of any vector $(v+W)$ is $(-v+W)$. $\blacksquare$

2 Exercise 40 in Sec 2.1

Let V be a vector space and W be a subspace of V. Define the mapping $\eta : V \to V/W$ by $\eta(v) = v + W$ for $v \in V$.

(a) Prove that η is a linear transformation from V onto v/W and that $N(\eta) = W$.
Sol. We have $\eta a v_1 + v_2 = a(v_1 + W) + (v_2 + W)$ and for each $v \in V$, $v + W = \eta(v)$. Also, it's trivial that $\eta(v) = v + W = W \iff v \in W$. $\blacksquare$

(b) Suppose that V is finite dimensional. Use (a) and the dimension theorem to derive a formula relating $\dim(V)$, $\dim(W)$ and $\dim(V/W)$. *Sol.* Using the theorem directly, we have that $\dim(V) = \dim(R(\eta))+ \dim(N(\eta)) = \dim(V/W) + \dim(W)$. $\blacksquare$

3 Exercise 24 in Sec 2.4

Let $T : V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping

$$\overline{T} = V/N(T) \to Z \text{ by } \overline{T}(v + N(T)) = T(v)$$

for any coset $v + N(T)$ in $V/N(T)$.

(a) Prove that $\overline{T}$ is well-defined; that is, prove that if $v + N(T) = v' + N(T)$, then $T(v) = T(v')$.
Sol. $T(v) - T(v') = T(v - v') = 0$, since $v - v' \in N(T)$ by the assumption. $\blacksquare$

(b) Prove that $\overline{T}$ is linear.
Sol. trivial.

(c) Prove that $\overline{T}$ is an isomorphism.
Sol. It suffices to show that $\overline{T}$ is injective and onto. For 1-1, we have that $\overline{T}(v + N(T)) = 0 \iff T(v) = 0 \iff v \in N(T) \iff (v + N(T)) = N(T)$. For onto, we have that for $z \in Z$, say $T(v) = z$ since T is onto. Then we have that $\overline{T}(v + N(T)) = z$. $\blacksquare$

(d) Prove that $T = \overline{T} \circ \eta$.
Sol. Trivially take the definition of these functions.

Remark. Note that $\overline{T}$ is called the canonical map in the context of abstract algebra.