Review of Calculus 3

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14 Partial Derivatives

14.3 Partial Derivatives

Definition The partial derivative of f with respect to x at (a, b), denoted by $f_x(a, b)$, is

$$\boxed{f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} = \frac{\partial f}{\partial y} = D_x f}.$$

Similarly, the partial derivative of f with respect to x at (a, b), denoted by $f_y(a, b)$, is

$$\boxed{f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h} = \frac{\partial f}{\partial y} = D_x f}.$$

When computing partial derivative of a function with respect to a variable, we regard other variables as constant (因為在定義當中只有偏微分變數有改變）, and the problem reduces to differentiation of single variable.

Theorem(Claurаut's Theorem) Suppose f is deined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$. (假設二次偏微連續，則改變偏微分順序不影響結果。)

Exercise (Textbook) If $g(x, y, z) = \sqrt{1 + xz} + \sqrt{1 - xy}$, find g_{xyz}.

14.4 Tangent Planes and Linear Approximation

Recall that the linear approximation (tangent line) of a differentiable function of single variable is

$$y = y_0 + f'(x_0)(x - x_0).$$

Similarly, suppose f **has continuous partial derivatives**. The linear approximation (tangent plane approx.) to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$\begin{aligned} z &= z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= z_0 + \nabla f \cdot \langle x - x_0, y - y_0\rangle \qquad \text{(Vector Representation)} \end{aligned}$$

The value $\nabla f \cdot \langle x - x_0, y - y_0\rangle = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ is called the differential.

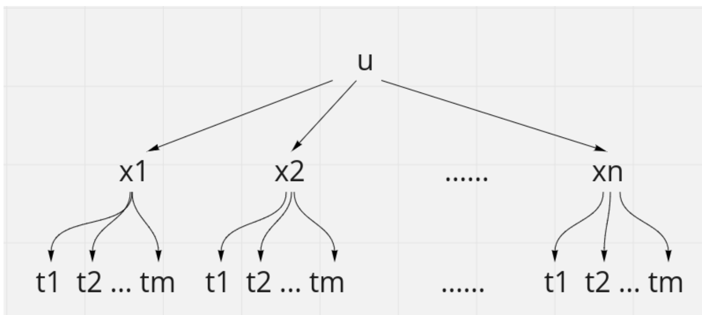
Theorem If the partial derivatives f_x and f_y exists and are continuous at (a, b), then f is differentiable at (a, b). Note that the converse is not true!

14.5 Chain Rules

Theorem (General Version of The Chain Rule) Suppose that u is a differentaible function of the n variables $x_1, \dots, x_n$ and each x_j is a differentiable function of the m variables $t_1, \dots, t_m$. Then u is a function of $t_1, \dots, t_m$ and for each $i = 1, \dots, m$, we have

$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}}.$$

It's helpful to draw the tree diagram as the following figure. To compute the partial derivative with respect to a variable t_i, we just compute the product of partial derivative of each path from u to t_i, and sum them up.



Exercise (106微乙) Assume $f(x, y)$ is a function of two variable and let $g(t) = f(1 + 2t, 3 + 4t)$. Use the first and the second order partial derivatives at $(1, 3)$ to represent $g'(0)$ and $g''(0)$.

14.6 Directional Derivatives; Gradient Vectors

Definition The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$\begin{aligned} D_u f(x_0, y_0) &= \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h} \qquad \text{(Vector form)} \end{aligned}$$

if the limite exists.

Definition The tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ can be written as

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0},$$

where $\nabla F(x_0, y_0, z_0)$ is the normal vector of the tangent plane.

Exercise Find the tangent plane of the surfaces at the specified points.

(a) (105微乙) $\ln xy + \sin yz = 0$ at $(1, 1, 0)$.

(b) (103微甲) $\frac{4}{\pi}\arctan\frac{z}{2} = x^2 + \int_{xy}^{z} xy\sqrt{1 + t^3}\, dt$ at the point $(1, 2, 2)$.

Theorem If f is differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$\begin{aligned} D_u f(x_0, y_0) &= f_x(a, b)a + f_y(x, y)b \\ &= \nabla f(x_0, y_0) \cdot \mathbf{u} \\ &\leq |\nabla f(x_0, y_0)||\mathbf{u}| = |\nabla f(x_0, y_0)|. \end{aligned}$$

Property Let f be differentaible multivariate variable and suppose $\nabla f(x) \neq 0$. Then

1. The directional derivative of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_u f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$, as the theorem above shows.
2. When the gradient vector is parallel to $\mathbf{u}$, the inequality is equality, i.e, $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that the maximum rate of change is $|\nabla f(\mathbf{x})|$.
3. If $\mathbf{u}$ is a direction along the level curve, then $\nabla f(x_0, y_0) \cdot \mathbf{u}$ is 0, i.e, $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.

14.7 Maximum and Minimum Values

Theorem Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (so (a, b) is a critical point of f). Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then (a, b) is a saddle point of f.

(d) If $D = 0$, then the test gives no information.

Tips For Finding Extremum:

1. Find all candidate points whose gradient is $\mathbf{0}$.
2. Compute the second-order partial derivatives of these points and use second derivative test to identify their types.
3. (Rarely Seen) For points that the test gives no information, use the definition (the function value near the points) to determine its type.

Exercise (107微甲) Find and classify all critical points of $f(x, y) = 4x^3 + 2xy^2 + \frac{2}{3}y^3 + 6x^2$.

14.8 Method of Lagrange Multipliers

Exercise (103微乙) Find the maximum and minimum (and the corresponding points) of $f(x, y) = xy$ under the constraint $x^2 + xy + y^2 = 1$.

15 Multiple Integrals

15.1 Multiple Integrals over Rectangles and General Regions (15.1, 15.2, 15.6)

Definition The double integral (triple integral is similar) of f over the rectangle R is

$$\boxed{\iint\limits_R f(x,y)\ dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\ \Delta A}$$

if this limit exists.
However, the above definition is difficult for evaluating the integral. With the continuity, Fubini's theorem helps us transform multiple integral to iterated integral.

Fubini's Theorem If f is continuous on the rectangel $R = [a,b] \times [c,d]$, then

$$\iint\limits_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy$$

Exercise (103微甲) Write the integral in 5 other orders.

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\ dzdydx$$

Exercise (Textbook) Evaluate the double integral $\iint\limits_R \sin^2 x\ dA$, where D is bounded by $y = \cos x$, $0 \le x \le \dfrac{\pi}{2}$, $y = 0$, $x = 0$.

15.2 Changes of Variables and Polar Coordinates (15.3, 15.9)

Theorem
Suppose that T is a C^1 one-to-one transformation whose Jacobian is nonzero and that T maps S in the uv-plane onto R in the xy-plane. Suppose that f is continuous on R and that R and S are either type I or type II plane regions. Then

$$\boxed{\iint\limits_R f(x,y)\ dA = \iint\limits_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|\ dudv}$$

Theorem (Integration of polar coordinate)
Let $x = r\cos\theta$ and $y = r\sin\theta$, then the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0 =$$

Then, if R is the polar rectangle given by $0 \le a \le r \le b$ and $\alpha \le \theta \le \beta$, then

$$\boxed{\iint\limits_R f(x,y)\ dxdy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ rdrd\theta}$$

Exercise Evaluate the following integral

1. $\iint\limits_{\Omega} \frac{x^2}{x^2+y^2}\, dA$, where $\Omega : 1 \leq x^2 + y^2 \leq 2$.
2. $\iint\limits_{\Omega} (y-x)(2x+y)\, dA$, where Ω is bounded by $y - x = 1$, $y - x = 2$, $2x + y = 0$, $2x + y = 2$.

11 Tayler Series

Theorem If f has a power series representation $\sum c_n(x-a)^n$, which has radius of convergence $R > 0$, then

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + \dots && \text{(Power Series Representation)} \\ &= f(a)\frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-2)^2 + \frac{f^{(3)}(a)}{3!}(x-3)^3 + \dots && \text{(Taylor Expansion)} \end{aligned}$$

When $|x-a| < R$, we can integrate or differentiate $f(x)$ term by term, or even replace x with some other functions of x (For example, take replace x with $2x$ in the above equation).

Warning There exists functions that are not equal to the sum of their Taylor expansions, if the function doesn't has a power series representation.

The following are some important Maclaurin series that you should be able to derive independently.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Exercise (106微乙) Express $\tan^{-1} x$ as a power series.

Exercise (103微乙) Find the Taylor expansion of the function $f(x) = \cos^{-1} x$ at $x = 0$.

Exercise (108微乙)

7. (14%) 若 $f(x) = \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \cdots + \frac{1}{n(n-1)}x^n + \cdots$, 在 $|x| < 1$.
   (a) (2%) 求 $f^{(10)}(0)$
   (b) (6%) 求 $f'(x)$ 和 $f''(x)$ 在 $x = 0$ 的泰勒展式，並認出它們是什麼基本函數。
   (c) (6%) 把 $f(x)$ 表示成基本函數。