Lecture Notes for 4/8 and 4/15

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1 Application in Probability - Convolution

Suppose that random variables X and Y are independent with p.d.f. $f_X(x)$ and $f_Y(y)$ respectively. Let $Z = X + Y$.

1.1

(a) Represent $P(Z \le z)$ as a double integral, $\int\int_S f_X(x) f_Y(y)\ dA$.
Sol. The region S is the space $\{(x, y) \mid x + y \le z\}$ in $\mathbb{R}^2$. Assume the the pdf of x and y are continuous, then by Fubini's theorem

$$\int\int_S f_X(x) f_Y(y)\ dA = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y)\ dx\ dy$$

(b) Try the change of variables $u = x + y$, $v = x$ and compute the Jacobian. Rewrite $P(Z \le z)$ as $\int_a^b \int_c^d g(u, v)\ dv\ du$.
Sol. First compute the Jacobian

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

Then observe the region after change of variable: $\{(x, y) \mid x + y \le z\} \to \{(u, v) \mid u \le z\}$. Hence

$$\begin{aligned} P(Z \le z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y)\ dx\ dy \\ &= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f_X(v) f_Y(u - v)\ dv\ du \end{aligned}$$

(c) Derive the pdf of Z. Show that $\frac{d}{dz} P(Z \le z) = \int_{-\infty}^{\infty} f_X(v) f_Y(z - v)\ dv$.
Sol. By Fundamental theorem of Calculus, we have

$$\begin{aligned} \frac{d}{dz}\left(\int_{-\infty}^{z} \int_{-\infty}^{\infty} f_X(v) f_Y(u - v)\ dv\ du\right) &= \int_{-\infty}^{\infty} f_X(v) f_Y(z - v)\ dv \\ &= \int_{-\infty}^{\infty} f_X(z - v) f_Y(v)\ dv \end{aligned}$$

(By another change of variabe)

1.3 Examples with exponential distributions

Suppose $f_{X_i}(x_i) = e^{-x_i}$ for $x_i \ge 0$ and $f_{X_i}(x_i) = 0$ for $x_i = 0$. (Such distribution is called exponential distribution with parameter $\lambda = 1$.) Find the pdf of $Z_2 = X_1 + X_2$ and $Z_n = X_1 + \cdots + X_n$.
Sol. Start from two variables, and then compute the pdf for three independent variables using the results of two variables.

2 Partial Derivative in Economics

2.2

The average product of labor is $\frac{F(L,K)}{L}$ and the average product of capital is $\frac{F(L,K)}{K}$.

1. Find all production functions such that MPL is proportion to the average product of labor i.e, $\frac{\partial F}{\partial L} = \alpha_L \frac{F(L,K)}{L}$ for some constant $\alpha_L > 0$.
   Sol. Let K be fixed. Then the problem is reduced to a linear differential equation function. Let $f(L) = F(L, K)$ for some fixed K. Then we want to find f such that $f'(L) = \alpha_L f(L)/L$.
   We first compute the integrating factor $I = e^{\int \frac{1}{-\alpha_L L} dL} = L^{-\alpha}$.
   Then $(Iy) = \int IQdL + c(K) = c(K)$ for some $c(K)$. Hence $y = L^{\alpha} c(K)$.

2. Show that the Cobb-Douglas production function $F(L, K) = AL^{\alpha_L} K^{\alpha_K}$ is the only function satisfying that $\frac{\partial F}{\partial L} = \alpha_L \frac{F(L,K)}{L}$ and $\frac{\partial F}{\partial K} = \alpha_K \frac{F(L,K)}{K}$.
   Sol. We have
   $$F(L, K) = C_1(K)L^{\alpha_L} = C_2(L)K^{\alpha_K}$$
   Hence the only possibility is that $C_1(K) = AK^{\alpha_K}$ and $C_2(L) = AL^{\alpha_L}$ for some constant A.