Lecture Notes for 3/11

He-Zhe Lin, March 10, 2021

Last Modified: March 17, 2021

1 Exercises on Method of Lagrange Multipliers

1.1 Exercise 14.56 (p.1034)

Fine the absolute maximum and minimum values of f on the set D, where $f(x, y) = 4xy^2 - x^2y^2 - xy^3$ and D is the closed triangular region in the xy-plane with vertices $(0, 0)$, $(0, 6)$ and $(6, 0)$.
Sol. First, we find the points whose gradient are zero.

$$f_x = 4y^2 - 2xy^2 - y^3 = y^2(2 - 2x - y)$$

$$f_y = 8xy - 2x^2y - 3xy^2 = xy(8 - 2x - 3y)$$

In the interior of D, we can find that the gradient is zero at $(1, 2)$. We further compute the second-order partial derivative

$$f_{xx} = -2y^2,\ f_{xy} = 8y - 4xy - 3y^2,\ f_{yy} = 8x - 2x^2 - 6xy$$

The determinant at $(1, 2)$ is $-8 \times (-6) - (-4)^2 > 0$ and $f_{xx}(1, 2) < 0$. Hence $(1, 2)$ is a local maximizer with maximum 4.
Then we compute the extermum at the boundary.

1. $x = 0, y \in [0, 6]$, then $f(x, y) = 0$.
2. $y = 0, x \in [0, 6]$, then $f(x, y) = 0$.
3. $x + y = 6$, we substitute $x = 6 - y$. the problem is to solve

$$g(y) = 4(6 - y)y^2 - (6 - y)^2y^2 - (6 - y)y^3 = 2y^3 - 12y^2,\ y \in [0, 6]$$

   Use closed interval method, we find $g'(0) = 0$ when $y = 4$. $g(0) = 0$, $g(4) = -64$, $g(6) = 0$.

In conclusion, the abs. minimum is -64, when $(x, y) = (2, 4)$ and the abs. maximum is 4, when $(x, y) = (1, 2)$.