Lecture Notes for 3/4

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1 Differentiability of Functions of Several Variables

Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).
Proof Please refer to Appendix F in the textbook.
Remark The above theorem is only a sufficient condition. This page shows a function differentiable at $(0, 0)$ with discontinuous partial derivatives.

2 Chain Rules

2.1 Use the above equation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

1. $x^2 + 2y^2 + 3z^2 = 1$.
2. $yz + x \ln y = z^2$.

2.2 Use chain rule to find the indicated partial derivatives.

1. $T = \frac{v}{2u+v}, u = pq\sqrt{r}, v = p\sqrt{q}r$ at $p = 2, q = 1, r = 4$.

3 Mean Value Theorem for Two Variables

Theorem Let $f : \Omega \to \mathbb{R}$. $\Omega \subset \mathbb{R}^2$ be an open set. Assume that f is differentiable in Ω. Given $a = (a_1, a_2), b = (b_1, b_2) \in \Omega$ and denote $L(a, b)$ as the line segment connecting a and b. Assume $L(a, b) \subset \Omega$. Then there exist $c = (c_1, c_2) \in L(a, b)$ such that

$$f(a_1, a_2) - f(b_1, b_2) = f_x(c_1, c_2)(a_1 - b_1) + f_y(c_1, c_2)(a_2 - b_2)$$

Proof
Let $u = (a_1 - b_1, a_2 - b_2)$ and $\omega(t) = (b_1 + t(a_1 - b_1), b_2 + t(a_2 - b_2))$, $t \in [0, 1]$. So that $\omega(0) = (b_1, b_2)$, $\omega(1) = (a_1, a_2)$.
We may consider the scalar function

$$F(t) = f(b_1 + t(a_1 - b_1), b_2 + t(a_2 - b_2))$$

So that $F(1) = f(a_1, a_2), F(0) = f(b_1, b_2)$.
Since f is differentiable (also continuous) on $L(a, b)$, F is continuous on $[0, 1]$ and $(0, 1)$. By 1-dimensional MVT, we have that there exist $\xi \in (0, 1)$ such that

$$F(1) - F(0) = F'(\xi)(1 - 0) \tag{*}$$

Using chain rule, we compute

$$\begin{aligned} F'(t) = f(x,y) &= f(b_1 + t(a_1 - b_1), b_2 + t(a_2 - b_2)) \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= f_x(x,y)(a_1 - b_1) + f_y(x,y)(a_2 - b_2) \end{aligned}$$

Let $c = (c_1, c_2) = (b_1 + \xi(a_1 - b_1), b_2 + \xi(a_2 - b_2))$. We finally obtain

$$F(1) - F(0) = f(a_1, a_2) - f(b_1, b_2) = f_x(c_1, c_2)(a_1 - b_1) + f_y(a_2 - b_2). \qquad \square$$