Calculus 1 - Midterm Review

2020.11.05

1.5 Inverse functions and logarithms

- A function is called 1-1 if $f(x_1) = f(x_2) \iff x_1 = x_2$.
- Definition of e:
    - The only real number x such that $\lim_{h\to 0} \frac{x^h - 1}{h} = 1$.
    - $\lim_{n\to\infty}(1+\frac{1}{n})^n$.
    - $\sum_{n=0}^{\infty} \frac{1}{n!}$.

2.2-2.4 The limit of a function

- $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.
- ex. (107) $\lim_{x\to -\infty} \left(\sqrt{x^2+2x-1} + x\right)$.
- ex. (106) $\lim_{x\to 0} \frac{\tan x}{\sqrt{1-\cos 3x}}$.
- ex. (106) $\lim_{x\to 0} \sin(\frac{1}{x^2}) \sin x$.

2.5, 2.8 Continuity / Differentiability

- **Definition** If $\lim_{x\to a} = f(a)$, then $f(x)$ is continuous at $x = a$.
- **Intermediate value theorem** If f is continuous on $[a, b]$ and N is a number between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = N$.
- **Definition** A function f is said to be differentiable at x if $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists.
- ex. (107)

3. (15 pts) Let $f(x) = \begin{cases} (1+x)^{\frac{1}{x}}, & \text{for } x \neq 0,\ x > -1, \\ a, & \text{for } x = 0. \end{cases}$

   (a) (3 pts) Find the value of a such that $f(x)$ is continuous at $x = 0$.

   (b) (4 pts) Find $\lim\limits_{x\to\infty} f(x)$.

   (c) (4 pts) Compute $f'(x)$, for $x \neq 0$.

   (d) (4 pts) Is $f(x)$ differentiable at $x = 0$? If $f(x)$ is differentiable at $x = 0$, then find $f'(0)$.

- ex. (105)

105. Suppose that $f(x) = \begin{cases} \sin x + b\ln(x+1) + c & \text{if } x \geq 0 \\ e^{x^2} & \text{if } x < 0 \end{cases}$.

(a) Find b, c such that $f(x)$ is continuous everywhere.

(b0 Find b, c such that $f(x)$ is differentiable everywhere.

(c) For b, c in (b), is $f'(x)$ continuous?

Ch 3 Differentiation

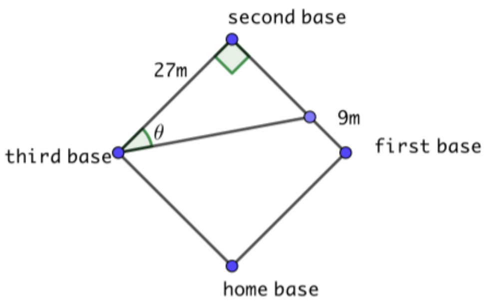
- Differentiation Method
    - Product/Quotient Rules
    - (Inverse) Trigonometric functions
    - Chain Rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.
    - Implicit differentiation: Let $F(x, y) = 0$, find $\frac{dy}{dx}$.
        - ex. (105) Let the curve $x^2y^2 + 2xy = 8$ be given. (a) Compute y' (b) Find tangent lines of the points with $y = 2$. (c) Find y'' at the points in (b).
        - ex. (103) If $xy + e^y = e$, find y' and the value of y, y', y'' where $x = 0$.
    - Logarithmic function

- (106)

  2. (15 points) Differentiate the following functions.
     - (a) (5 points) $f(x) = \frac{\sin x}{1+\cos x}$.
     - (b) (5 points) $f(x) = \log_2 \sqrt{x} + \tan^{-1}(x^3)$.
     - (c) (5 points) $f(x) = x^{\cos x}$.

- (107)

  5. (12 pts) A baseball diamond is a square with side 27 m. A player runs from the first base to the second base at a rate of 5 m/s.
     - (a) (4 pts) At what rate is the player's distance from the third base changing when the player is 9m from the first base?
     - (b) (4 pts) At what rate is the angle θ changing at the moment in part (a)?
     - (c) (4 pts) The player slides into the second base at a rate of 4.5 m/s. At what rate is the angle θ changing as the player touches the second base?



- Linear Approximation:
  - $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ when x near x_0.

4.1 Minimum and Maximum values

- Theorem: If f defined on $[a, b]$ and has a local extrema at $c \in (a, b)$, then $f'(c) = 0$ or $f'(c)$ DNE.
- Definition: The interior point $c \in dom(f)$ is called a critical point of f if $f'(c) = 0$ or $f'(c)$ DNE.

4.2 Mean Value Theorem

- If f is differentiable on (a, b) and continuous on $[a, b]$, then there is at least a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- ex. (101) Show that $|\tan \frac{x}{2} - \tan \frac{y}{2}| \geq \frac{|x-y|}{2}$ for $x, y \in (-\pi, \pi)$.

Graphing a function

- Local minimum: $f(x) \geq f(c)$ when x is near to c (on some open interval containing c).
- Local maximum: $f(x) \leq f(c)$ when x is near to c (on some open interval containing c).
- Increasing/Decreasing Test:
  - If $f'(x) > 0$ on an interval, then f is increasing on that interval.
  - If $f'(x) < 0$ on an interval, then f is decreasing on that interval.
- Concavity Test:
  - If $f''(x) > 0$ on I, then f is concave upward on I.
  - If $f''(x) < 0$ on I, then f is concave downward on I.
- First Derivative Test:
  - Suppose c is a critical number. If $f'(c)$ changes from positive to negative at c, then f has a local maximum.
  - If $f'(c)$ is positive to the left and right of c, then $f(c)$ is neither positive or negative.
- Second Derivative Test:
  - Suppose f'' is continuous near c. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
  - Note that $f''(c) = 0$ tells us nothing. An famous example is x^3 at $x = 0$.
- Horizontal asymptote:

- $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

- Vertical asymptote:
  - $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ is either ∞ or $-\infty$.

ex. $\frac{\ln|x|}{x}$ for $x \neq 0$