109-2 Calculus Final Review (2021.01.06)

Chapter 5

Definition (Definite integral) Divide the interval $[a, b]$ into n subintervals $\{a = x_1 < \cdots < x_n = b\}$ (切成長方形的分割點) and choose sample points $x_i^* \in [x_{i-1}, x_i]$ (小長方形的高)。The definite integral of f from a to b is defined by the Reimann sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\,\Delta x_i$$

- *Exercise* (106甲) Find the limits $\lim_{n\to\infty} \sum_{i=1}^n \frac{n}{n^2+4i^2}$. (Hint: 試著拆成上面「長方形的高 乘以 長方形的寬的形式」)

Theorem (Fundamental theorem of Calculus) Suppose f is continuous on an interval containing a. Then the following holds:
i. Let $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$. （令 F 代表 f 從 a 到 x 曲線下面積，則 F 的變化率會等於 f 的高度）
ii. If $G(x)$ is any antiderivative of $f(x)$ on I, then $\int_a^b f(x)\,dx = G(x)|_a^b$.

- *Exercise:* $F(x) = \int_a^x f(x)\,dx$, express $F'(x)$. (注意 Chain Rule !)

Chapter 6

Definition (Area between curves) Area between curves $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ is defined as

$$\int_a^b |f(x) - g(x)|\,dx$$

- *Remark* 用積分的定義去想，切成很多小長方形，每塊長度為 $|f(x) - g(x)|$ ，高度為 dx.

Definition (Disk Method) Rotate the region bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$ about the x-axis gives the volume

$$V = \int_a^b \pi f(x)^2\,dx$$

- *Remark* 用切片去想，每片圓盤底面積是$\pi f(x)^2$，高是 dx.

Definition (Cylindrical Shell Method) Rotate the region bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$ about the y-axis gives the volume

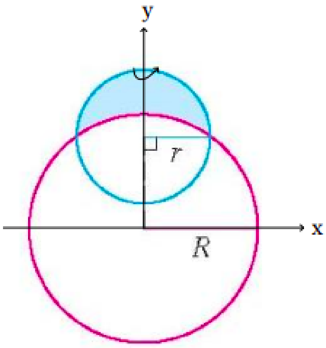
$$V = 2\pi \int_a^b x f(x)\,dx$$

- *Remark* 用很多圓柱殼想，圓柱殼的表面積是 $2\pi x dx$ （周長乘厚度dx），柱高 $f(x)$.

- *Exercise* (102 甲) Let Ω be the region bounded by $y = \cos x$, $y = 0$, $x = 0$ and $x = \frac{\pi}{2}$.
  (a) Find the volume of the solid obtained by revolving Ω of x-axis.
  (b) Find the volume of the solid obtained by revolving Ω of y-axis.

- *Exercise* (108 甲)

4. (10 pts) Consider the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R, where $0 < r < R$. Rotate the region about the y-axis. Find the resulting volume.



Chapter 7 Integration Techniques

- Substitution rules （記得令變數變換，一定要算 du）
- Integration by parts (which has reviewed in the problem above :-)
  - 看到兩種類型的函數相乘，如多項式×反三角、多項式×ln、多項式×指數、指數×三角函數，可考慮使用 integration by parts，藉由把其中一項微分，一項積分讓積分變簡單。
  - *Exercise* (101 甲) Evaluate $\int_0^1 x(\tan^{-1} x)^2\, dx$. (要用兩次 integration by parts)
- Trigonometric integrals (三角函數的乘法 $\cos^m x \sin^n x$, $\tan^m x \sec^n x$)
  - **記得變數變換後不要忘記** du。
- Trigonometric substitution（根號型）
  - $\sqrt{a^2 - x^2}$，Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$, $\theta \in [-\pi/2, \pi/2]$
  - $\sqrt{a^2 + x^2}$，Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$, $\theta \in (-\pi/2, \pi/2)$
  - $\sqrt{x^2 - a^2}$，Let $x = a\sec\theta$, $dx = a\sec\tan\theta\, d\theta$, $\theta \in (0, \pi/2) \cup (\pi, -\pi/3)$.
  - 變數變換之後可以做化簡。
  - *Exercise* (106) Evaluate the integral $\int \frac{xdx}{\sqrt{25-8x+x^2}}$.
- Integration by partial fractions
  - 先因式分解一次式（務必先確認是對的分解再繼續，不然白作工），再展開，會遇到以下形式：
    - $ax + b : \frac{A}{ax+b}$
    - $(ax+b)^2 : \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2}$
    - $ax^2 + bx + c : \frac{Ax+B}{ax^2+bx+c}$ e.g. $\frac{2x+5}{x^2+4x+5}$
    - $(ax^2+bx+c)^2 : \frac{Ax+b}{ax+bx+c} + \frac{Ax+b}{(ax^2+bx+c)^2}$
- Improper integrals
  - 積分範圍無限：$\int_a^\infty f(x)\, dx = \lim\limits_{b\to\infty} \int_a^b f(x)\, dx$.
  - 積分有暇點（函數跑到無限大）：

    If $\lim_{x\to a} f(x) = \pm\infty$, then $\int_a^b f(x)\, dx = \lim_{c\to a^+} \int_c^b f(x)\, dx$.
  - $\int_{-1}^1 \frac{1}{x^3}\, dx$ 不能說因為 $\frac{1}{x^3}$ 是 odd function，所以積分為 0。因為有暇點，必須根據瑕積分的定義算出來是否收斂/發散。
  - 作答時，不必急於寫下等號，先算出定積分的函數，再套用極限定義，優雅流暢的寫下作法。
  - 如果只要判斷積分斂散，可以使用 Comparison test.

Chapter 9

- Separable Differential Equations
- Linear Equations

  $$y' + P(x)y = Q(x)$$

  Let the integrating factor $I(x) = e^{\int P(x)dx}$. Then solve

  $$(I(x)y)' = I(x)Q$$

  so that $y(x) = \frac{1}{I(x)}(\int I(x)Q(x)\,dx + C)$

Exercise Solve $x\frac{dy}{dx} - y = x^2 \sin x$ with $y(\pi) = 0$.

Chapter 10.1, 10.2

參數化：x, y 都是 t 的函數（用一個變數表達 x 和 y）。

- Slope:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

- Arc Length: 把曲線切成很多小段，每一段可以用直線近似，長度是 $\sqrt{\Delta x^2 + \Delta y^2}$

  $$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

- Area:

  $$A = \int_a^b y\,dx = \int_\alpha^\beta y(t)x'(t)\,dt$$

- Surface Area: 想成切片，每片表面積是周長乘以 arc length
  - Rotate about x axis:

    $$S = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

  - Rotate about y axis:

    $$S = \int_\alpha^\beta 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Exercise Find the arc length of the curve $x = t\sin 2t,\ y = t\cos 2t,\ 0 \le t \le 1$.