Lesson 8. Reducibility

CSIE 3110 – Formal Languages and Automata Theory

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Recall

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HALT is undecidable because it is a more "general" language than HALT_0.

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This technique is called *reductions*.

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Suppose we can show how to “reduce” problem K to problem L.

Intuitively, this reduction means problem L is more “general” than problem K.

That is, problem L is “harder” than problem K.

So, if problem K is undecidable, then problem L is undecidable too.

The general strategy to deal with a problem/language

L

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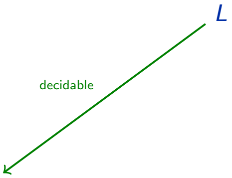

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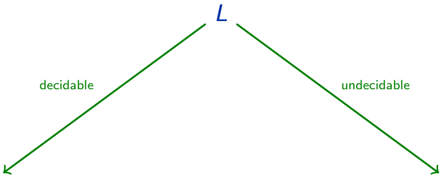

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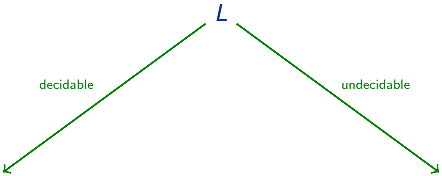

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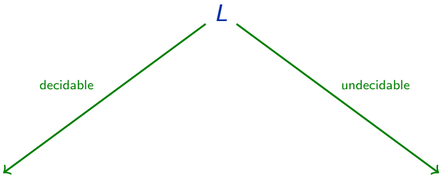

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The focus of this lesson

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The focus of this lesson

Two types of reductions: *Mapping* reductions and *Turing* reductions.

Computable functions

Let $F : \Sigma^* \to \Sigma^*$ be a function from Σ^* to Σ^*.

(Def.) A Turing machine $\mathcal{M}$ computes the function F, if $\mathcal{M}$ is a 2-tape Turing machine that accepts every word $w \in \Sigma^*$ and when it halts, the content of its second tape is $F(w)$.

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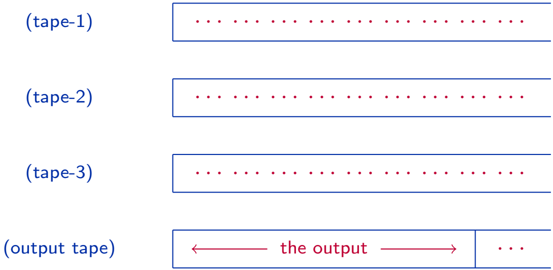
(Def.) A function $F : \Sigma^* \to \Sigma^*$ is *computable*, if there is a Turing machine that computes it.

Computable functions by multi-tape Turing machines

The Turing machine $\mathcal{M}$ that computes F can be any multi-tape Turing machine with a designated output tape that contains the output string.



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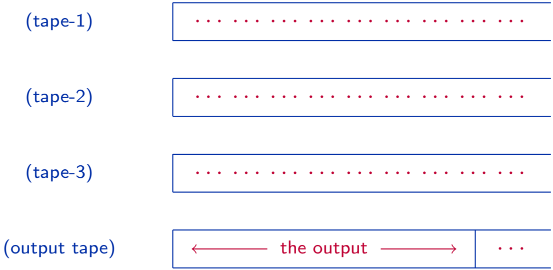

(Note) Any function that can be computed by a multi-tape Turing machine can also be computed by a 2-tape Turing machine.

Mapping reductions

(Def.) A language L_1 is *mapping reducible* to another language L_2, denoted by:

$$L_1 \;\leqslant_m\; L_2$$

if there is a computable function F such that for every $w \in \Sigma^*$:

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Intuitively $L_1 \leqslant_m L_2$ means "L_2 is (computationally) more general than L_1".

It also means that a Turing machine that decides L_2 can be used to decide L_1.

Turing reductions

(Def.) A language L_1 is *Turing reducible* to another language L_2, denoted by:

$$L_1 \leqslant_T L_2$$

if there is a Turing machine $\mathcal{M}_2$ that decides L_2, then there is a Turing machine $\mathcal{M}_1$ that decides L_1 using $\mathcal{M}_2$ as a "subroutine."

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(Def.) We call $\mathcal{M}_1$ a Turing machine *with oracle access to L_2*.

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On input w:

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$\vdots$ *some algorithm*

Output v

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- $w \in L_1$ if and only if $v \in L_2$. $\Rightarrow$ Very important!
- Inside the algorithm we do not assume/use anything about L_2.
- View it this way: If L_2 is decidable by, say, $\mathcal{M}_2$, then in the algorithm we can only use $\mathcal{M}_2$ once(!).
  The answer provided by $\mathcal{M}_2$ must also be the answer to whether $w \in L_1$.

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Note that:

$$\lfloor\mathcal{M}\rfloor \in \mathsf{HALT}_0 \qquad \text{if and only if} \qquad \lfloor\mathcal{M}\rfloor\$\lfloor\mathcal{M}\rfloor \in \mathsf{HALT}$$

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- $L_4 := \{\lfloor\mathcal{M}\rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$.

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- $L_3 := \{\lfloor \mathcal{M} \rfloor \ \mid \ \mathcal{M} \text{ accepts the word } 1101\}$.
- $L_4 := \{\lfloor \mathcal{M} \rfloor \ \mid \ L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$.
- $L_5 := \{\lfloor \mathcal{M} \rfloor \ \mid \ L(\mathcal{M}) \text{ is a regular language}\}$.

Some variants of the halting problem

(Def.) $L(\mathcal{M})$ denotes the set of all words accepted by the Turing machine $\mathcal{M}$.

The following languages are all undecidable.

- $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$.
  That is, $\lfloor \mathcal{M} \rfloor \in L_0$ if and only if $\mathcal{M}$ does not accept any word.

- $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$.

Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable

We show that HALT $\leqslant_m \overline{L}_0$, where $\overline{L}_0$ is the complement of L_0.

Proof that $L_0 := \{\lfloor\mathcal{M}\rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable

We show that HALT $\leqslant_m \overline{L}_0$, where $\overline{L}_0$ is the complement of L_0.

On input $\lfloor\mathcal{M}\rfloor\$w$:

- Construct the following Turing machine denoted by $\mathcal{K}_{\mathcal{M},w}$:

  On input u:
  – Run $\mathcal{M}$ on w.
  – If $\mathcal{M}$ accepts w, ACCEPT.
  – If $\mathcal{M}$ rejects w, REJECT.

  (Note: ACCEPT and REJECT above are inside $\mathcal{K}_{\mathcal{M},w}$.)

- Output $\lfloor\mathcal{K}_{\mathcal{M},w}\rfloor$.

Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable

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- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT,

Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable

We show that HALT $\leqslant_m \overline{L}_0$, where $\overline{L}_0$ is the complement of L_0.

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- Construct the following Turing machine denoted by $\mathcal{K}_{\mathcal{M},w}$:

  On input u:
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  – If $\mathcal{M}$ accepts w, ACCEPT.
  – If $\mathcal{M}$ rejects w, REJECT.

  (Note: ACCEPT and REJECT above are inside $\mathcal{K}_{\mathcal{M},w}$.)

- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \Sigma^*$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in \overline{L}_0$.

Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable

We show that HALT $\leqslant_m \overline{L}_0$, where $\overline{L}_0$ is the complement of L_0.

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  (Note: ACCEPT and REJECT above are inside $\mathcal{K}_{\mathcal{M},w}$.)
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If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \Sigma^*$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in \overline{L}_0$.

If $\lfloor \mathcal{M} \rfloor \$ w \notin$ HALT,

Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable

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If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \Sigma^*$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in \overline{L}_0$.

If $\lfloor \mathcal{M} \rfloor \$ w \notin$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \emptyset$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \notin \overline{L}_0$.

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  (Note: ACCEPT and REJECT above are inside $\mathcal{K}_{\mathcal{M},w}$.)

- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \Sigma^*$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in \overline{L}_0$.

If $\lfloor \mathcal{M} \rfloor \$ w \notin$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \emptyset$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \notin \overline{L}_0$.

Thus,

$\lfloor \mathcal{M} \rfloor \$ w \in$ HALT if and only if $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in \overline{L}_0$

Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable

We show that HALT $\leqslant_m \overline{L}_0$, where $\overline{L}_0$ is the complement of L_0.

On input $\lfloor \mathcal{M} \rfloor \$ w$:

- Construct the following Turing machine denoted by $\mathcal{K}_{\mathcal{M},w}$:

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  (Note: ACCEPT and REJECT above are inside $\mathcal{K}_{\mathcal{M},w}$.)

- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \Sigma^*$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in \overline{L}_0$.

If $\lfloor \mathcal{M} \rfloor \$ w \notin$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \emptyset$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \notin \overline{L}_0$.

Thus,

$$\lfloor \mathcal{M} \rfloor \$ w \in \text{HALT} \quad \text{if and only if} \quad \lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in \overline{L}_0$$

So, HALT $\leqslant_m \overline{L}_0$.

Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable – illustration

On input $\lfloor \mathcal{M} \rfloor \$ w$:

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Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable – illustration

On input $\lfloor \mathcal{M} \rfloor \$ w$:

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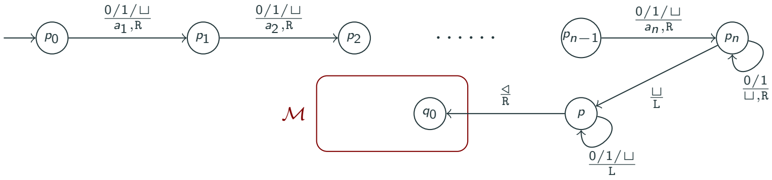
  On input u:
  – Run $\mathcal{M}$ on w.
  – If $\mathcal{M}$ accepts w, ACCEPT.
  – If $\mathcal{M}$ rejects w, REJECT.

Add the following: (where $w = a_1 a_2 \cdots a_n$)



Proof that $L_0 := \{\lfloor\mathcal{M}\rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable – illustration

On input $\lfloor\mathcal{M}\rfloor\$w$:

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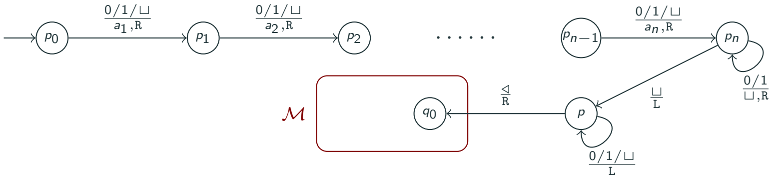


- Make p_0 the initial state of $\mathcal{K}_{\mathcal{M},w}$.
- The accept state of $\mathcal{K}_{\mathcal{M},w}$ is the accept state of $\mathcal{M}$.
- The reject state of $\mathcal{K}_{\mathcal{M},w}$ is the reject state of $\mathcal{M}$.

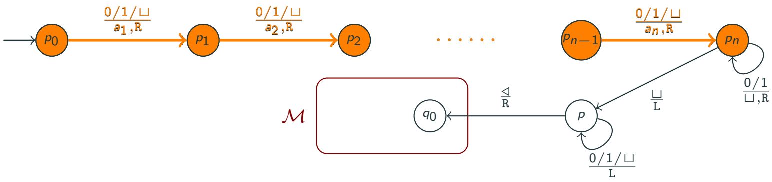
Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable – illustration

On input $\lfloor \mathcal{M} \rfloor \$ w$:

- Construct the following Turing machine denoted by $\mathcal{K}_{\mathcal{M},w}$:

  On input u:
  – Run $\mathcal{M}$ on w.
  – If $\mathcal{M}$ accepts w, ACCEPT.
  – If $\mathcal{M}$ rejects w, REJECT.

Add the following: (where $w = a_1 a_2 \cdots a_n$)



Rewrite the content of the tape to be w.

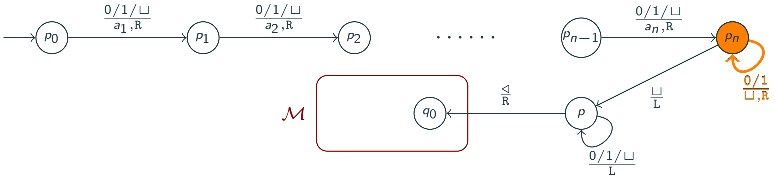
Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable – illustration

On input $\lfloor \mathcal{M} \rfloor \$ w$:

- Construct the following Turing machine denoted by $\mathcal{K}_{\mathcal{M},w}$:

  On input u:
  – Run $\mathcal{M}$ on w.
  – If $\mathcal{M}$ accepts w, ACCEPT.
  – If $\mathcal{M}$ rejects w, REJECT.

Add the following: (where $w = a_1 a_2 \cdots a_n$)



"Erase" the remaining of the input v when $|v| > |w|$.

Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable – illustration

On input $\lfloor \mathcal{M} \rfloor \$ w$:

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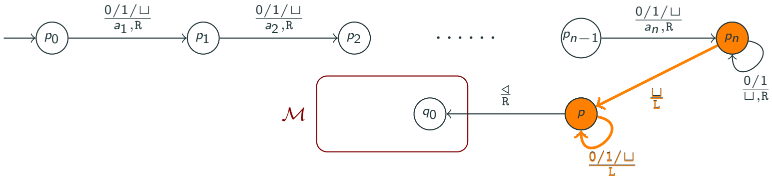
  On input u:
  – Run $\mathcal{M}$ on w.
  – If $\mathcal{M}$ accepts w, ACCEPT.
  – If $\mathcal{M}$ rejects w, REJECT.

Add the following: (where $w = a_1 a_2 \cdots a_n$)



Move the head back to the beginning of the tape.

Proof that $L_0 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \emptyset\}$ is undecidable – illustration

On input $\lfloor \mathcal{M} \rfloor \$ w$:

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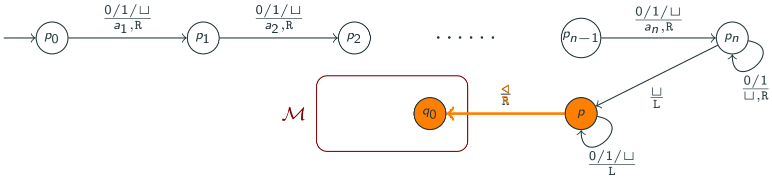

When the head reaches the left-end marker $\triangleleft$, it moves right.

It enters the state q_0 of $\mathcal{M}$ (i.e., to run $\mathcal{M}$ on w).

Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable

Let $\mathcal{A}$ be a TM that decides the language $\{a^n b^n | n \geqslant 0\}$.

Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable

Let $\mathcal{A}$ be a TM that decides the language $\{a^n b^n | n \geqslant 0\}$.

We show that HALT $\leqslant_m L_4$.

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- Construct the following Turing machine denoted by $\mathcal{K}_{\mathcal{M},w}$:

  On input u:
  - Run $\mathcal{A}$ on u. (to check if $u \in \{a^n b^n | n \geqslant 0\}$.)
  - If $\mathcal{A}$ rejects u, REJECT.
  - If $\mathcal{A}$ accepts u:
    * Run $\mathcal{M}$ on w.
    * If $\mathcal{M}$ accepts w, ACCEPT.
    * If $\mathcal{M}$ rejects w, REJECT.
- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

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- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

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- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \{a^n b^n | n \geqslant 0\}$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in L_4$.

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    - If $\mathcal{M}$ rejects w, REJECT.
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If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \{a^n b^n | n \geqslant 0\}$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in L_4$.

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Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable

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  - If $\mathcal{A}$ rejects u, REJECT.
  - If $\mathcal{A}$ accepts u:
    - Run $\mathcal{M}$ on w.
    - If $\mathcal{M}$ accepts w, ACCEPT.
    - If $\mathcal{M}$ rejects w, REJECT.
- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \{a^n b^n | n \geqslant 0\}$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in L_4$.

If $\lfloor \mathcal{M} \rfloor \$ w \notin$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \emptyset$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \notin L_4$.

Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable

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  - If $\mathcal{A}$ accepts u:
    - $*$ Run $\mathcal{M}$ on w.
    - $*$ If $\mathcal{M}$ accepts w, ACCEPT.
    - $*$ If $\mathcal{M}$ rejects w, REJECT.
- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \{a^n b^n | n \geqslant 0\}$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in L_4$.

If $\lfloor \mathcal{M} \rfloor \$ w \notin$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \emptyset$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \notin L_4$.

Thus,

$$\lfloor \mathcal{M} \rfloor \$ w \in \text{HALT} \quad \text{if and only if} \quad \lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in L_4$$

Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable

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    - If $\mathcal{M}$ accepts w, ACCEPT.
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- Output $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor$.

If $\lfloor \mathcal{M} \rfloor \$ w \in$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \{a^n b^n | n \geqslant 0\}$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in L_4$.

If $\lfloor \mathcal{M} \rfloor \$ w \notin$ HALT, then $L(\mathcal{K}_{\mathcal{M},w}) = \emptyset$, so, $\lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \notin L_4$.

Thus,

$$\lfloor \mathcal{M} \rfloor \$ w \in \text{HALT} \quad \text{if and only if} \quad \lfloor \mathcal{K}_{\mathcal{M},w} \rfloor \in L_4$$

So, HALT $\leqslant_m L_4$.

Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable – Illustration

On input $\lfloor \mathcal{M} \rfloor \$ w$:

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    * If $\mathcal{M}$ accepts w, ACCEPT.
    * If $\mathcal{M}$ rejects w, REJECT.

Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable – Illustration

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    - ∗ If $\mathcal{M}$ accepts w, ACCEPT.
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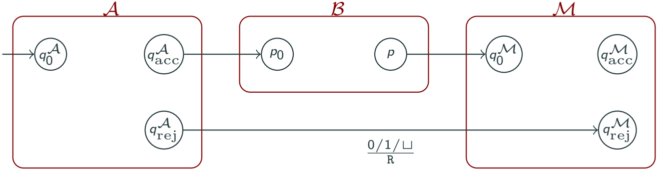
$\mathcal{K}_{\mathcal{M},w}$:



Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable – Illustration

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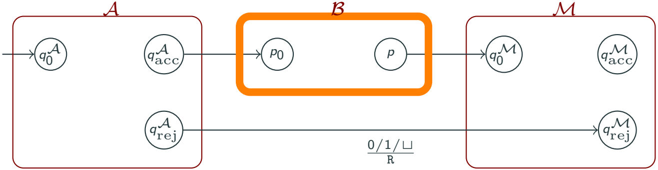
$\mathcal{K}_{\mathcal{M},w}$:



Turing machine $\mathcal{B}$ writes w on the tape and enters $q_0^{\mathcal{M}}$ (to run $\mathcal{M}$ on w).

Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable – Illustration

On input $\lfloor \mathcal{M} \rfloor \$ w$:

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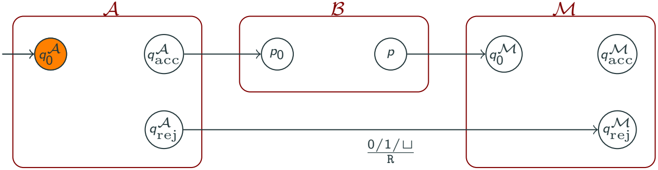

$q_0^{\mathcal{A}}$ is the initial state.

Proof that $L_4 := \{\lfloor \mathcal{M} \rfloor \mid L(\mathcal{M}) = \{a^n b^n \mid n \geqslant 0\}\}$ is undecidable – Illustration

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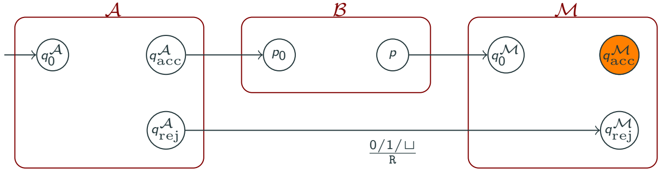

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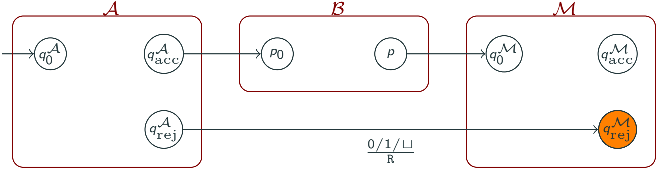

$q_{\mathrm{rej}}^{\mathcal{M}}$ is the reject state

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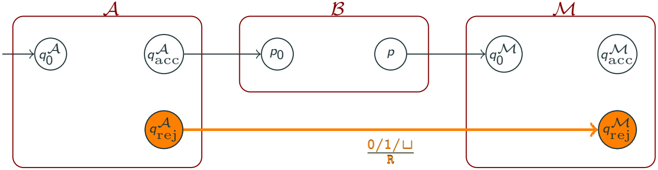

Add a transition so that from $q_{\mathrm{rej}}^{\mathcal{A}}$ the TM enters $q_{\mathrm{rej}}^{\mathcal{M}}$.

Rice's theorem

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P is a *property*, if for every Turing machines $\mathcal{M}_1$ and $\mathcal{M}_2$, if:

$$L(\mathcal{M}_1) \; = \; L(\mathcal{M}_2)$$

then:

$$\text{either} \qquad \lfloor\mathcal{M}_1\rfloor, \lfloor\mathcal{M}_2\rfloor \; \in P \qquad \text{or} \qquad \lfloor\mathcal{M}_1\rfloor, \lfloor\mathcal{M}_2\rfloor \; \notin P$$

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The criteria for $\lfloor \mathcal{M} \rfloor$ to be in P depends on the language $L(\mathcal{M})$, and *not* on the string $\lfloor \mathcal{M} \rfloor$ itself.

(Def.) A property P is called a *trivial* property, if:

either $\quad P = \emptyset \quad$ or $\quad P$ contains all the descriptions of Turing machines

Rice's theorem – continued

Theorem 8.6 (Rice's theorem)
For a property P, if P is not a trivial property, then P is undecidable.

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Such $\mathcal{A}$ exists since P is not trivial.

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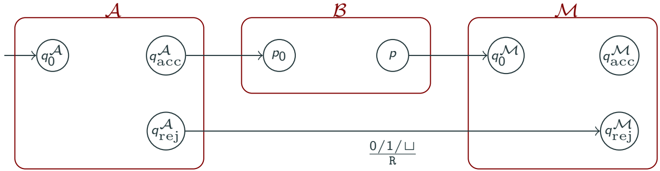

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By similar reasoning as the proof of the undecidability of L_4:

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Thus, we have proved Rice's theorem for the case where P does not contain $\lfloor \mathcal{M} \rfloor$ where $L(\mathcal{M}) = \emptyset$

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Table of contents

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- A rule, say, $S \rightarrow 0X11$ is encoded as $\langle 0 \rangle \ \rightarrow \ 0\langle 3 \rangle 11$.
  (Assuming that S is represented as 0 and X as 3).

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Theorem 8.8

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We will show that HALT $\leqslant_m$ CFL-Intersection.

We assume that HALT contains only $\lfloor \mathcal{M} \rfloor \$ w$ where $\mathcal{M}$ is a 1-tape Turing machine and $\mathcal{M}$ accepts w.

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After adding those states, the following holds for every word w:

- If $\mathcal{M}$ accepts w, then the run is finite and has odd length.
- If $\mathcal{M}$ does not w, then the run is infinite.

Some observations – continued

Recall that the states of a Turing machines $\mathcal{M}$ are represented as numbers written in binary form. Thus, the run (1) can be viewed as a string over the alphabet $\{\vdash, 0, 1, \tilde{\sqcup}, [,]\}$, where we write $[i]$ to represent the state in the configuration.

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(Def.) We call the string: $C_0 \ \vdash \ C_1^r \ \vdash \ C_2 \ \vdash \ C_3^r \ \vdash \ \cdots \ \vdash \ C_n^r$

the reverse representation of the run: $C_0 \ \vdash \ C_1 \ \vdash \ C_2 \ \vdash \ C_3 \ \vdash \ \cdots \ \vdash \ C_n$.

The construction of $\mathcal{G}_1$ and $\mathcal{G}_2$

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(b) u_0 is the initial configuration of $\mathcal{M}$ on w.

The construction of $\mathcal{G}_1$ and $\mathcal{G}_2$

A string $u_0 \ \vdash \ u_1 \ \vdash \ u_2 \ \vdash \ u_3 \ \vdash \ \cdots \ \vdash \ u_n$ is the reverse representation of the run of $\mathcal{M}$ on w, if:

(a) n is an odd number, i.e., the symbol $\vdash$ appears even number of times.

(b) u_0 is the initial configuration of $\mathcal{M}$ on w.

(c) $u_{i-1} \vdash u_i^r$, for each odd i in between 1 and n.

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There is an algorithm where on input $\lfloor \mathcal{M} \rfloor$, it constructs a CFG $\mathcal{G}_1$ such that $\mathcal{G}_1$ generates the strings that satisfies conditions (a), (b) and (c).

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(See Note 8 for the details.)

The reduction HALT $\leqslant_m$ CFL-Intersection

On input $\lfloor \mathcal{M} \rfloor \$ w$, do the following.

- Add some new states to $\mathcal{M}$ so that:
  $\mathcal{M}$ accepts w iff the run of $\mathcal{M}$ on w is finite and has odd length.
- Construct $\mathcal{G}_1$ that generates words satisfying conditions (a), (b) and (c).
- Construct $\mathcal{G}_2$ that generates words satisfying conditions (d) and (e).
- Output $\lfloor \mathcal{G}_1 \rfloor \$ \lfloor \mathcal{G}_2 \rfloor$.

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$L(\mathcal{G}_1) \cap L(\mathcal{G}_2)$ contains the reverse representation of the accepting run of $\mathcal{M}$ on w.

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$L(\mathcal{G}_1) \cap L(\mathcal{G}_2)$ contains the reverse representation of the accepting run of $\mathcal{M}$ on w.

Thus,

$$\lfloor \mathcal{M} \rfloor \$ w \in \text{HALT} \qquad \text{if and only if} \qquad L(\mathcal{G}_1) \cap L(\mathcal{G}_2) \neq \emptyset$$

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Hence, CFL-Intersection is undecidable.

CFL universality

CFL-Universality

Input: A CFG $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ where $\Sigma = \{0, 1\}$.

Task: Output `True`, if $L(\mathcal{G}) = \Sigma^*$. Otherwise, output `False`.

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Similar to CFL-Intersection, the problem CFL-Universality can be viewed as language.

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Similar to CFL-Intersection, the problem CFL-Universality can be viewed as language.

Theorem 8.9
The problem CFL-Universality *is undecidable.*

Proof that CFL-Universality is undecidable

The proof is similar to Theorem 8.8.

Proof that CFL-Universality **is undecidable**

The proof is similar to Theorem 8.8.

We describe an algorithm that does the following.

On input $\lfloor \mathcal{M} \rfloor \$ w$:

- Construct a CFG $\mathcal{G}$ such that:
  $\mathcal{G}$ generates all strings that are *not*(!) the run of $\mathcal{M}$ on w.

Proof that CFL-Universality is undecidable

The proof is similar to Theorem 8.8.

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If $\lfloor \mathcal{M} \rfloor \$ w \notin$ HALT, then $L(\mathcal{G}) = \Sigma^*$.

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Thus,

$$\lfloor \mathcal{M} \rfloor \$ w \in \mathsf{HALT} \qquad \text{if and only if} \qquad L(\mathcal{G}) \neq \Sigma^*$$

The construction of the CFG $\mathcal{G}$

A word $u_0 \vdash u_1 \vdash u_2 \vdash u_3 \cdots \vdash u_n$ is not the reverse representation of the run $\mathcal{M}$ on w, if at least one of the following holds.

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A word $u_0 \vdash u_1 \vdash u_2 \vdash u_3 \cdots \vdash u_n$ is not the reverse representation of the run $\mathcal{M}$ on w, if at least one of the following holds.

(C1) The symbol $\vdash$ appears even number of times.

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(C2) u_0 is not the initial configuration.

The construction of the CFG $\mathcal{G}$

A word $u_0 \ \vdash \ u_1 \ \vdash \ u_2 \ \vdash \ u_3 \ \cdots \ \vdash \ u_n$ is not the reverse representation of the run $\mathcal{M}$ on w, if at least one of the following holds.

(C1) The symbol $\vdash$ appears even number of times.

(C2) u_0 is not the initial configuration.

(C3) For some $0 \leqslant i \leqslant n$, the string u_i is not a configuration.
It does not contain a state or the states appear at least twice or the brackets [and] do not appear "properly" or inside the bracket [and] is not a state of $\mathcal{M}$.

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A word $u_0 \vdash u_1 \vdash u_2 \vdash u_3 \cdots \vdash u_n$ is not the reverse representation of the run $\mathcal{M}$ on w, if at least one of the following holds.

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(C3) For some $0 \leqslant i \leqslant n$, the string u_i is not a configuration.
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(C4) For some $0 \leqslant i \leqslant n-1$, the string $u_i \vdash u_i$ is not according to the transitions of $\mathcal{M}$.

(C5) For some $o \leqslant i \leqslant n-1$, the string u_i is not the reverse of u_{i+1} (disregarding the state symbol and the symbols next to the state in both u_i and u_{i+1}).

The construction of the CFG $\mathcal{G}$

A word $u_0 \vdash u_1 \vdash u_2 \vdash u_3 \cdots \vdash u_n$ is not the reverse representation of the run $\mathcal{M}$ on w, if at least one of the following holds.

(C1) The symbol $\vdash$ appears even number of times.

(C2) u_0 is not the initial configuration.

(C3) For some $0 \leqslant i \leqslant n$, the string u_i is not a configuration.
It does not contain a state or the states appear at least twice or the brackets [and] do not appear "properly" or inside the bracket [and] is not a state of $\mathcal{M}$.

(C4) For some $0 \leqslant i \leqslant n-1$, the string $u_i \vdash u_i$ is not according to the transitions of $\mathcal{M}$.

(C5) For some $o \leqslant i \leqslant n-1$, the string u_i is not the reverse of u_{i+1} (disregarding the state symbol and the symbols next to the state in both u_i and u_{i+1}).

(C6) The last string u_n does not contain q_{acc}.

The construction of the CFG $\mathcal{G}$ – continued

We can construct one CFG $\mathcal{G}_i$ that generates all the strings that satisfy one condition (Ci), where $1 \leqslant i \leqslant 6$.

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It is useful to recall that CFL are closed union.

The construction of the CFG $\mathcal{G}$ – continued

We can construct one CFG $\mathcal{G}_i$ that generates all the strings that satisfy one condition (Ci), where $1 \leqslant i \leqslant 6$.

It is useful to recall that CFL are closed union.

The final CFG $\mathcal{G}$ generates $L(\mathcal{G}_1) \cup \cdots \cup L(\mathcal{G}_6)$.

The reduction HALT $\leqslant_T$ CFL-Universality

The following algorithm assumes that there is an algorithm for checking whether $L(\mathcal{G}) = \Sigma^*$.

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The following algorithm assumes that there is an algorithm for checking whether $L(\mathcal{G}) = \Sigma^*$.

On input $\lfloor \mathcal{M} \rfloor \$ w$, do the following.

- Construct the CFG $\mathcal{G}$ that generates words where at least one of (C1)–(C6) holds.

The reduction HALT $\leqslant_T$ CFL-Universality

The following algorithm assumes that there is an algorithm for checking whether $L(\mathcal{G}) = \Sigma^*$.

On input $\lfloor \mathcal{M} \rfloor \$ w$, do the following.

- Construct the CFG $\mathcal{G}$ that generates words where at least one of (C1)–(C6) holds.
- If $L(\mathcal{G}) = \Sigma^*$, then REJECT.
  If $L(\mathcal{G}) \neq \Sigma^*$, then ACCEPT.

The reduction HALT $\leqslant_T$ CFL-Universality

The following algorithm assumes that there is an algorithm for checking whether $L(\mathcal{G}) = \Sigma^*$.

On input $\lfloor \mathcal{M} \rfloor \$ w$, do the following.

- Construct the CFG $\mathcal{G}$ that generates words where at least one of (C1)–(C6) holds.
- If $L(\mathcal{G}) = \Sigma^*$, then REJECT.
  If $L(\mathcal{G}) \neq \Sigma^*$, then ACCEPT.

The algorithm is correct due to:

$$\lfloor \mathcal{M} \rfloor \$ w \in \text{HALT} \qquad \text{if and only if} \qquad L(\mathcal{G}) \neq \Sigma^*$$

To conclude:

CFL-Intersection and CFL-Universality are both undecidable.

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CFL-Intersection and CFL-Universality are both undecidable.

Consider the following problem.

CFL-Subset

Input: Two CFG $\mathcal{G}_1 = \langle \Sigma, V_1, R_1, S_1 \rangle$ and $\mathcal{G}_2 = \langle \Sigma, V_2, R_2, S_2 \rangle$, where $\Sigma = \{0, 1\}$.

Task: Output `True`, if $L(\mathcal{G}_1) \subseteq L(\mathcal{G}_2)$. Otherwise, output `False`.

To conclude:

CFL-Intersection and CFL-Universality are both undecidable.

Consider the following problem.

CFL-Subset

Input: Two CFG $\mathcal{G}_1 = \langle \Sigma, V_1, R_1, S_1 \rangle$ and $\mathcal{G}_2 = \langle \Sigma, V_2, R_2, S_2 \rangle$, where $\Sigma = \{0, 1\}$.
Task: Output `True`, if $L(\mathcal{G}_1) \subseteq L(\mathcal{G}_2)$. Otherwise, output `False`.

The following is a direct consequence of the undecidability of CFL-Universality.

Corollary 8.10
The problem CFL-Subset *is undecidable.*

End of Lesson 8