Lesson 4. Push-down automata

CSIE 3110 – Formal Languages and Automata Theory

Tony Tan
Department of Computer Science and Information Engineering
College of Electrical Engineering and Computer Science
National Taiwan University

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In PDA the stack can only store symbols.

A PDA works as follows.

- It moves from one state to another state (like an NFA).
- The move depends on the input symbol it currently reads and the symbol on top of the stack.
- During the move, it can pop the top symbol in the stack and push some other symbol into it.

Push-down automata (PDA) – The formal definition

(Def.) A *push-down automaton* (PDA) is a system $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$, where each of the component is as follows.

- Σ is a finite alphabet, called the *input* alphabet, whose elements are called *input symbols*.
- Γ is a finite alphabet, called the *stack* alphabet, whose elements are called *stack symbols*.
- Q is a finite set of states, $q_0 \in Q$ is the initial state. and $F \subseteq Q$ is the set of accepting states.
- δ is the set of transitions of the form: $(p, x, \mathrm{pop}(y)) \to (q, \mathrm{push}(z))$ where $x \in \Sigma \cup \{\varepsilon\}$ and $y, z \in \Gamma \cup \{\varepsilon\}$.

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(Note) The stack can only contain stack symbols.

Illustration of the transition $(p, x, \mathbf{pop}(y)) \rightarrow (q, \mathbf{push}(z))$ in PDA

$$p \xrightarrow[\text{push}(z)]{x,\ \text{pop}(y)} q$$

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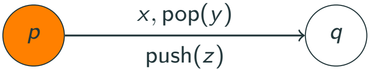

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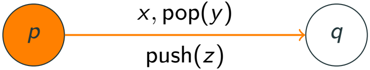

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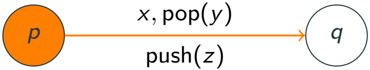

The input: $\ldots\ldots\ldots x \ldots\ldots\ldots$

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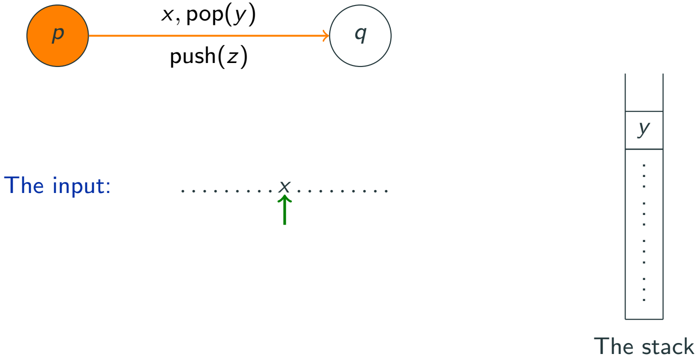

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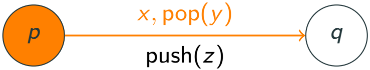

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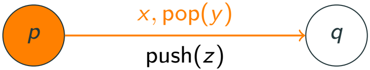

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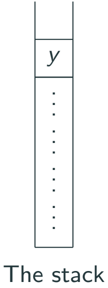

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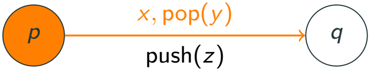

The input: . . . . x . . . .

The stack

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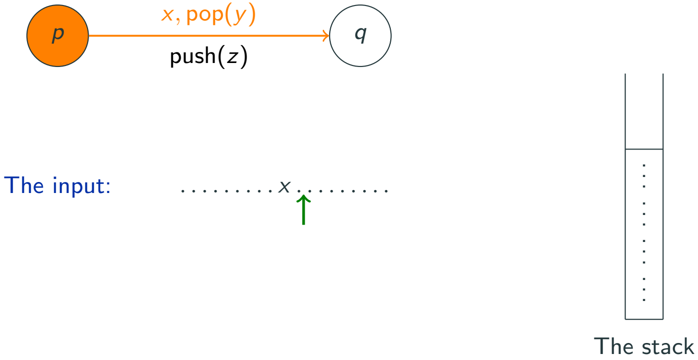

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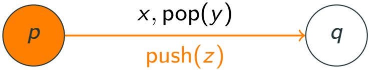

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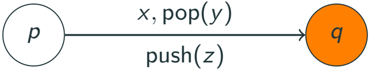

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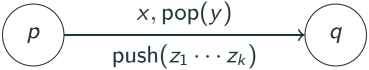
Illustration of PDA transitions – continued



The PDA “pushes” a string onto its stack:

- Read the symbol x.
- Pop y (if possible) from the stack.
- Push z_1 into the stack, then z_2, then z_3 and so on until z_k.
- Move into state q.

Illustration of PDA transitions – continued

$$p \xrightarrow[\text{push}(z)]{x,\ \text{pop}(\varepsilon)} q$$

When $y = \varepsilon$, the PDA does not check the content the stack.

- Read the symbol x.
- Push z into the stack.
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Illustration of PDA transitions – continued

$$p \xrightarrow[\text{push}(\varepsilon)]{x,\ \text{pop}(y)} q$$

When $z = \varepsilon$, the PDA does not push anything to the stack.

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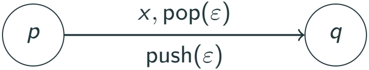
Illustration of PDA transitions – continued



When $y = z = \varepsilon$, the PDA does not do anything to the stack.

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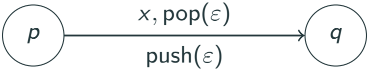

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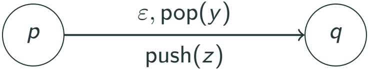
(Note) We can view NFA as a special case of PDA where the transitions are all of the form: $(p, x, \text{pop}(\varepsilon)) \to (q, \text{push}(\varepsilon))$.

Illustration of PDA transitions – continued



When $x = \varepsilon$, the PDA does not read any input symbol.

- Pop y (if possible) from the stack.
- Push z into the stack.
- Move into state q.

The criteria for the acceptance of a word by a PDA

Suppose $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$ is a PDA.

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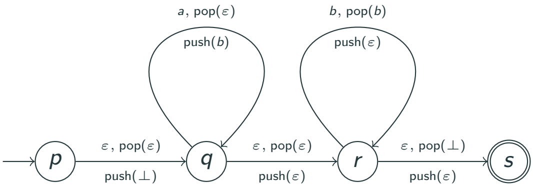
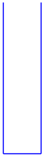
The language accepted by a PDA $\mathcal{A}$ is denoted by $L(\mathcal{A})$:

$$L(\mathcal{A}) \quad := \quad \{w \in \Sigma^* \mid \text{there is an accepting run of } \mathcal{A} \text{ on } w\}$$

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.





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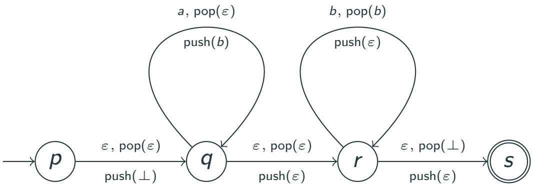
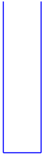


Input word: ε



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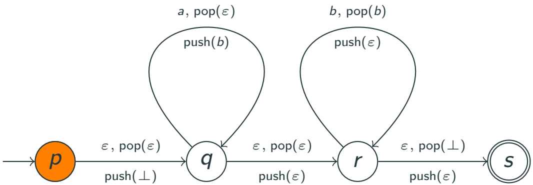

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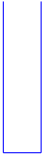

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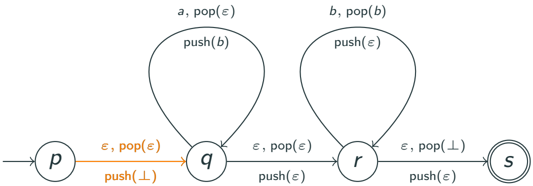

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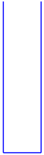

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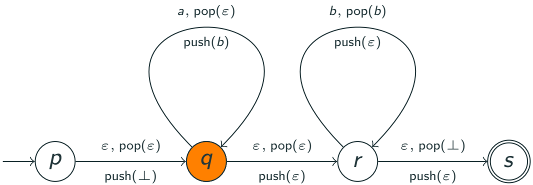

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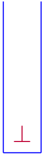

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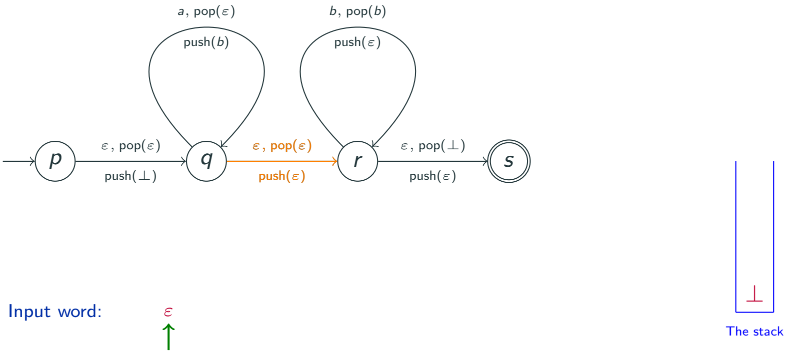

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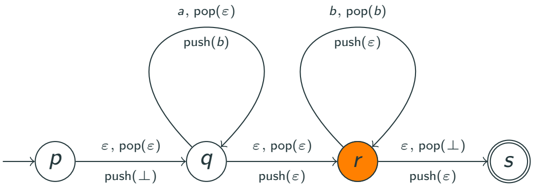

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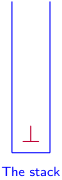

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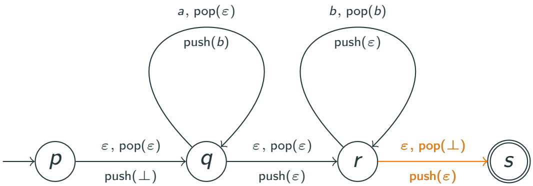

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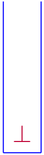

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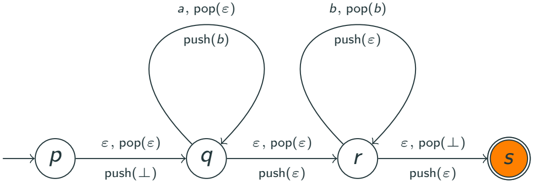

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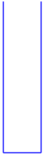

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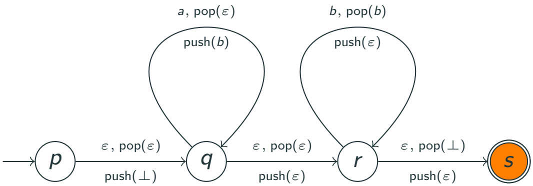

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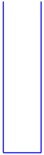

This is an accepting run of $\mathcal{A}$! So, $\varepsilon \in L(\mathcal{A})$.

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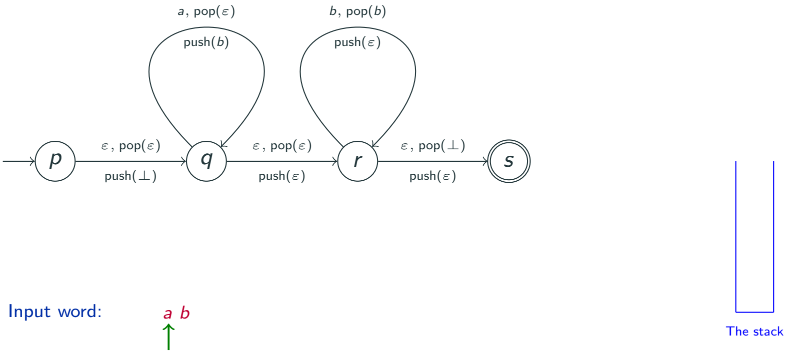

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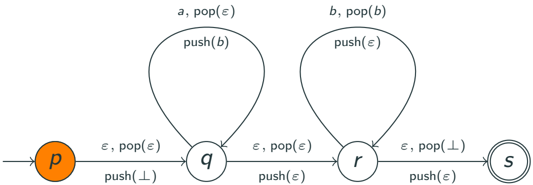

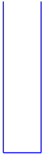

The stack

Input word: $a\ b$

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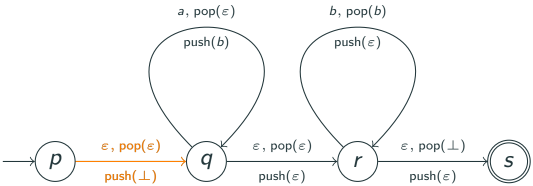

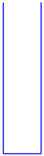

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$\uparrow$

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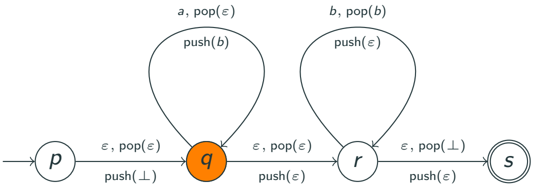

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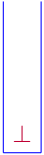

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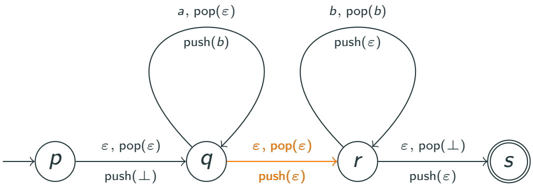
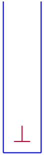
An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.

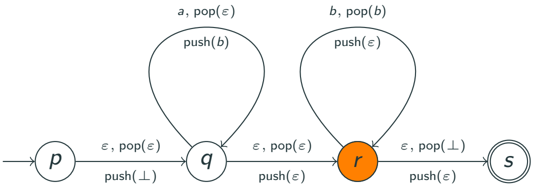


Input word: $a\ b$

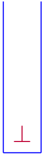


An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



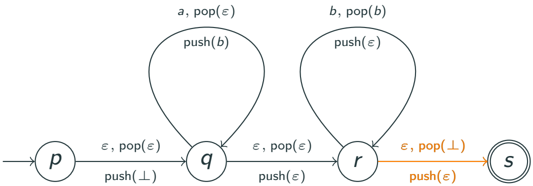
Input word: $a\ b$

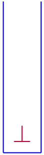


The stack

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \bot\}$.





The stack

Input word: $a\ b$

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

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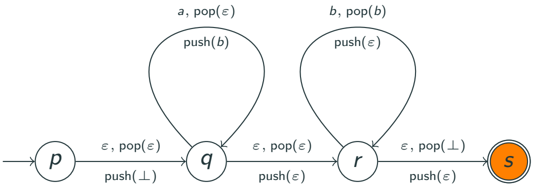
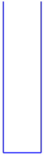


Input word: $a\ b$



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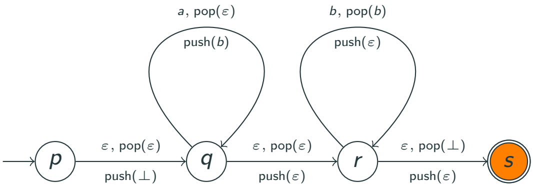
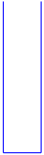




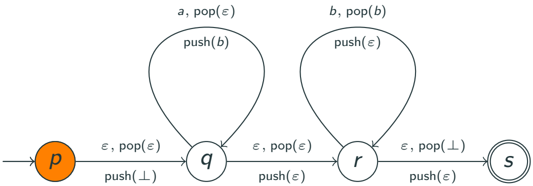
Input word: $a\ b$

The stack

This is not an accepting run of $\mathcal{A}$, because the PDA does not finish reading the input word.

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.

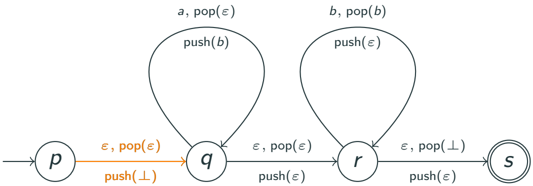


Input word: $a\ b$

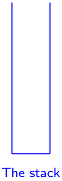
The stack

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ b$



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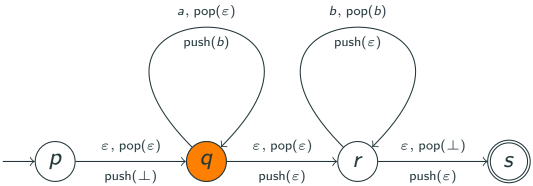
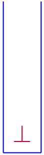
$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



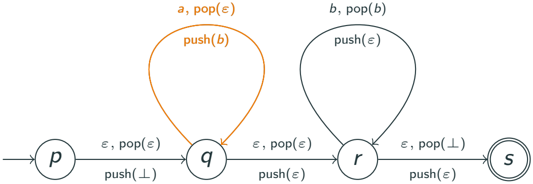


The stack

Input word: $a\ b$

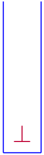
An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ b$

$\uparrow$



The stack

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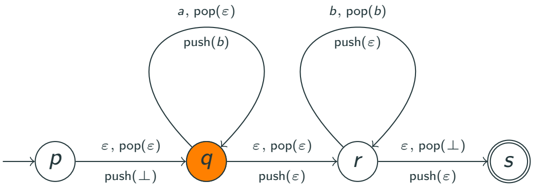
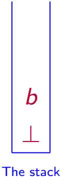


Input word: $a\ b$



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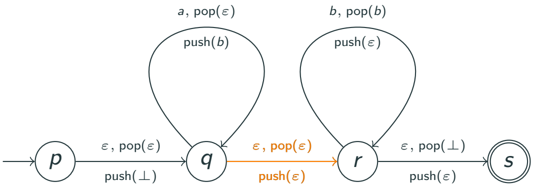

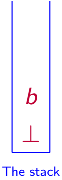

Input word: $a\ b$

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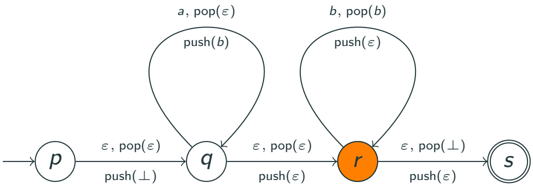

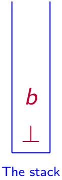

Input word: $a\ b$

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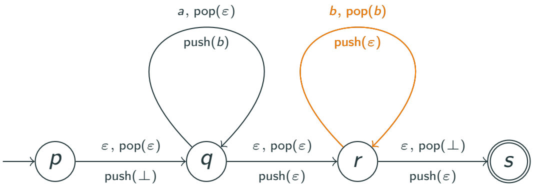
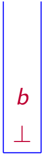


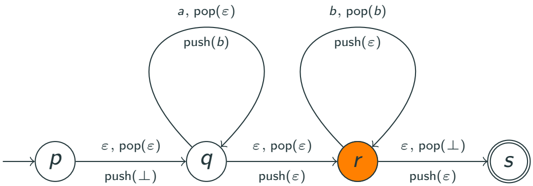


The stack

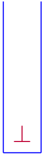
Input word: $a\ b$

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ b$



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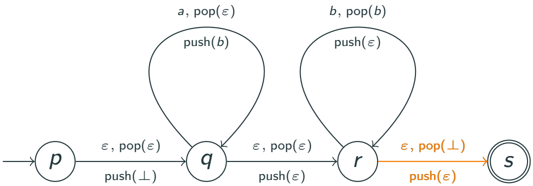
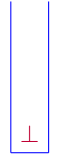


Input word: $a\ b$



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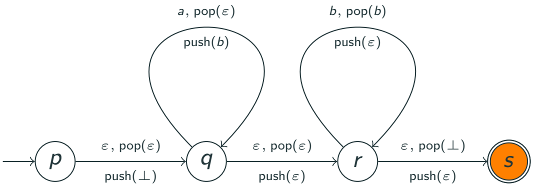
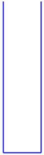


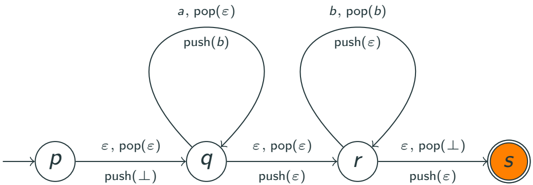
Input word: $a \ b$

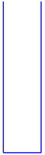


The stack

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



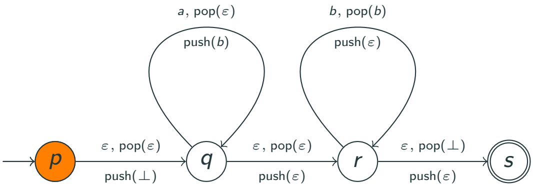


Input word: $a\ b$

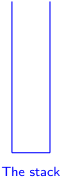
This is an accepting run of $\mathcal{A}$, because the PDA finishes reading the input word and ends in an accepting state.

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.

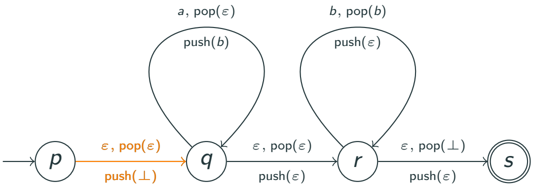


Input word: $a\ a\ a\ b\ b\ b$

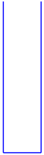


An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ a\ a\ b\ b\ b$



The stack

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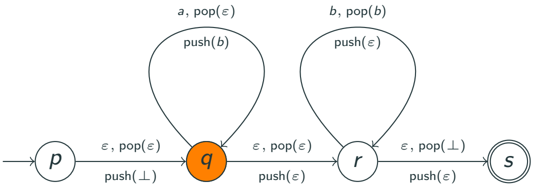
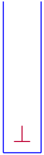
$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ a\ a\ b\ b\ b$



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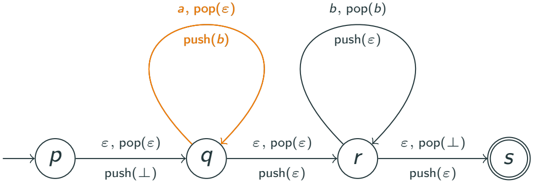
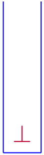
$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ a\ a\ b\ b\ b$



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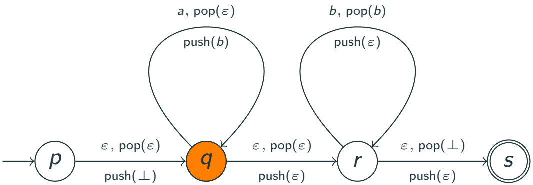
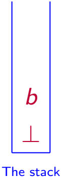
$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ a\ a\ b\ b\ b$



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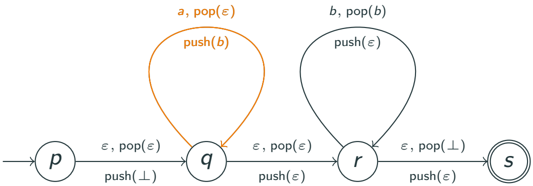
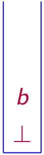
$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ a\ a\ b\ b\ b$



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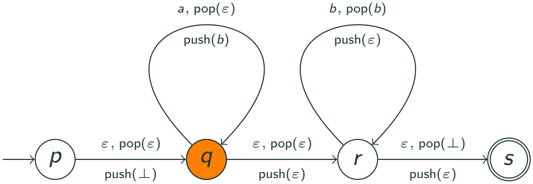
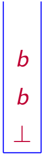
$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



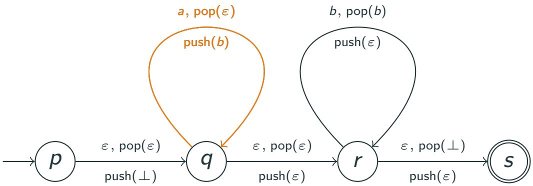
Input word: $a\ a\ a\ b\ b\ b$



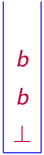
The stack

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ a\ a\ b\ b\ b$



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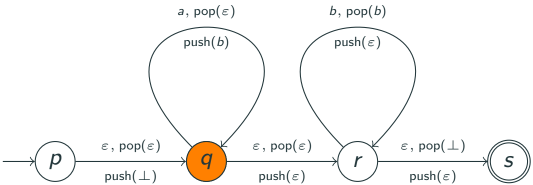
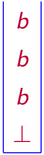
$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



Input word: $a\ a\ a\ b\ b\ b$



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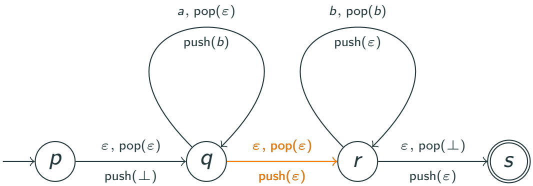
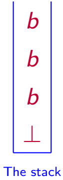
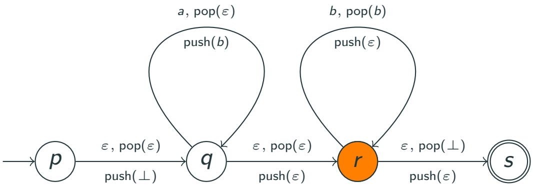




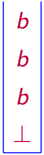
Input word: $a\ a\ a\ b\ b\ b$

An example: A PDA $\mathcal{A}$ for $L = \{a^n b^n | n \geqslant 0\}$

$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \bot\}$.



Input word: $a\ a\ a\ b\ b\ b$



The stack

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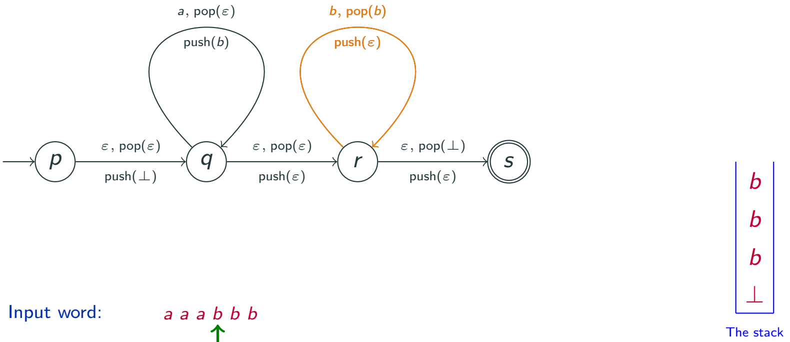

Input word: $a\ a\ a\ b\ b\ b$

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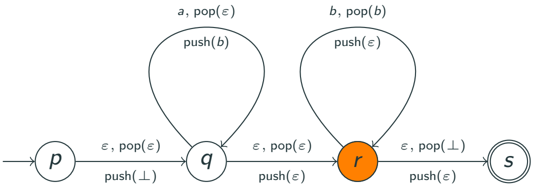

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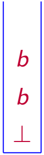

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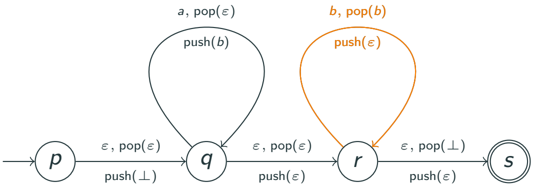
$\Sigma = \{a, b\}$ and $\Gamma = \{a, b, \perp\}$.



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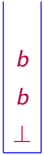

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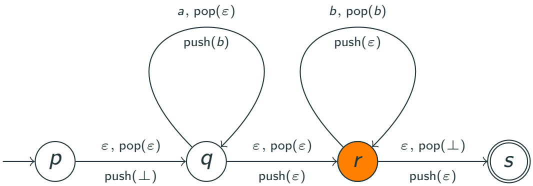

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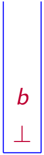

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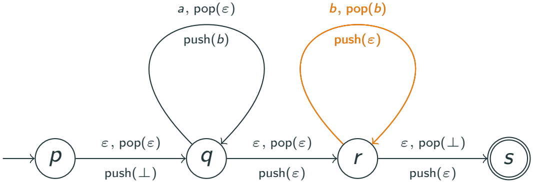

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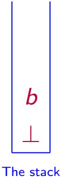

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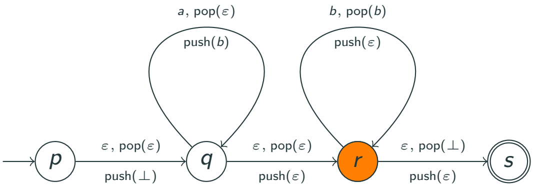

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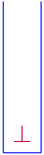

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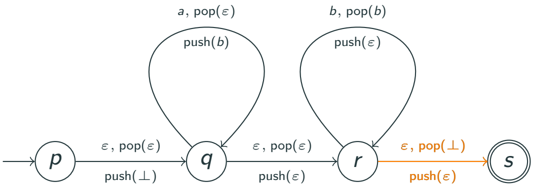

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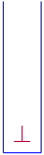

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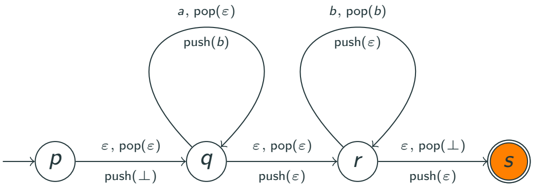

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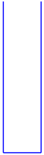

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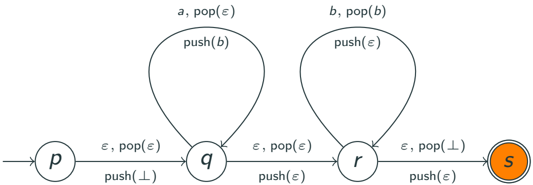

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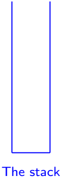

This is an accepting run of $\mathcal{A}$, because the PDA finishes reading the input word and ends in an accepting state.

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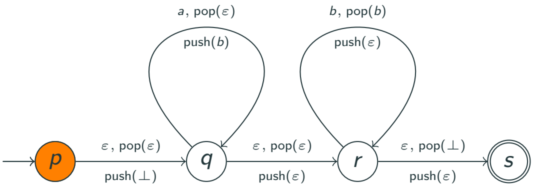

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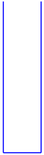

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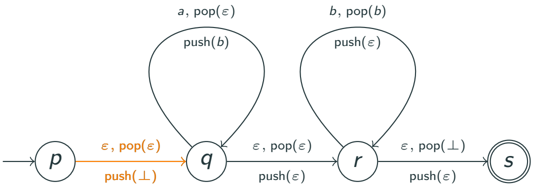

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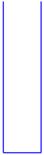

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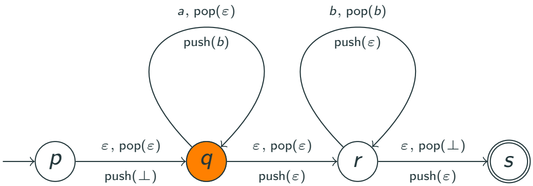

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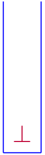

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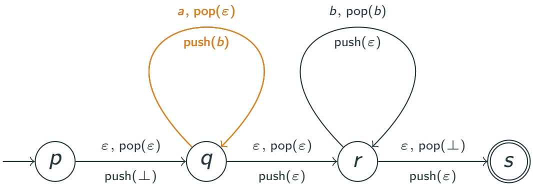

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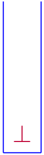

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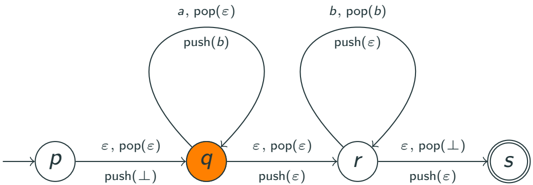

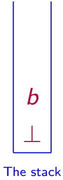

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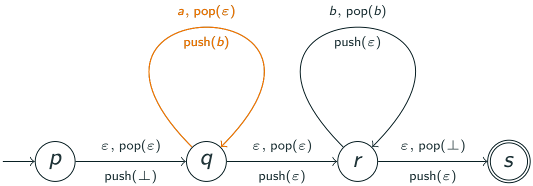

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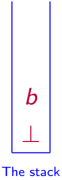

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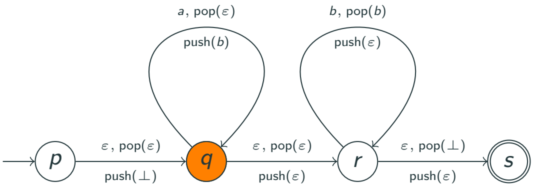

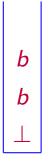

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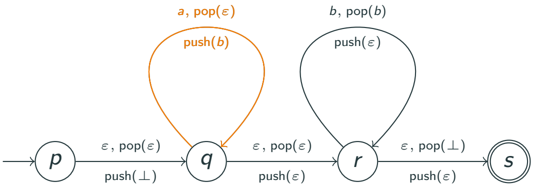

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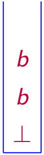

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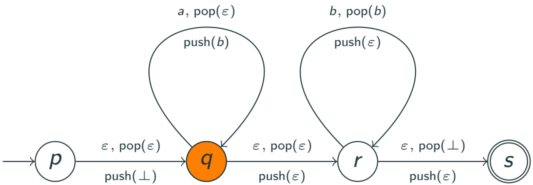

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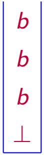

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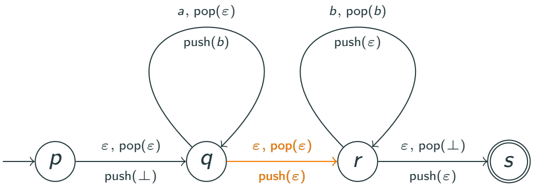

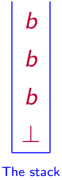

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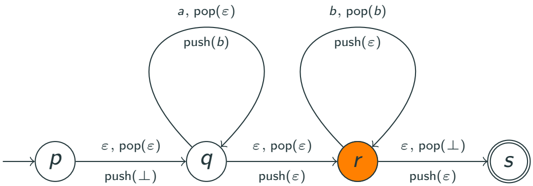

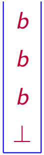

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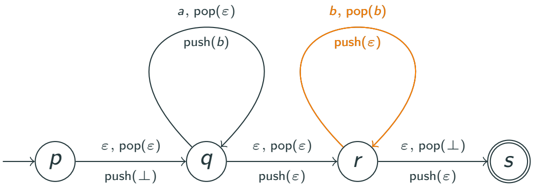

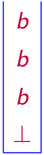

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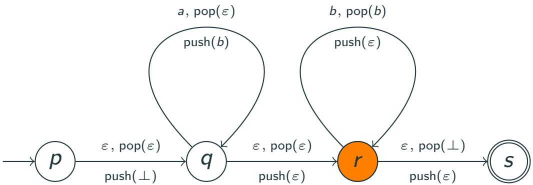

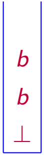

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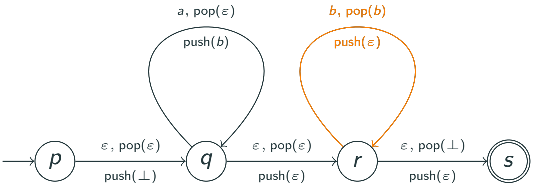

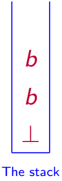

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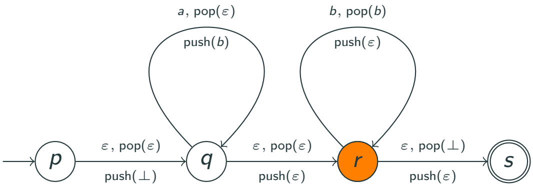

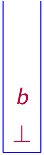

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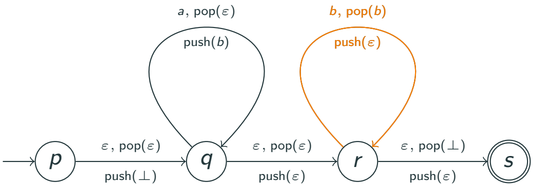

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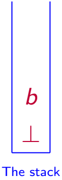

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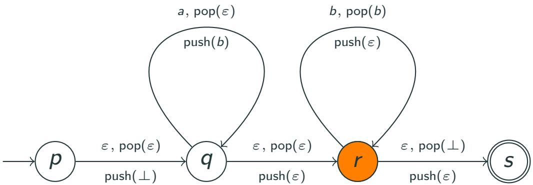

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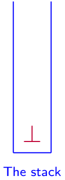

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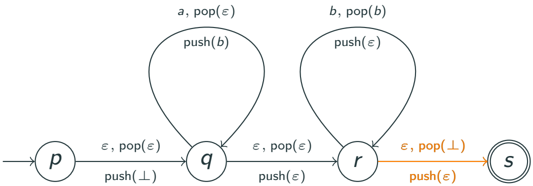

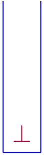

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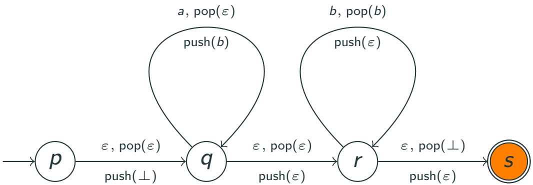

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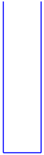

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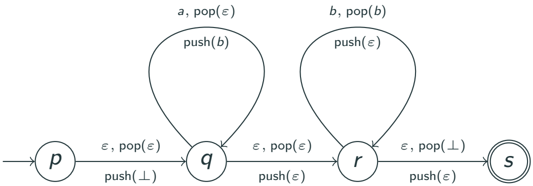

Note: $w \in L(\mathcal{A})$ if and only if w is of the form $a^n b^n$.

The equivalence between CFG and PDA

Theorem 4.1

- *For every CFG $\mathcal{G}$, there is a PDA $\mathcal{A}$ such that $L(\mathcal{A}) = L(\mathcal{G})$.*
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From CFG to PDA

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For every CFG $\mathcal{G}$, there is a PDA $\mathcal{A}$ such that $L(\mathcal{A}) = L(\mathcal{G})$.

The idea is to use a PDA to "simulate" the derivations of the words generated by the CFG.

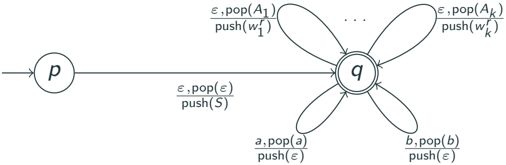
Proof of Theorem 4.1: From CFG to PDA

Let $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ where $\Sigma = \{a, b\}$ and $R = \{A_1 \to w_1, \ldots, A_k \to w_k\}$.

Proof of Theorem 4.1: From CFG to PDA

Let $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ where $\Sigma = \{a, b\}$ and $R = \{A_1 \to w_1, \ldots, A_k \to w_k\}$.

Consider the following PDA with $\Sigma = \{a, b\}$ and $\Gamma = V \cup \Sigma$:



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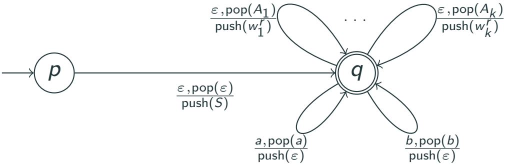

- The stack contains the strings to derive.

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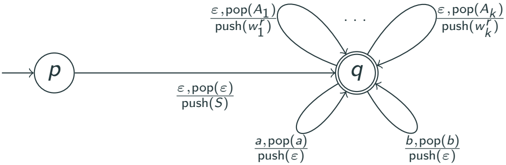

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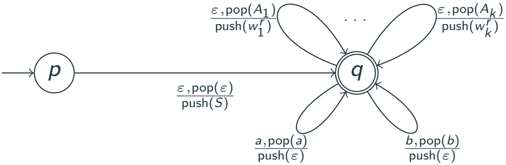

- The stack contains the strings to derive.
- The transition $(q, \varepsilon, \mathrm{pop}(A_i)) \to (q, \mathrm{push}(w_i^r))$ simulates the rule $A_i \to w_i$.
- The transitions $(q, a, \mathrm{pop}(\varepsilon)) \to (q, \mathrm{push}(\varepsilon))$ and $(q, b, \mathrm{pop}(\varepsilon)) \to (q, \mathrm{push}(\varepsilon))$ enforce the input word to be the same as the terminals that are already derived.

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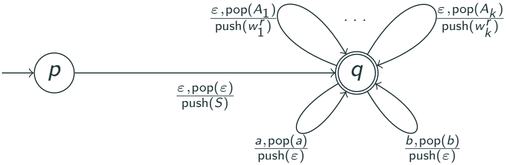
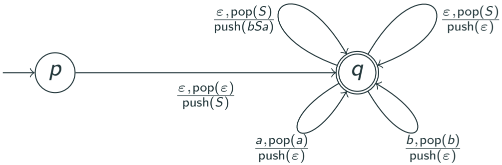


- The stack contains the strings to derive.
- The transition $(q, \varepsilon, \text{pop}(A_i)) \to (q, \text{push}(w_i^r))$ simulates the rule $A_i \to w_i$.
- The transitions $(q, a, \text{pop}(\varepsilon)) \to (q, \text{push}(\varepsilon))$ and $(q, b, \text{pop}(\varepsilon)) \to (q, \text{push}(\varepsilon))$ enforce the input word to be the same as the terminals that are already derived.
- The transition $(p, \varepsilon, \text{pop}(\varepsilon)) \to (q, \text{push}(S))$ enforces the words are derived only from the start variable S.

An example: From CFG to PDA

Let $\mathcal{G}$ be the CFG with rules: $S \to aSb \mid \varepsilon$.

The PDA is:



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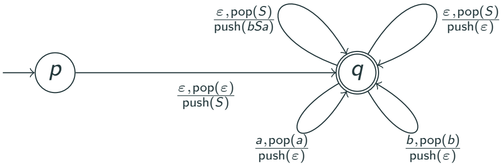
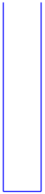


Input word: $a\ a\ a\ b\ b\ b$



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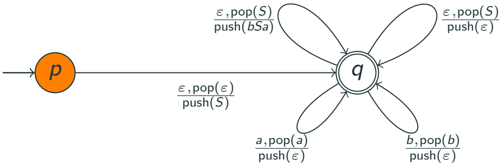

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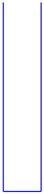

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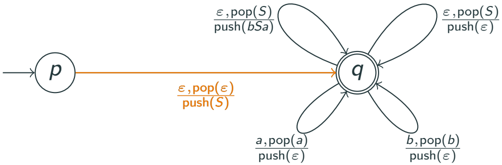
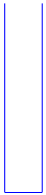


Input word: $a\ a\ a\ b\ b\ b$



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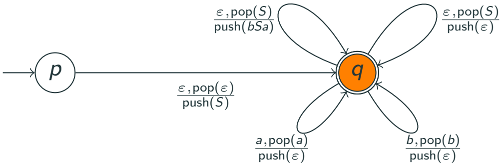
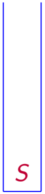


Input word: $a\ a\ a\ b\ b\ b$



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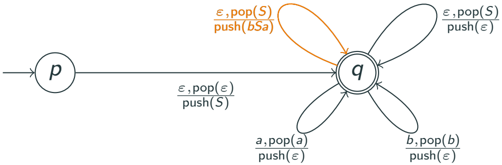
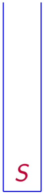


Input word: $a\ a\ a\ b\ b\ b$



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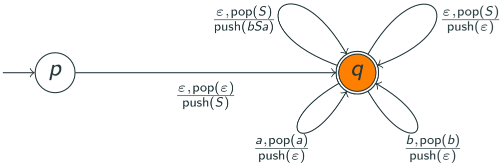

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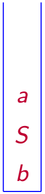

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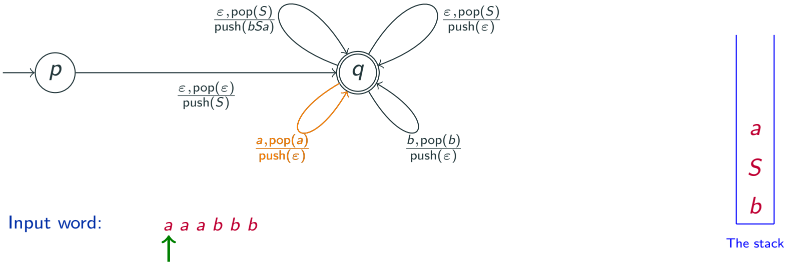

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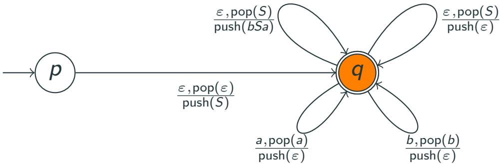

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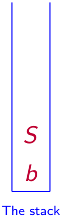

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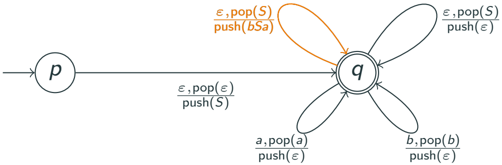

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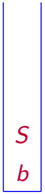

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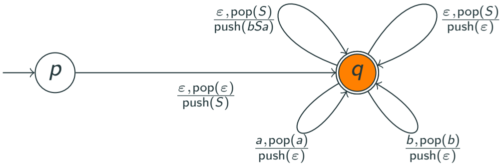

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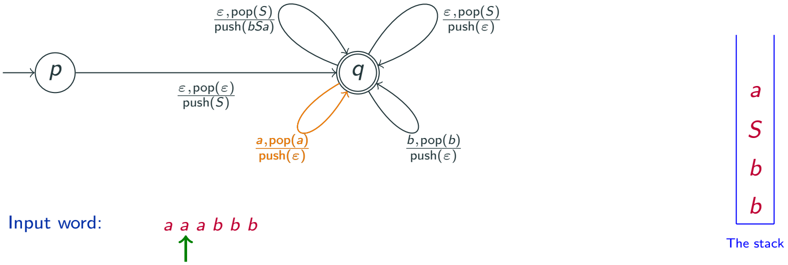

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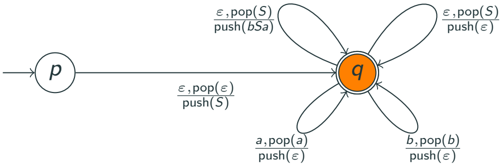

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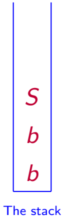

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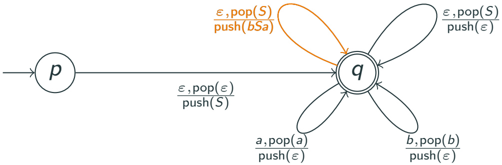

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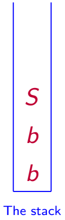

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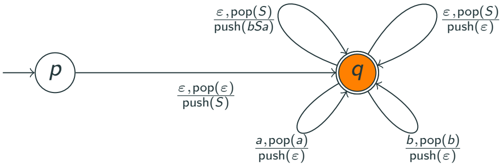

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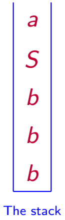

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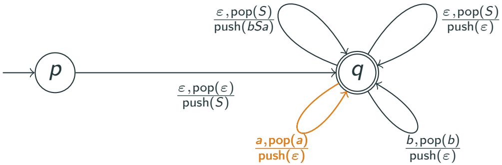

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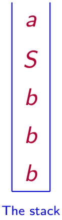

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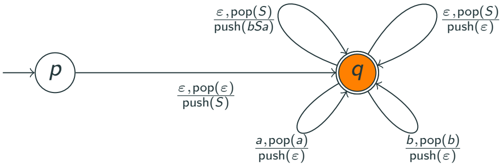
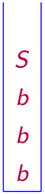


Input word: $a\ a\ a\ b\ b\ b$



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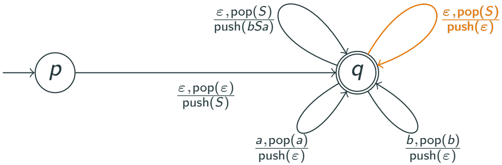

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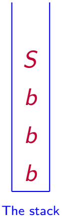

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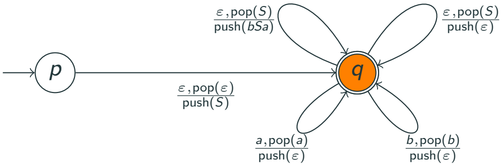

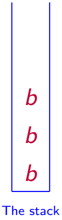

Input word: $a\ a\ a\ b\ b\ b$

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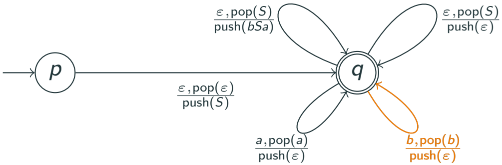

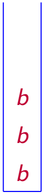

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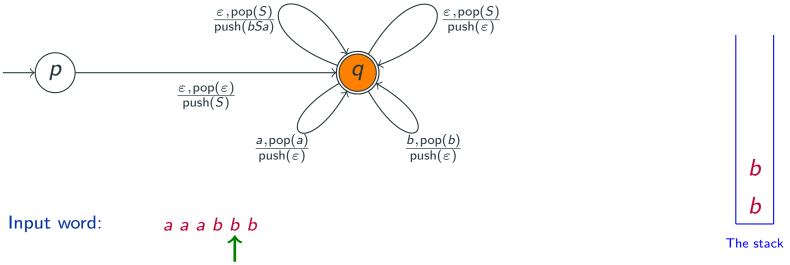

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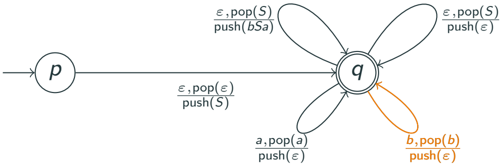

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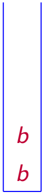

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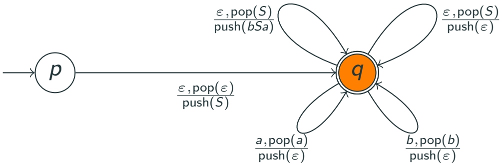
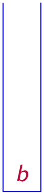


Input word: $a\ a\ a\ b\ b\ b$



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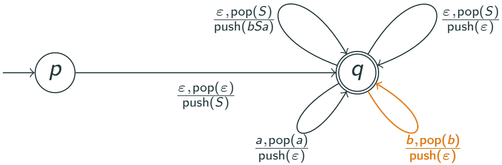
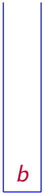


Input word: $a\ a\ a\ b\ b\ b$



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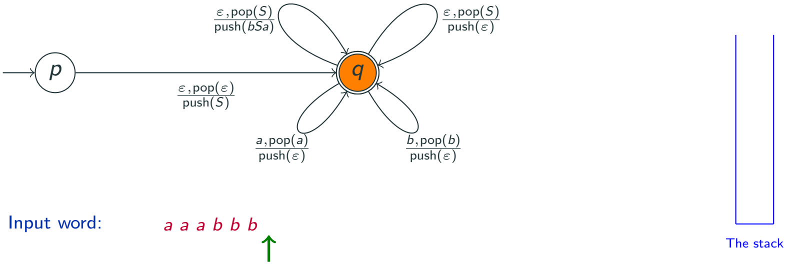

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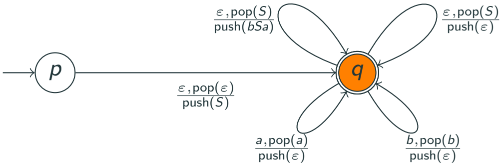
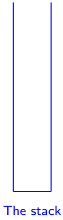


Input word: *a a a b b b*



So, the word *aaabbb* is accepted by the PDA.

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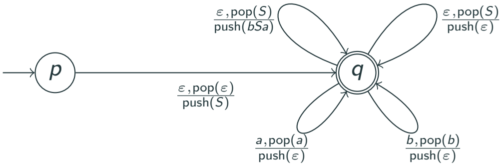
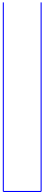


Input word: $a\ a\ a\ b\ b\ b\ a\ b$



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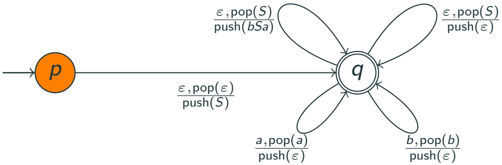
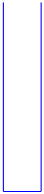


Input word: $a\ a\ a\ b\ b\ b\ a\ b$

$\uparrow$

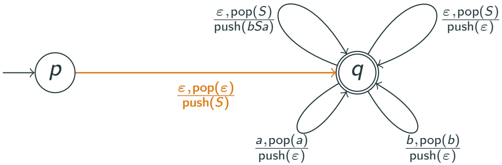


The stack

An example: From CFG to PDA

Let $\mathcal{G}$ be the CFG with rules: $S \to aSb \mid \varepsilon$.

The PDA is:



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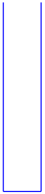

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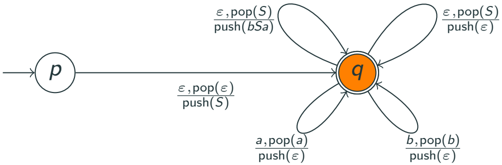

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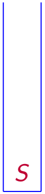

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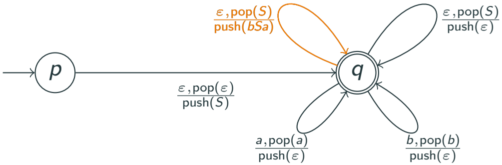

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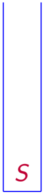

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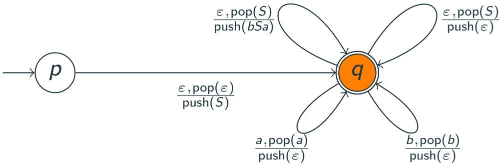

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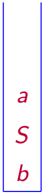

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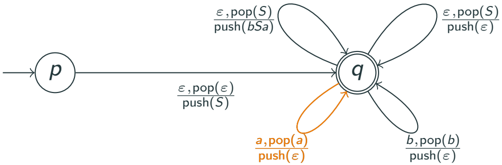

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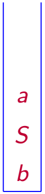

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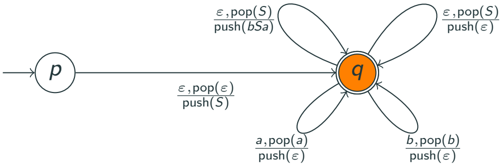

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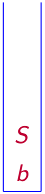

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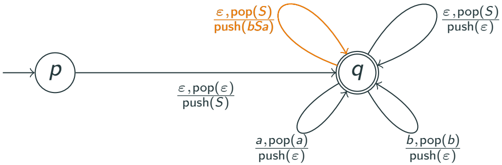

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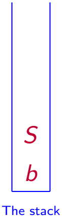

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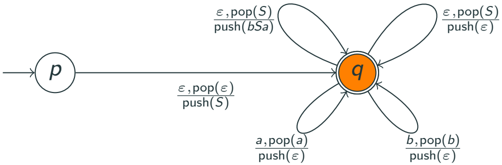

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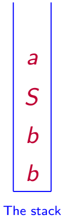

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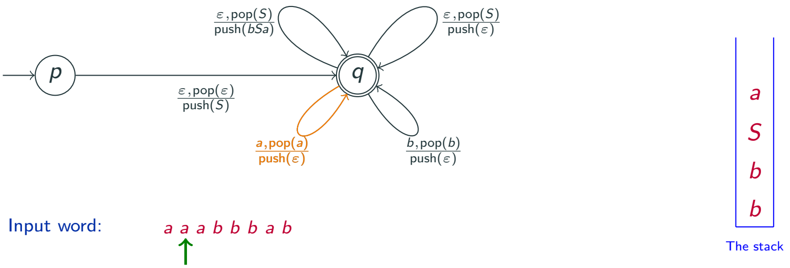

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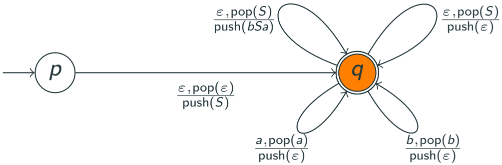

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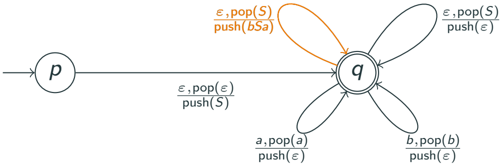

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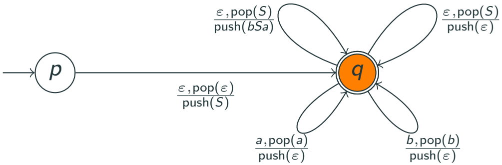

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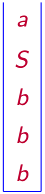

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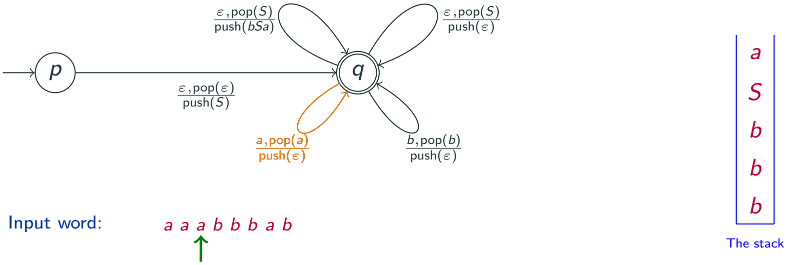

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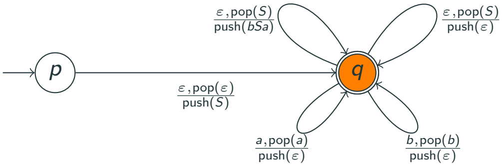

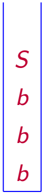

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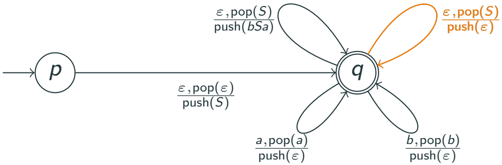

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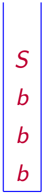

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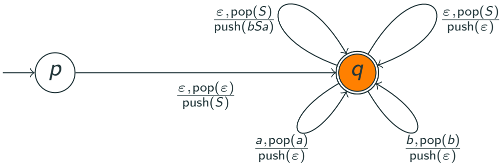

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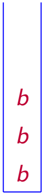

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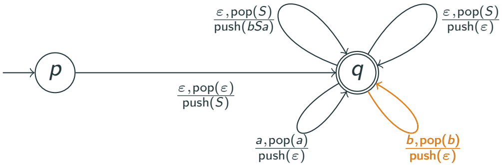

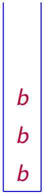

The stack

Input word: $a\ a\ a\ b\ b\ b\ a\ b$

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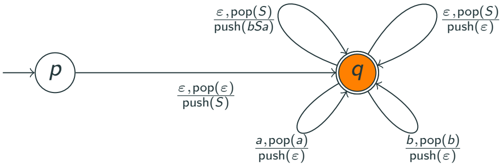

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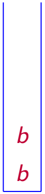

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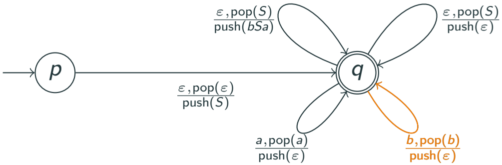

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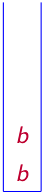

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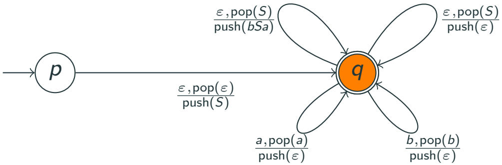

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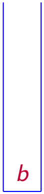

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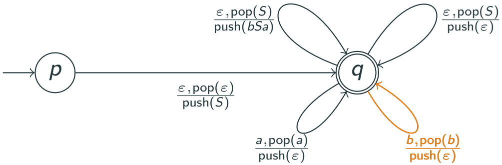

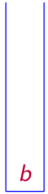

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Input word: $a\ a\ a\ b\ b\ b\ a\ b$

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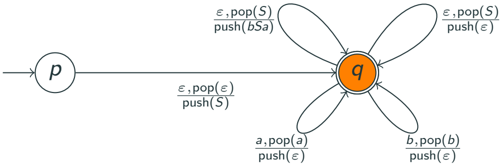

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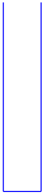

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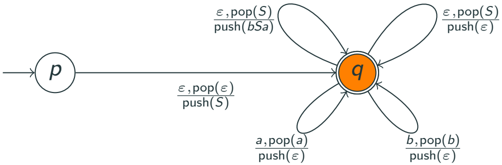
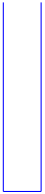




The stack

Input word: $a\ a\ a\ b\ b\ b\ a\ b$

The word *aaabbbab* is not accepted by the PDA, because the PDA cannot finish reading it.

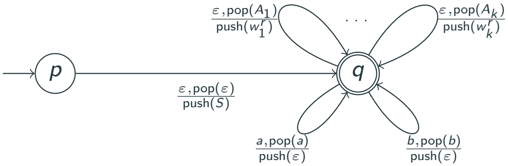
Proof of Theorem 4.1: From CFG to PDA – continued

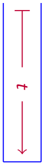
Let $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ where $\Sigma = \{a, b\}$ and $R = \{A_1 \to w_1, \ldots, A_k \to w_k\}$.

Proof of Theorem 4.1: From CFG to PDA – continued

Let $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ where $\Sigma = \{a, b\}$ and $R = \{A_1 \to w_1, \ldots, A_k \to w_k\}$.

Consider the following PDA with $\Sigma = \{a, b\}$ and $\Gamma = V \cup \Sigma \cup \{\bot\}$:





The stack

Input word: $\longleftarrow u \longrightarrow\longleftarrow v \longrightarrow$

Proof of Theorem 4.1: From CFG to PDA – continued

Let $\mathcal{G} = \langle \Sigma, V, R, S \rangle$ where $\Sigma = \{a, b\}$ and $R = \{A_1 \to w_1, \ldots, A_k \to w_k\}$.

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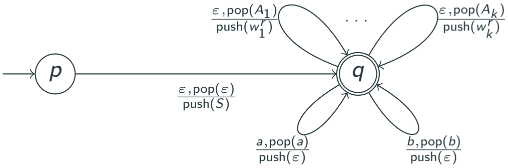

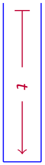

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Input word:

(Claim.) $t \Rightarrow^* v$ if and only if $(q, t) \vdash_v^* (q, \varepsilon)$.

Proof of Theorem 4.1: From CFG to PDA – continued

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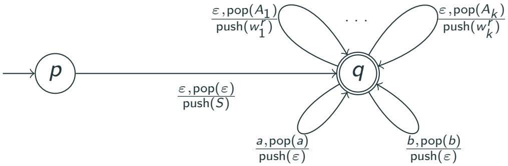

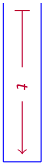

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(Claim.) $t \Rightarrow^* v$ if and only if $(q, t) \vdash_v^* (q, \varepsilon)$.

In particular, when $t = S$ and $u = \varepsilon$, the following holds:
$S \Rightarrow^* v$ if and only if $(q, S) \vdash_v^* (q, \varepsilon)$.

Table of contents

From PDA to CFG

Theorem (The second item of Theorem 4.1)

For every PDA $\mathcal{A}$, there is a CFG $\mathcal{G}$ such that $L(\mathcal{A}) = L(\mathcal{G})$.

From PDA to CFG

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For every PDA $\mathcal{A}$, there is a CFG $\mathcal{G}$ such that $L(\mathcal{A}) = L(\mathcal{G})$.

The idea is to use variables in CFG to "simulate" the runs of the PDA.

Proof of Theorem 4.1: From PDA to CFG

Let $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$ be a PDA.

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Without loss of generality, we can assume the following.

- It has only one final state, say q_f. That is, $F = \{q_f\}$.

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Without loss of generality, we can assume the following.

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- The stack is empty before accepting an input word.
  More precisely, on every word w, if $\mathcal{A}$ accepts w, there is an accepting run of $\mathcal{A}$ on w from the initial configuration (q_0, ε) to a final configuration (q_f, ε) where the content of the stack is empty.

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  More precisely, on every word w, if $\mathcal{A}$ accepts w, there is an accepting run of $\mathcal{A}$ on w from the initial configuration (q_0, ε) to a final configuration (q_f, ε) where the content of the stack is empty.
- In each transition, $\mathcal{A}$ either pushes a symbol into the stack or pops one from the stack, but it cannot do both.

Proof of Theorem 4.1: From PDA to CFG

Let $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$ be a PDA.

Without loss of generality, we can assume the following.

- It has only one final state, say q_f. That is, $F = \{q_f\}$.
- The stack is empty before accepting an input word.
  More precisely, on every word w, if $\mathcal{A}$ accepts w, there is an accepting run of $\mathcal{A}$ on w from the initial configuration (q_0, ε) to a final configuration (q_f, ε) where the content of the stack is empty.
- In each transition, $\mathcal{A}$ either pushes a symbol into the stack or pops one from the stack, but it cannot do both.
  More precisely, every transition can only be of the forms:

$$
\begin{array}{rcl}
(p, x, \mathrm{pop}(y)) & \rightarrow & (q, \mathrm{push}(\varepsilon)) \\
(p, x, \mathrm{pop}(\varepsilon)) & \rightarrow & (q, \mathrm{push}(z))
\end{array}
$$

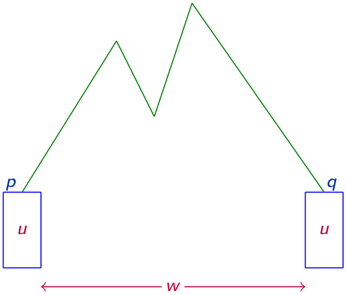
Proof of Theorem 4.1: From PDA to CFG – continued

Let $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$ be a PDA.

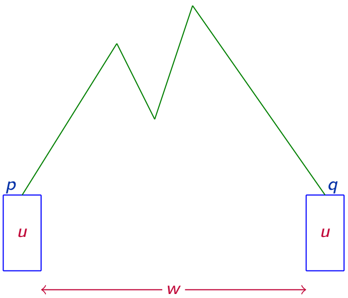
Consider the following CFG $\mathcal{G} = \langle \Sigma, V, R, S \rangle$:

- $V = \{A_{p,q} \mid p, q \in Q\}$.
- A_{q_0,q_f} is the start variable.
- The set of rules R will be defined later.

The intuitive meaning of the variable $A_{p,q}$



The intuitive meaning of the variable $A_{p,q}$



Then, $A_{p,q} \Rightarrow^* w$.

Defining the rules in R – part. 1

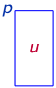
For every transition:

$$(p, a, \mathrm{pop}(\varepsilon)) \to (r, \mathrm{push}(z)) \quad \text{and} \quad (s, b, \mathrm{pop}(z)) \to (q, \mathrm{push}(\varepsilon))$$

the following rule is in R:

$$A_{p,q} \ \to \ a \ A_{r,s} \ b$$

Defining the rules in R – part. 1



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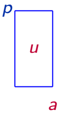

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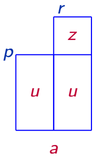

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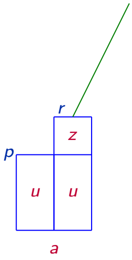

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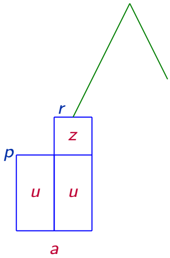

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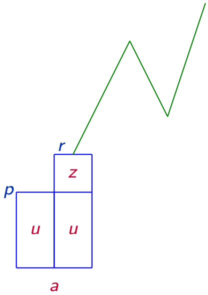

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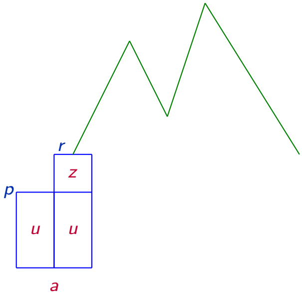

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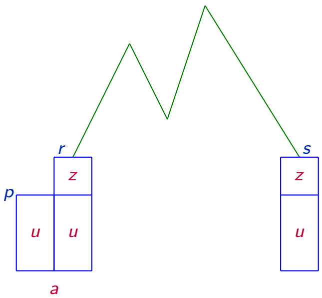

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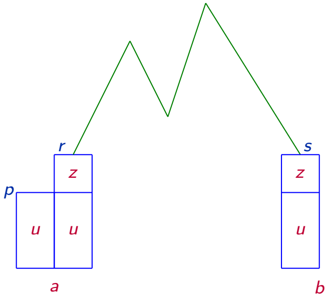

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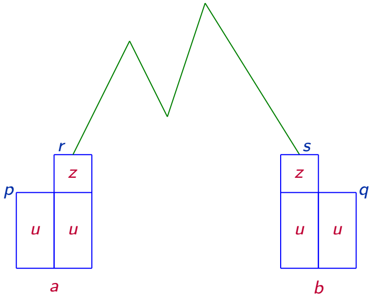

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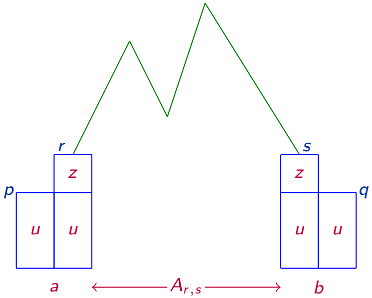

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Defining the rules in R – part. 2

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$$(p, a, \mathrm{pop}(\varepsilon)) \;\rightarrow\; (r, \mathrm{push}(\varepsilon))$$

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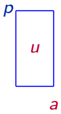

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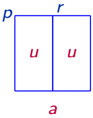

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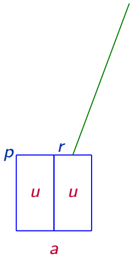

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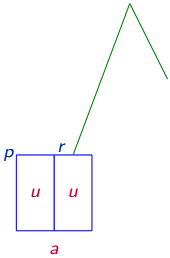

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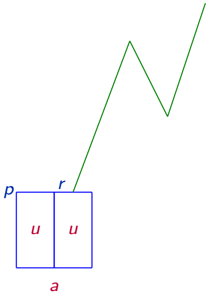

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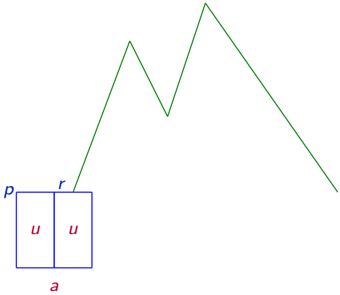

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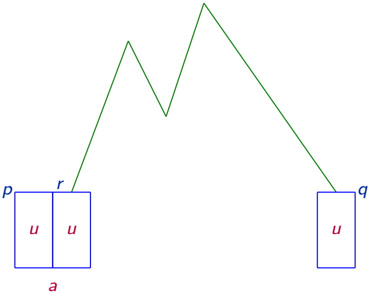

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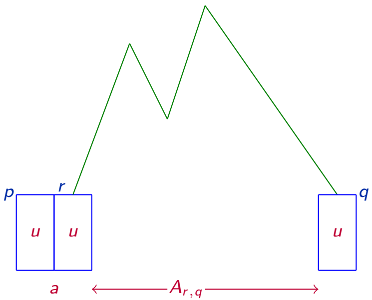

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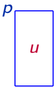
Defining the set of rules in R – part. 3

For every $r \in Q$, the following rule is in R:

$$A_{p,q} \;\rightarrow\; A_{p,r}\; A_{r,q}$$

Defining the set of rules in R – part. 3



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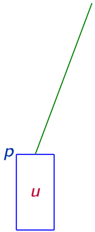
Defining the set of rules in R – part. 3



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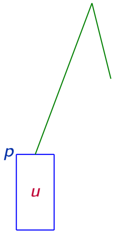

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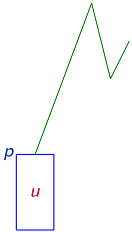

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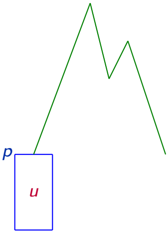

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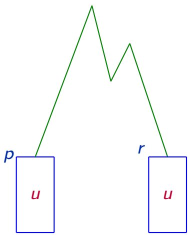

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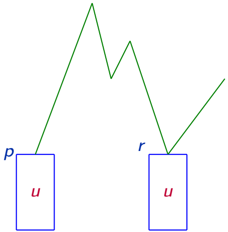
Defining the set of rules in R – part. 3



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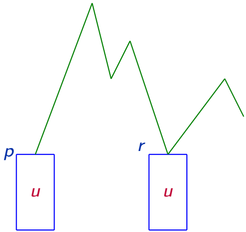

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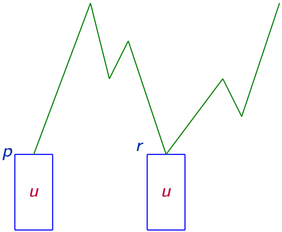

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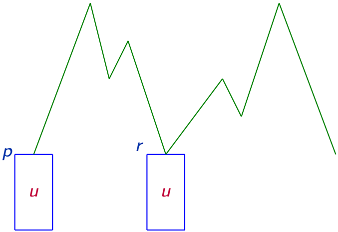
Defining the set of rules in R – part. 3



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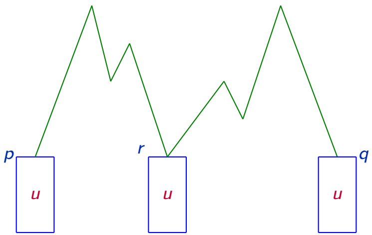
Defining the set of rules in R – part. 3



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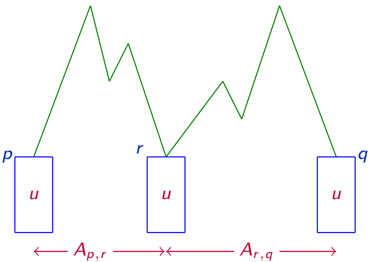
Defining the set of rules in R – part. 3



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Defining the set of rules in R – part. 4

For every $p \in Q$, the following rule is in R:

$$A_{p,p} \rightarrow \varepsilon$$

Defining the set of rules in R – part. 4

For every $p \in Q$, the following rule is in R:

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Proof of Theorem 4.1: From PDA to CFG – continued

For a PDA $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$, its equivalent CFG is:

Proof of Theorem 4.1: From PDA to CFG – continued

For a PDA $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$, its equivalent CFG is:

$\mathcal{G} = \langle \Sigma, V, R, S \rangle$, where $V = \{A_{p,q} \mid p, q \in Q\}$ and A_{q_0,q_f} is the start variable.

Proof of Theorem 4.1: From PDA to CFG – continued

For a PDA $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$, its equivalent CFG is:

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The rules in R are as follows.

- For every transition $(p, a, \mathrm{pop}(\varepsilon)) \to (r, \mathrm{push}(z))$ and $(s, b, \mathrm{pop}(z)) \to (q, \mathrm{push}(\varepsilon))$, the rule $A_{p,q} \; \to \; a \; A_{r,s} \; b$ is in R.
- For every transition $(p, a, \mathrm{pop}(\varepsilon)) \; \to \; (r, \mathrm{push}(\varepsilon))$, the rule $A_{p,q} \; \to \; a \; A_{r,q}$ is in R.
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Proof of Theorem 4.1: From PDA to CFG – continued

For a PDA $\mathcal{A} = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$, its equivalent CFG is:

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The rules in R are as follows.

- For every transition $(p, a, \text{pop}(\varepsilon)) \to (r, \text{push}(z))$ and $(s, b, \text{pop}(z)) \to (q, \text{push}(\varepsilon))$, the rule $A_{p,q} \to a\ A_{r,s}\ b$ is in R.
- For every transition $(p, a, \text{pop}(\varepsilon)) \to (r, \text{push}(\varepsilon))$, the rule $A_{p,q} \to a\ A_{r,q}$ is in R.
- For every $r \in Q$, the rule $A_{p,q} \to A_{p,r}\ A_{r,q}$ is in R.
- For every $p \in Q$, the rule $A_{p,p} \to \varepsilon$ is in R.

Claim

$L(\mathcal{A}) = L(\mathcal{G})$.

From PDA to CFG

Theorem (The second item of Theorem 4.1)

For every PDA $\mathcal{A}$, there is a CFG $\mathcal{G}$ such that $L(\mathcal{A}) = L(\mathcal{G})$.

To conclude:

CFG and PDA express exactly the same class of languages.

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Theorem 4.1

- *For every CFG $\mathcal{G}$, there is a PDA $\mathcal{A}$ such that $L(\mathcal{A}) = L(\mathcal{G})$.*
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End of Lesson 4