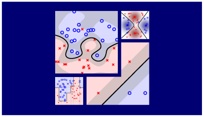
Machine Learning Techniques
(機器學習技法)



Lecture 11: Gradient Boosted Decision Tree

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Roadmap

1. Embedding Numerous Features: Kernel Models
2. Combining Predictive Features: Aggregation Models

Lecture 10: Random Forest

bagging of **randomized C&RT** trees with **automatic validation** and **feature selection**

Lecture 11: Gradient Boosted Decision Tree

- Adaptive Boosted Decision Tree
- Optimization View of AdaBoost
- Gradient Boosting
- Summary of Aggregation Models

3. Distilling Implicit Features: Extraction Models

From Random Forest to AdaBoost-DTree

function RandomForest($\mathcal{D}$)
For $t = 1, 2, \ldots, T$

1. request size-N' data $\tilde{\mathcal{D}}_t$ by bootstrapping with $\mathcal{D}$
2. obtain tree g_t by Randomized-DTree($\tilde{\mathcal{D}}_t$)

return $G = \text{Uniform}(\{g_t\})$

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1. reweight data by $\mathbf{u}^{(t)}$

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need: weighted DTree($\mathcal{D}, \mathbf{u}^{(t)}$)

Weighted Decision Tree Algorithm

Weighted Algorithm

minimize (regularized) $E_{\text{in}}^{\mathbf{u}}(h) = \frac{1}{N} \sum_{n=1}^{N} u_n \cdot \text{err}(y_n, h(\mathbf{x}_n))$

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weights $\mathbf{u}$ expressed by bootstrap-sampled copies

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AdaBoost-DTree: often via
AdaBoost + **sampling** $\propto \mathbf{u}^{(t)}$ + DTree($\tilde{\mathcal{D}}_t$)
without modifying DTree

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AdaBoost: **votes** $\alpha_t = \ln \blacklozenge_t = \ln \sqrt{\frac{1-\epsilon_t}{\epsilon_t}}$ with weighted error rate ϵ_t

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AdaBoost-DTree: often via AdaBoost + **sampling** $\propto \mathbf{u}^{(t)}$ + **pruned** DTree($\tilde{\mathcal{D}}$)

AdaBoost with Extremely-Pruned Tree

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DTree (C&RT) with **height** ≤ 1

learn branching criteria

$$b(\mathbf{x}) = \underset{\text{decision stumps } h(\mathbf{x})}{\operatorname{argmin}} \sum_{c=1}^{2} |\mathcal{D}_c \text{ with } h| \cdot \text{impurity}(\mathcal{D}_c \text{ with } h)$$

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AdaBoost-Stump
= **special case** of AdaBoost-DTree

Fun Time

When running AdaBoost-DTree with sampling and getting a decision tree g_t such that g_t achieves zero error on the sampled data set $\tilde{D}_t$. Which of the following is possible?

1. $\alpha_t < 0$
2. $\alpha_t = 0$
3. $\alpha_t > 0$
4. all of the above

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Reference Answer: 4

While g_t achieves zero error on $\tilde{\mathcal{D}}_t$, g_t may not achieve zero weighted error on $(\mathcal{D}, \mathbf{u}^{(t)})$ and hence ϵ_t can be anything, even $\geq \frac{1}{2}$. Then, α_t can be ≤ 0.

Example Weights of AdaBoost

$$u_n^{(t+1)} = \begin{cases} u_n^{(t)} \cdot \blacklozenge_t & \textbf{if incorrect} \\ u_n^{(t)} / \blacklozenge_t & \textbf{if correct} \end{cases}$$

$$= u_n^{(t)} \cdot \blacklozenge_t$$

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AdaBoost: $u_n^{(T+1)} \propto \exp(-y_n(\boxed{\text{voting score on } \mathbf{x}_n}))$

Voting Score and Margin

linear blending = LinModel + hypotheses as transform + ~~constraints~~

$$G(\mathbf{x}_n) = \text{sign}\left(\overbrace{\sum_{t=1}^{T} \underbrace{\alpha_t}_{w_i} \underbrace{g_t(\mathbf{x}_n)}_{\phi_i(\mathbf{x}_n)}}^{\text{voting score}}\right)$$

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claim: AdaBoost **decreases** $\sum_{n=1}^{N} u_n^{(t)}$

AdaBoost Error Function

claim: AdaBoost decreases $\sum_{n=1}^{N} u_n^{(t)}$ and thus somewhat **minimizes**

$$\sum_{n=1}^{N} u_n^{(T+1)} = \frac{1}{N} \sum_{n=1}^{N} \exp\left(-y_n \sum_{t=1}^{T} \alpha_t g_t(\mathbf{x}_n)\right)$$

AdaBoost Error Function

claim: AdaBoost decreases $\sum_{n=1}^{N} u_n^{(t)}$ and thus somewhat **minimizes**

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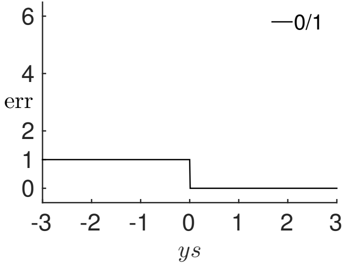

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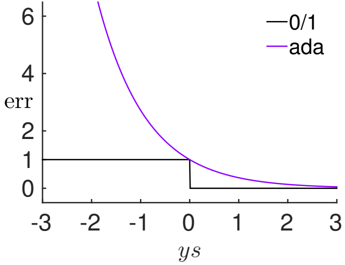

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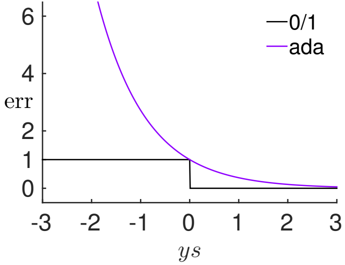

$\widehat{\mathrm{err}}_{\mathrm{ADA}}$: **algorithmic error measure** by **convex upper bound** of $\mathrm{err}_{0/1}$

Gradient Descent on AdaBoost Error Function

recall: gradient descent (**remember? :-)**), at iteration t

$$\min_{\|\mathbf{v}\|=1} E_{\text{in}}(\mathbf{w}_t + \eta\mathbf{v}) \approx \underbrace{E_{\text{in}}(\mathbf{w}_t)}_{\text{known}} + \underbrace{\eta}_{\text{given positive}} \mathbf{v}^T \underbrace{\nabla E_{\text{in}}(\mathbf{w}_t)}_{\text{known}}$$

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at iteration t, to find g_t, solve

$$\min_{h} \quad \widehat{E}_{\text{ADA}} = \frac{1}{N}\sum_{n=1}^{N} \exp\left(-y_n\left(\sum_{\tau=1}^{t-1}\alpha_\tau g_\tau(\mathbf{x}_n) + \eta h(\mathbf{x}_n)\right)\right)$$

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$$\overset{\text{taylor}}{\approx} \sum_{n=1}^{N} u_n^{(t)} \left(\qquad\qquad \right)$$

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\end{aligned}$$

good h: minimize $\sum_{n=1}^{N} u_n^{(t)} \left(-y_n h(\mathbf{x}_n)\right)$

Learning Hypothesis as Optimization

finding good h (function direction) $\Leftrightarrow$ minimize $\sum_{n=1}^{N} u_n^{(t)} \left(-y_n h(\mathbf{x}_n)\right)$

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$\mathcal{A}$: **good** $g_t = h$ for 'gradient descent'

Deciding Blending Weight as Optimization

AdaBoost finds g_t by approximately $\min_{h} \widehat{E}_{\text{ADA}} = \sum_{n=1}^{N} u_n^{(t)} \exp\left(-y_n \eta h(\mathbf{x}_n)\right)$

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Deciding Blending Weight as Optimization

AdaBoost finds g_t by approximately $\min_{h} \widehat{E}_{\text{ADA}} = \sum_{n=1}^{N} u_n^{(t)} \exp\left(-y_n \eta h(\mathbf{x}_n)\right)$

after finding g_t, how about $\min_{\eta} \widehat{E}_{\text{ADA}} = \sum_{n=1}^{N} u_n^{(t)} \exp\left(-y_n \eta g_t(\mathbf{x}_n)\right)$

- optimal η_t somewhat **'greedily faster'** than fixed (small) η —called **steepest** descent for optimization
- two cases inside summation:
  - $y_n = g_t(\mathbf{x}_n)$: $u_n^{(t)} \exp(-\eta)$ (correct)
  - $y_n \neq g_t(\mathbf{x}_n)$: $u_n^{(t)} \exp(+\eta)$ (incorrect)
- $\widehat{E}_{\text{ADA}} = \left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left((1-\epsilon_t)\exp(-\eta) + \epsilon_t \exp(+\eta)\right)$

by solving $\frac{\partial \widehat{E}_{\text{ADA}}}{\partial \eta} = 0$, **steepest** $\eta_t = \ln \sqrt{\frac{1-\epsilon_t}{\epsilon_t}} =$

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by solving $\frac{\partial \widehat{E}_{\text{ADA}}}{\partial \eta} = 0$, **steepest** $\eta_t = \ln \sqrt{\frac{1-\epsilon_t}{\epsilon_t}} = \alpha_t$, **remember? :-)**

Deciding Blending Weight as Optimization

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by solving $\frac{\partial \widehat{E}_{\text{ADA}}}{\partial \eta} = 0$, **steepest** $\eta_t = \ln \sqrt{\frac{1-\epsilon_t}{\epsilon_t}} = \alpha_t$, **remember? :-)**
—AdaBoost: steepest descent with approximate functional gradient

Fun Time

With $\widehat{E}_{\text{ADA}} = \left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left((1-\epsilon_t) \exp(-\eta) + \epsilon_t \exp(+\eta) \right)$, which of the following is $\frac{\partial \widehat{E}_{\text{ADA}}}{\partial \eta}$ that can be used for solving the optimal η_t?

1. $\left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left(+ (1-\epsilon_t) \exp(-\eta) + \epsilon_t \exp(+\eta) \right)$
2. $\left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left(+ (1-\epsilon_t) \exp(-\eta) - \epsilon_t \exp(+\eta) \right)$
3. $\left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left(- (1-\epsilon_t) \exp(-\eta) + \epsilon_t \exp(+\eta) \right)$
4. $\left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left(- (1-\epsilon_t) \exp(-\eta) - \epsilon_t \exp(+\eta) \right)$

Fun Time

With $\widehat{E}_{\text{ADA}} = \left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left((1-\epsilon_t)\exp(-\eta) + \epsilon_t \exp(+\eta)\right)$, which of the following is $\frac{\partial \widehat{E}_{\text{ADA}}}{\partial \eta}$ that can be used for solving the optimal η_t?

1. $\left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left(+(1-\epsilon_t)\exp(-\eta) + \epsilon_t \exp(+\eta)\right)$
2. $\left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left(+(1-\epsilon_t)\exp(-\eta) - \epsilon_t \exp(+\eta)\right)$
3. $\left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left(-(1-\epsilon_t)\exp(-\eta) + \epsilon_t \exp(+\eta)\right)$
4. $\left(\sum_{n=1}^{N} u_n^{(t)}\right) \cdot \left(-(1-\epsilon_t)\exp(-\eta) - \epsilon_t \exp(+\eta)\right)$

Reference Answer: 3

Differentiate $\exp(-\eta)$ and $\exp(+\eta)$ with respect to η and you should easily get the result.

Gradient Boosting for Arbitrary Error Function

AdaBoost

$$\min_{\eta} \min_{h} \frac{1}{N} \sum_{n=1}^{N} \exp\left(-y_n \left(\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n) + \eta h(\mathbf{x}_n)\right)\right)$$

with binary-output hypothesis h

Gradient Boosting for Arbitrary Error Function

AdaBoost

$$\min_{\eta} \min_{h} \frac{1}{N} \sum_{n=1}^{N} \exp\left(-y_n \left(\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n) + \eta h(\mathbf{x}_n)\right)\right)$$

with binary-output hypothesis h

GradientBoost

Gradient Boosting for Arbitrary Error Function

AdaBoost

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with binary-output hypothesis h

GradientBoost

$$\min_{\eta} \min_{h} \frac{1}{N} \sum_{n=1}^{N} \mathrm{err}\left(\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n) + \eta h(\mathbf{x}_n), y_n\right)$$

Gradient Boosting for Arbitrary Error Function

AdaBoost

$$\min_{\eta} \boxed{\min_{h} \frac{1}{N} \sum_{n=1}^{N} \exp\left(-y_n \left(\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n) + \eta h(\mathbf{x}_n)\right)\right)}$$

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with any hypothesis h (usually real-output hypothesis)

Gradient Boosting for Arbitrary Error Function

AdaBoost

$$\min_{\eta} \min_{h} \frac{1}{N} \sum_{n=1}^{N} \exp\left(-y_n \left(\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n) + \eta h(\mathbf{x}_n)\right)\right)$$

with binary-output hypothesis h

GradientBoost

$$\min_{\eta} \min_{h} \frac{1}{N} \sum_{n=1}^{N} \mathrm{err}\left(\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n) + \eta h(\mathbf{x}_n), y_n\right)$$

with any hypothesis h (usually real-output hypothesis)

GradientBoost: allows **extension to different** err for regression/soft classification/etc.

GradientBoost for Regression

$$\min_{\eta} \min_{h} \frac{1}{N} \sum_{n=1}^{N} \text{err}\Big(\underbrace{\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n)}_{s_n} + \eta h(\mathbf{x}_n), y_n\Big) \text{ with } \text{err}(s, y) = (s - y)^2$$

GradientBoost for Regression

$$\min_{\eta} \min_{h} \frac{1}{N} \sum_{n=1}^{N} \text{err}\left(\underbrace{\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n)}_{s_n} + \eta h(\mathbf{x}_n), y_n\right) \text{ with } \text{err}(s, y) = (s - y)^2$$

$$\min_{h} \ldots \overset{\text{taylor}}{\approx} \min_{h} \quad \frac{1}{N} \sum_{n=1}^{N} \underbrace{\text{err}\left(s_n, y_n\right)}_{\text{constant}} + \frac{1}{N} \sum_{n=1}^{N} \eta h(\mathbf{x}_n)$$

GradientBoost for Regression

$$\min_{\eta} \min_{h} \frac{1}{N} \sum_{n=1}^{N} \text{err}\left(\underbrace{\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n)}_{s_n} + \eta h(\mathbf{x}_n), y_n \right) \text{ with } \text{err}(s, y) = (s - y)^2$$

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$$= \min_{h} \quad \text{constants} + \frac{\eta}{N} \sum_{n=1}^{N} h(\mathbf{x}_n) \cdot$$

GradientBoost for Regression

$$\min_{\eta} \boxed{\min_{h} \frac{1}{N}\sum_{n=1}^{N} \operatorname{err}\Big(\underbrace{\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n)}_{s_n} + \eta h(\mathbf{x}_n), y_n\Big)} \text{ with } \operatorname{err}(s, y) = (s-y)^2$$

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$$= \min_{h} \quad \text{constants} + \frac{\eta}{N}\sum_{n=1}^{N} h(\mathbf{x}_n) \cdot \boxed{2(s_n - y_n)}$$

GradientBoost for Regression

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$$= \min_{h} \text{constants} + \frac{\eta}{N} \sum_{n=1}^{N} h(\mathbf{x}_n) \cdot 2(s_n - y_n)$$

naïve solution $h(\mathbf{x}_n) = - \quad (s_n - y_n)$
if no constraint on h

GradientBoost for Regression

$$\min_{\eta} \boxed{\min_{h} \frac{1}{N}\sum_{n=1}^{N} \text{err}\Big(\underbrace{\sum_{\tau=1}^{t-1}\alpha_\tau g_\tau(\mathbf{x}_n)}_{s_n} + \eta h(\mathbf{x}_n), y_n\Big)} \text{ with } \text{err}(s,y) = (s-y)^2$$

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$$= \min_{h} \text{constants} + \frac{\eta}{N}\sum_{n=1}^{N} h(\mathbf{x}_n) \cdot 2(s_n - y_n)$$

naïve solution $h(\mathbf{x}_n) = -\infty \cdot (s_n - y_n)$
if no constraint on h

Learning Hypothesis as Optimization

$$\min_{h} \quad \text{constants} + \frac{\eta}{N} \sum_{n=1}^{N} 2h(\mathbf{x}_n)(s_n - y_n)$$

Learning Hypothesis as Optimization

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Learning Hypothesis as Optimization

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$$\min_{h} \qquad \text{constants} + \frac{\eta}{N} \sum_{n=1}^{N} \left(2h(\mathbf{x}_n)(s_n - y_n) + (h(\mathbf{x}_n))^2\right)$$

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Learning Hypothesis as Optimization

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- magnitude of h does not matter: because η will be optimized next
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$$\begin{aligned}
\min_{h} \quad & \text{constants} + \frac{\eta}{N} \sum_{n=1}^{N} \left(2h(\mathbf{x}_n)(s_n - y_n) + (h(\mathbf{x}_n))^2\right) \\
= \quad & \text{constants} + \frac{\eta}{N} \sum_{n=1}^{N} \left(\text{constant} + \left(h(\mathbf{x}_n) - (y_n - s_n)\right)^2\right)
\end{aligned}$$

- solution of penalized approximate functional gradient: squared-error regression on $\{(\mathbf{x}_n, \underbrace{y_n - s_n})\}$

Learning Hypothesis as Optimization

$$\min_h \quad \text{constants} + \frac{\eta}{N} \sum_{n=1}^{N} 2h(\mathbf{x}_n)(s_n - y_n)$$

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- solution of penalized approximate functional gradient: squared-error regression on $\{(\mathbf{x}_n, \underbrace{y_n - s_n}_{\textbf{residual}})\}$

GradientBoost for regression:
find $g_t = h$ by regression with **residuals**

Deciding Blending Weight as Optimization

after finding $g_t = h$,

$$\min_{\eta} \cancel{\min_{h}} \frac{1}{N} \sum_{n=1}^{N} \text{err}\Big(\underbrace{\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n)}_{s_n} + \eta g_t(\mathbf{x}_n), y_n\Big) \text{ with } \text{err}(s, y) = (s - y)^2$$

Deciding Blending Weight as Optimization

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$$\min_{\eta} \quad \frac{1}{N} \sum_{n=1}^{N} (s_n + \eta g_t(\mathbf{x}_n) - y_n)^2$$

Deciding Blending Weight as Optimization

after finding $g_t = h$,

$$\min_{\eta} \cancel{\min_{h}} \frac{1}{N} \sum_{n=1}^{N} \text{err}\Big(\underbrace{\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n)}_{s_n} + \eta g_t(\mathbf{x}_n), y_n\Big) \text{ with } \text{err}(s, y) = (s - y)^2$$

$$\min_{\eta} \quad \frac{1}{N} \sum_{n=1}^{N} (s_n + \eta g_t(\mathbf{x}_n) - y_n)^2 = \frac{1}{N} \sum_{n=1}^{N} (\qquad\qquad - \eta g_t(\mathbf{x}_n))^2$$

Deciding Blending Weight as Optimization

after finding $g_t = h$,

$$\min_{\eta} \cancel{\min_{h}} \frac{1}{N} \sum_{n=1}^{N} \text{err}\Big(\underbrace{\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n)}_{s_n} + \eta g_t(\mathbf{x}_n), y_n\Big) \text{ with } \text{err}(s, y) = (s - y)^2$$

$$\min_{\eta} \quad \frac{1}{N} \sum_{n=1}^{N} (s_n + \eta g_t(\mathbf{x}_n) - y_n)^2 = \frac{1}{N} \sum_{n=1}^{N} ((y_n - s_n) - \eta g_t(\mathbf{x}_n))^2$$

Deciding Blending Weight as Optimization

after finding $g_t = h$,

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—one-variable linear regression on $\{(g_t\text{-transformed input}, \textbf{residual})\}$

Deciding Blending Weight as Optimization

after finding $g_t = h$,

$$\min_{\eta} \cancel{\min_{h}} \frac{1}{N} \sum_{n=1}^{N} \text{err}\Big(\underbrace{\sum_{\tau=1}^{t-1} \alpha_\tau g_\tau(\mathbf{x}_n)}_{s_n} + \eta g_t(\mathbf{x}_n), y_n\Big) \text{ with } \text{err}(s, y) = (s - y)^2$$

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—one-variable linear regression on $\{(g_t\text{-transformed input}, \textbf{residual})\}$

GradientBoost for regression: $\alpha_t =$ optimal η
by g_t-transformed linear regression

Putting Everything Together

Gradient Boosted Decision Tree (GBDT)

for $t = 1, 2, \ldots, T$

return $G(\mathbf{x}) = \sum_{t=1}^{T} \alpha_t g_t(\mathbf{x})$

Putting Everything Together

Gradient Boosted Decision Tree (GBDT)

$s_1 = s_2 = \ldots = s_N = 0$
for $t = 1, 2, \ldots, T$

1. obtain g_t by $\mathcal{A}(\{(\mathbf{x}_n, y_n - s_n)\})$ where $\mathcal{A}$ is a (squared-error) regression algorithm

return $G(\mathbf{x}) = \sum_{t=1}^{T} \alpha_t g_t(\mathbf{x})$

Putting Everything Together

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$s_1 = s_2 = \ldots = s_N = 0$

for $t = 1, 2, \ldots, T$

1. obtain g_t by $\mathcal{A}(\{(\mathbf{x}_n, y_n - s_n)\})$ where $\mathcal{A}$ is a (squared-error) regression algorithm
—**how about sampled and pruned C&RT?**

return $G(\mathbf{x}) = \sum_{t=1}^{T} \alpha_t g_t(\mathbf{x})$

Putting Everything Together

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1. obtain g_t by $\mathcal{A}(\{(\mathbf{x}_n, y_n - s_n)\})$ where $\mathcal{A}$ is a (squared-error) regression algorithm
—**how about sampled and pruned C&RT?**
2. compute $\alpha_t = \text{OneVarLinearRegression}(\{(g_t(\mathbf{x}_n), y_n - s_n)\})$

return $G(\mathbf{x}) = \sum_{t=1}^{T} \alpha_t g_t(\mathbf{x})$

Putting Everything Together

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3. update $s_n \leftarrow s_n + \alpha_t g_t(\mathbf{x}_n)$

return $G(\mathbf{x}) = \sum_{t=1}^{T} \alpha_t g_t(\mathbf{x})$

Putting Everything Together

Gradient Boosted Decision Tree (GBDT)

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2. compute $\alpha_t = \text{OneVarLinearRegression}(\{(g_t(\mathbf{x}_n), y_n - s_n)\})$
3. update $s_n \leftarrow s_n + \alpha_t g_t(\mathbf{x}_n)$

return $G(\mathbf{x}) = \sum_{t=1}^{T} \alpha_t g_t(\mathbf{x})$

GBDT: 'regression sibling' of AdaBoost-DTree
—**popular in practice**

Fun Time

Which of the following is the optimal η for

$$\min_{\eta} \quad \frac{1}{N}\sum_{n=1}^{N}((y_n - s_n) - \eta g_t(\mathbf{x}_n))^2$$

1. $(\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s_n)) \quad \cdot \quad (\sum_{n=1}^{N} g_t^2(\mathbf{x}_n))$
2. $(\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s_n)) \quad / \quad (\sum_{n=1}^{N} g_t^2(\mathbf{x}_n))$
3. $(\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s_n)) \quad + \quad (\sum_{n=1}^{N} g_t^2(\mathbf{x}_n))$
4. $(\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s_n)) \quad - \quad (\sum_{n=1}^{N} g_t^2(\mathbf{x}_n))$

Fun Time

Which of the following is the optimal η for

$$\min_{\eta} \quad \frac{1}{N}\sum_{n=1}^{N}\left((y_n - s_n) - \eta g_t(\mathbf{x}_n)\right)^2$$

1. $\left(\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s_n)\right) \cdot \left(\sum_{n=1}^{N} g_t^2(\mathbf{x}_n)\right)$
2. $\left(\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s_n)\right) / \left(\sum_{n=1}^{N} g_t^2(\mathbf{x}_n)\right)$
3. $\left(\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s_n)\right) + \left(\sum_{n=1}^{N} g_t^2(\mathbf{x}_n)\right)$
4. $\left(\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s_n)\right) - \left(\sum_{n=1}^{N} g_t^2(\mathbf{x}_n)\right)$

Reference Answer: 2

Derived within Lecture 9 of ML Foundations, **remember? :-)**

Map of Blending Models

blending: aggregate after getting **diverse** g_t

Map of Blending Models

blending: aggregate **after getting** **diverse** g_t

uniform

simple
voting/averaging of g_t

Map of Blending Models

blending: aggregate after getting **diverse** g_t

uniform	non-uniform
simple voting/averaging of g_t	linear model on g_t-transformed inputs

Map of Blending Models

blending: aggregate after getting **diverse** g_t

uniform	non-uniform	conditional
simple voting/averaging of g_t	linear model on g_t-transformed inputs	nonlinear model on g_t-transformed inputs

Map of Blending Models

blending: aggregate after getting **diverse** g_t

uniform	non-uniform	conditional
simple voting/averaging of g_t	linear model on g_t-transformed inputs	nonlinear model on g_t-transformed inputs

uniform for 'stability';

Map of Blending Models

blending: aggregate after getting **diverse** g_t

uniform	non-uniform	conditional
simple voting/averaging of g_t	linear model on g_t-transformed inputs	nonlinear model on g_t-transformed inputs

uniform for 'stability';
non-uniform/conditional **carefully** for 'complexity'

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Bagging

diverse g_t by bootstrapping;

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Bagging

diverse g_t by bootstrapping; uniform vote by nothing :-)

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Bagging

diverse g_t by bootstrapping;
uniform vote by nothing :-)

AdaBoost

diverse g_t by reweighting;

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Bagging

diverse g_t by bootstrapping;
uniform vote by nothing :-)

AdaBoost

diverse g_t by reweighting;
linear vote by steepest search

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Bagging

diverse g_t by bootstrapping;
uniform vote by nothing :-)

AdaBoost

diverse g_t by reweighting;
linear vote by steepest search

Decision Tree

conditional vote by branching

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Bagging

diverse g_t by bootstrapping;
uniform vote by nothing :-)

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diverse g_t by reweighting;
linear vote by steepest search

Decision Tree

diverse g_t by data splitting;
conditional vote by branching

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Bagging

diverse g_t by bootstrapping;
uniform vote by nothing :-)

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diverse g_t by reweighting;
linear vote by steepest search

Decision Tree

diverse g_t by data splitting;
conditional vote by branching

GradientBoost

diverse g_t by residual fitting;

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Bagging

diverse g_t by bootstrapping;
uniform vote
by nothing :-)

AdaBoost

diverse g_t
by reweighting;
linear vote
by steepest search

Decision Tree

diverse g_t
by data splitting;
conditional vote
by branching

GradientBoost

diverse g_t
by residual fitting;
linear vote
by steepest search

Map of Aggregation-Learning Models

learning: aggregate as well as getting **diverse** g_t

Bagging

diverse g_t by bootstrapping;
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AdaBoost

diverse g_t by reweighting;
linear vote by steepest search

Decision Tree

diverse g_t by data splitting;
conditional vote by branching

GradientBoost

diverse g_t by residual fitting;
linear vote by steepest search

boosting-like algorithms most popular

Map of Aggregation of Aggregation Models

Bagging

AdaBoost

Decision Tree

GradientBoost

Map of Aggregation of Aggregation Models

Bagging

AdaBoost

Decision Tree

Random Forest

randomized bagging
+ ‘strong’ DTree

GradientBoost

Map of Aggregation of Aggregation Models

Bagging

AdaBoost

Decision Tree

Random Forest

randomized bagging
+ ‘strong’ DTree

AdaBoost-DTree

AdaBoost
+ ‘weak’ DTree

GradientBoost

Map of Aggregation of Aggregation Models

Bagging

AdaBoost

Decision Tree

Random Forest

randomized bagging
+ ‘strong’ DTree

AdaBoost-DTree

AdaBoost
+ ‘weak’ DTree

GradientBoost

GBDT

GradientBoost
+ ‘weak’ DTree

Map of Aggregation of Aggregation Models

Bagging

AdaBoost

Decision Tree

Random Forest

randomized bagging
+ 'strong' DTree

AdaBoost-DTree

AdaBoost
+ 'weak' DTree

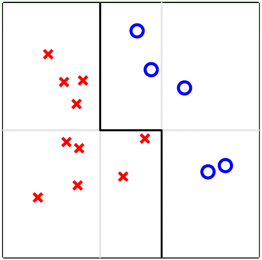
GradientBoost

GBDT

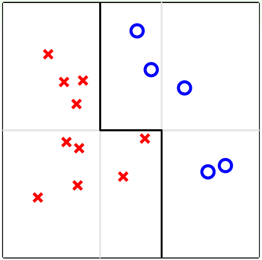
GradientBoost
+ 'weak' DTree

all three frequently used in practice

Specialty of Aggregation Models



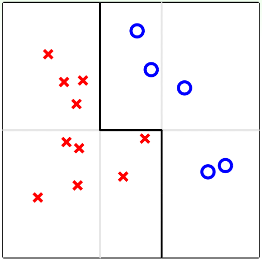
Specialty of Aggregation Models



cure underfitting

- $G(\mathbf{x})$ 'strong'

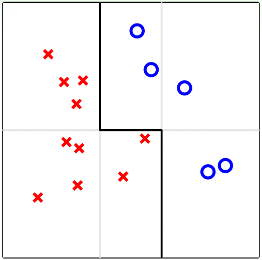
Specialty of Aggregation Models

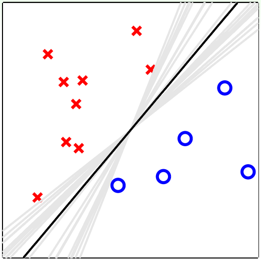


cure underfitting

- $G(\mathbf{x})$ 'strong'
- aggregation
  $\Longrightarrow$ **feature transform**

Specialty of Aggregation Models





cure underfitting

- $G(\mathbf{x})$ 'strong'
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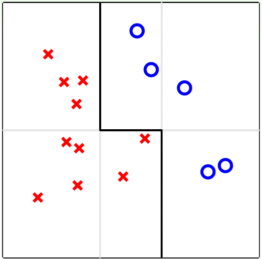
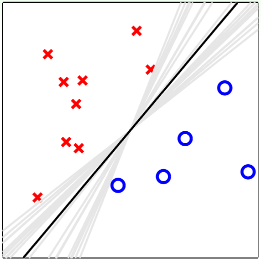
  $\Longrightarrow$ **feature transform**

Specialty of Aggregation Models





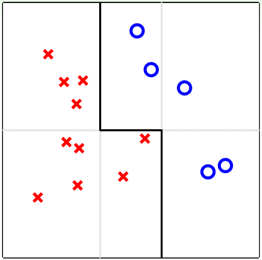
cure underfitting

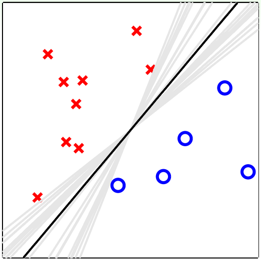
- $G(\mathbf{x})$ 'strong'
- aggregation
  $\Longrightarrow$ **feature transform**

cure overfitting

- $G(\mathbf{x})$ 'moderate'

Specialty of Aggregation Models





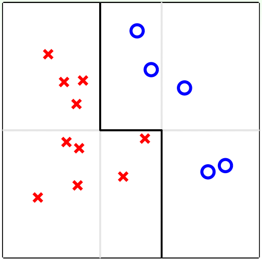
cure underfitting

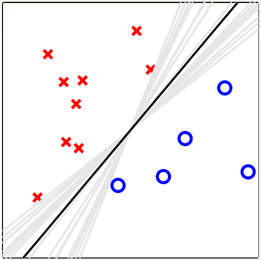
- $G(\mathbf{x})$ 'strong'
- aggregation
  $\Longrightarrow$ **feature transform**

cure overfitting

- $G(\mathbf{x})$ 'moderate'
- aggregation
  $\Longrightarrow$ **regularization**

Specialty of Aggregation Models





cure underfitting

- $G(\mathbf{x})$ 'strong'
- aggregation $\Longrightarrow$ **feature transform**

cure overfitting

- $G(\mathbf{x})$ 'moderate'
- aggregation $\Longrightarrow$ **regularization**

proper aggregation (a.k.a. 'ensemble')
$\Longrightarrow$ **better performance**

Fun Time

Which of the following aggregation model learns diverse g_t by reweighting and calculates linear vote by steepest search?

1. AdaBoost
2. Random Forest
3. Decision Tree
4. Linear Blending

Fun Time

Which of the following aggregation model learns diverse g_t by reweighting and calculates linear vote by steepest search?

1. AdaBoost
2. Random Forest
3. Decision Tree
4. Linear Blending

Reference Answer: 1

Congratulations on being an **expert** in aggregation models! **:-)**

Summary

1. Embedding Numerous Features: Kernel Models
2. Combining Predictive Features: Aggregation Models

Lecture 11: Gradient Boosted Decision Tree

- Adaptive Boosted Decision Tree
  sampling and pruning for 'weak' trees
- Optimization View of AdaBoost
  functional gradient descent on exponential error
- Gradient Boosting
  iterative steepest residual fitting
- Summary of Aggregation Models
  some cure underfitting; some cure overfitting

3. Distilling Implicit Features: Extraction Models
   - **next: extract features other than hypotheses**