Problem 1. (Bipartite vs. Nonbipartite Expanders) Show that constructing bipartite expanders is equivalent to constructing (standard, nonbipartite) expanders. That is, show how given an explicit construction of one of the following, you can obtain an explicit construction of the other:

(a). $D$-regular $(\alpha N, A)$ expanders on $N$ vertices for infinitely many $N$, where $\alpha > 0$, $A > 1$, and $D$ are constants independent of $N$.

(b). $D$-regular (on both sides) $(\alpha N, A)$ bipartite expanders with $N$ vertices on each side for infinitely many $N$, where $\alpha > 0$, $A > 1$, and $D$ are constants independent of $N$.

(Your transformations need not preserve the constants.)

Problem 2. (Unbalanced Vertex Expanders and Data Structures) Consider a $(K, (1-\varepsilon)D)$ bipartite vertex expander $G$ with $N$ left vertices, $M$ right vertices, and left degree $D$.

(a). For a set $S$ of left vertices, a $y \in N(S)$ is called a unique neighbor of $S$ if $y$ is incident to exactly one edge from $S$. Prove that every left-set $S$ of size at most $K$ has at least $(1-2\varepsilon)D|S|$ unique neighbors.

(b). For a set $S$ of size at most $K/2$, prove that at most $|S|/2$ vertices outside $S$ have at least $\delta D$ neighbors in $N(S)$, for $\delta = O(\varepsilon)$.

Now we’ll see a beautiful application of such expanders to data structures. Suppose we want to store a small subset $S$ of a large universe $[N]$ such that we can test membership in $S$ by probing just 1 bit of our data structure. A trivial way to achieve this is to store the characteristic vector of $S$, but this requires $N$ bits of storage. The hashing-based data structures mentioned earlier in the
course only require storing $O(|S|)$ words, each of $O(\log N)$ bits, but testing membership requires reading an entire word (rather than just one bit.)

Our data structure will consist of $M$ bits, which we think of as a \{0,1\}-assignment to the right vertices of our expander. This assignment will have the following property.

**Property II:** For all left vertices $x$, all but a $\delta = O(\varepsilon)$ fraction of the neighbors of $x$ are assigned the value $\chi_S(x)$ (where $\chi_S(x) = 1$ iff $x \in S$).

(c). Show that if we store an assignment satisfying Property II, then we can probabilistically test membership in $S$ with error probability $\delta$ by reading just one bit of the data structure.

(d). Show that an assignment satisfying Property II exists provided $|S| \leq K/2$. (Hint: first assign 1 to all of $S$'s neighbors and 0 to all its nonneighbors, then try to correct the errors.)

It turns out that the needed expanders exist with $M = O(K \log N)$ (for any constant $\varepsilon$), so the size of this data structure matches the hashing-based scheme while admitting 1-bit probes.

However, note that such bipartite vertex expanders do not follow from explicit spectral expanders as given in class, because the latter do not provide vertex expansion beyond $D/2$ nor do they yield highly imbalanced expanders (with $M \ll N$) as needed here. But later in the course, we will see how to explicitly construct expanders that are quite good for this application (specifically, with $M = K^{1.01} \cdot \text{polylog} N$).

**Problem 3. (Spectral Graph Theory)** Let $M$ be the random-walk matrix for a $d$-regular undirected graph $G = (V,E)$ on $n$ vertices. We allow $G$ to have self-loops and multiple edges. Recall that the uniform distribution (or all-ones vector) is an eigenvector of $M$ of eigenvalue $\lambda_1 = 1$.

Prove the following statements. (Hint: for intuition, it may help to think about what the statements mean for the behavior of the random walk on $G$.)

(a). All eigenvalues of $M$ have absolute value at most 1.

(b). $G$ is disconnected $\iff$ 1 is an eigenvalue of multiplicity at least 2.

(c). Suppose $G$ is connected. Then $G$ is bipartite $\iff$ $-1$ is an eigenvalue of $M$.

(d). $G$ connected $\Rightarrow$ all eigenvalues of $M$ other than $\lambda_1$ are $\leq 1 - 1/\text{poly}(n,d)$. To do this, it may help to first show that the second largest eigenvalue of $M$ (not necessarily in absolute value) equals

$$
\max_x \langle Ax, x \rangle = 1 - \frac{1}{d} \cdot \min_x \sum_{(i,j) \in E} (x_i - x_j)^2,
$$

where the maximum/minimum is taken over all vectors $x$ of length 1 such that $\sum_i x_i = 0$, and $\langle x, y \rangle = \sum_i x_i y_i$ is the standard inner product. For intuition, consider restricting the above maximum/minimum to $x \in \{+\alpha,-\beta\}^n$ for $\alpha, \beta > 0$.

(e). $G$ connected and nonbipartite $\Rightarrow$ all eigenvalues of $M$ (other than 1) have absolute value at most $1 - 1/\text{poly}(n,d)$.

(f*) Extra credit: Establish the (tight) bound $1 - \Omega(1/d \cdot D \cdot n)$ in Part (d), where $D$ is the diameter of the graph, and show that a simple graph satisfies $D \leq O(n/d)$. (The $1 - \Omega(1/d \cdot D \cdot n)$ bound also holds for Part (e), even though I'm not asking you to prove it.)
Problem 4. (Error Reduction For Free*)  Show that if a problem has a \textbf{BPP} algorithm with constant error probability, then it has a \textbf{BPP} algorithm with error probability \(1/n\) which uses \textit{exactly} the same number of random bits.