Partition of Integers into Even Summands

- We ask for the number of partitions of $m \in \mathbb{Z}^+$ into positive even integers.
- The desired number is the coefficient of $x^m$ in
  \[(1 + x^2 + x^4 + \cdots)(1 + x^4 + x^8 + \cdots)\]
  \[(1 + x^6 + x^{12} + \cdots)\cdots(1 + x^{2\lfloor m/2 \rfloor} + x^{4\lfloor m/2 \rfloor} + \cdots).\]
- We are more economical than in “partition of integers into odd summands” by removing higher-order terms.

Partition of Integers into Distinct Summands with Upper Bounds

- We ask for the number of partitions of $m \in \mathbb{Z}^+$ into distinct positive integers at most $n$.
- The desired number is the coefficient of $x^m$ in
  \[(1 + x)(1 + x^2)(1 + x^3)\cdots(1 + x^n).\]
- No known closed-form formula.
- Applications in computational finance,\(^\text{a}\)
- Can calculate all $n(n+1)/2$ coefficients in time $o(n^3)$?
  \(^\text{a}\)Lyuu (2002).

Partition of Integers into Even Numbers of Each Summand

- We ask for the number of partitions of $m \in \mathbb{Z}^+$ into positive integer summands where each summand appears an even number of times.
- The desired number remains the coefficient of $x^m$ in
  \[(1 + x^2 + x^4 + \cdots)(1 + x^4 + x^8 + \cdots)\]
  \[(1 + x^6 + x^{12} + \cdots)\cdots(1 + x^{2\lfloor m/2 \rfloor} + x^{4\lfloor m/2 \rfloor} + \cdots).\]
- 1 appears an even number of times, 2 appears an even number of times, etc.

An Example

- Note that
  \[(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6)\]
  \[= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 4x^7 + \cdots + x^{21}.\]
- So there are 4 ways to partition 7 into distinct positive integers at most 6.
  - Indeed, the partitions are: 6 + 1, 5 + 2, 4 + 3, 4 + 2 + 1.
- In fact, we solved 21 problems: The coefficient of $x^i$, where $1 \leq i \leq 21$, represents the number of ways to partition $i$ into distinct positive integers at most 6.
Partition of Integers into Summands \( \not\equiv 0 \mod k \)

- The desired number is the coefficient of \( x^m \) in
  \[
  (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \cdots \\
  (1 + x^{k-1} + x^{2(k-1)} + \cdots) \\
  (1 + x^{k+1} + x^{2(k+1)} + \cdots) \cdots.
  \]

- In other words, terms \( (1 + x^{ik} + x^{2ik} + \cdots) \), where \( k \in \mathbb{Z}^+ \), are missing.

The Proof (continued)

- Now, the desired number is the coefficient of \( x^m \) in
  \[
  \frac{1}{1-x} \frac{1}{1-x^2} \cdots \frac{1}{1-x^k} \frac{1}{1-x^{k+1}} \cdots
  \]

  
  \[
  = \prod_{i=1}^{\infty} (1 - x^{ik}) = \prod_{i=1}^{\infty} \frac{1 - x^{ik}}{1 - x^i} \\
  = (1 + x + x^2 + \cdots + x^{k-1})(1 + x^2 + x^4 + \cdots + x^{2(k-1)}) \cdots.
  \]

The Proof (concluded)

- The count is the same as that of partitions into positive integers where no summands appear \( > k - 1 \) times.

- This generalizes results on p. 425 and p. 431.a

- With \( k = 2 \), we have an alternative proof of Euler’s theorem (p. 432).b

\( ^a \) An alternative proof uses the principle of inclusion and exclusion.
\( ^b \) Observation by Mr. Ansel Lin (893902003) on November 29, 2004.

Partition of Integers vs. Integer Solution of Linear Equations

- These two issues are often related in subtle ways.

- To wit, what is the number of partitions of \( m \in \mathbb{N} \) into \( n \) nonnegative integers where the order of summands is relevant?

- Each nonnegative integer solution of
  \[
  x_1 + x_2 + \cdots + x_n = m
  \]
  corresponds to a valid partition.

- The answer is thus \( \binom{n+m-1}{m} \) from p. 413.
Partition of Integers as Linear Equation

- What is the number of integer solutions of
  \[ x_1 + x_2 + \cdots + x_n = n, \]
  where \( 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \)?
- A solution corresponds to a partition of \( n \) into positive integers (the order of summands is irrelevant).
- \((0, 0, 0, 1, 2, 3) \iff 6 = 1 + 2 + 3.\)
- From Eq (54) on p. 421, the number equals the coefficient of \( x^n \) in
  \[ \frac{1}{(1 - x)(1 - x^2) \cdots (1 - x^n)}. \]

---

Partition of Integers as Linear Equation (continued)

- Alternatively, each solution corresponds to a partition of integer \( m \) into \( \leq n \) positive integers.
- Yet alternatively, each solution corresponds to a partition of \( m \) into summands at most \( n \).
  - See the Ferrers graph on the next page.
- The desired number hence equals the coefficient of \( x^m \) in
  \[ \frac{1}{(1 - x)(1 - x^2) \cdots (1 - x^n)}. \]

---

The Ferrers Graph

\[ 9 = 4 + 4 + 1 \]

\[ 9 = 0 + 0 + 2 + 2 + 2 + 3 \]

---

*Professor Andrews, private communication, October 2001.*
Partition of Integers as Linear Equation (continued)

• For instance,
\[
\frac{1}{(1-x)(1-x^2)(1-x^3)} = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + 10x^8 + 12x^9 + 14x^{10} + 16x^{11} + 19x^{12} + \cdots.
\]

• There are 7 ways to partition 6 into 3 nonnegative integers:

0 + 0 + 6, 0 + 1 + 5,
0 + 2 + 4, 0 + 3 + 3,
1 + 1 + 4, 1 + 2 + 3, 2 + 2 + 2.

Partition of Integers as Linear Equation (continued)

• What is the number of integer solutions to
\[
x_1 + 2x_2 + 3x_3 + \cdots + nx_n = m,
\]
where \(x_i \geq 0\)?

• Well, it is the same partition-of-integers problem considered at Eq. (57) on p. 442.
  - As on p. 427, each solution corresponds to a partition \(m\) into \(x_1 1s, x_2 2s, \text{etc.}\)

• The desired number is therefore the coefficient of \(x^m\) in
\[
\frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \frac{1}{1-x^n}
\]
from Eq. (58) on p. 443.

• In summary, if \(p_n(m)\) denotes the number of partitions of \(m\) into summands at most \(n\), then Eq. (60) is the generating function for \(\{ p_n(m) \}_{m=0,1,\ldots}\).

• If \(q_n(m)\) denotes the number of partitions of \(m\) into at most \(n\) summands, then \(p_n(m) = q_n(m)\) (recall p. 443).

• Furthermore, \(p(m) = p_n(m)\) for all \(n > m\) (p. 420).

Partition of Integers as Linear Equation (continued)

• Recall that the number of nonnegative integer solutions to Eq. (59) on p. 446,
\[
x_1 + 2x_2 + 3x_3 + \cdots + nx_n = m,
\]
is the coefficient of \(x^m\) in
\[
\frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \frac{1}{1-x^n}
\]
by Eq. (60) on p. 447.

• We now arrive at the answer in a more “direct” way.
Partition of Integers as Linear Equation (continued)

- Define
  \[ x_n = w_1, \]
  \[ x_{n-1} = w_2 - w_1, \]
  \[ x_{n-2} = w_3 - w_2, \]
  \[ \vdots \]
  \[ x_1 = w_n - w_{n-1}. \]

- The above mapping is bijective.

Recurrence Relations for \( p_n(m) \) and \( q_n(m) \) (p. 447)

**Lemma 66** \( q_n(m) = q_{n-1}(m) + q_n(m-n) \).

- Recall that \( q_n(m) \) denotes the number of partitions of \( m \) into at most \( n \) summands.
- Consider \( m \) identical objects and \( n \) identical containers.
- There are \( q_n(m) - q_{n-1}(m) \) ways to partition the \( m \) objects into \( n \) containers with none left empty.
- Or, we can distribute one object into each container and then the remaining \( m-n \) objects, in \( q_n(m-n) \) ways.
- Hence \( q_n(m) - q_{n-1}(m) = q_n(m-n) \).

Partition of Integers with Exact Number of Summands and Largest Summand

- Let \( p(m,n) \) denote the number of partitions of \( m \) with largest summand equal to \( n \).
- Let \( q(m,n) \) denote the number of partitions of \( m \) into exactly \( n \) summands.
- Similar to \( p_n(m) = q_n(m) \) on p. 447,
  \[ p(m,n) = q(m,n). \]
  \[ - p(m,n) = p_n(m) - p_{n-1}(m) = q_n(m) - q_{n-1}(m) = q(m,n). \]
We knew that
\[ p(m, n) = p_n(m) - p_{n-1}(m) \]
and the generating function for \( \{ p_n(m) \}_{m=0,1,...} \) is
\[ \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \frac{1}{1-x^n} \quad (61) \]
(p. 447).

Thus the generating function for \( \{ p(m, n) \}_{m=0,1,...} \) is
\[ \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \frac{1}{1-x^n} - \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdots \frac{1}{1-x^n-1} = x^n \cdot \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdots \frac{1}{1-x^n}. \quad (62) \]

The coefficient of \( x^m \) in Eq. (62) on p. 453 equals that of \( x^{m-n} \) in Eq. (61) on (p. 447).

From Eq. (55) on p. 421, \( p(m-n) \) equals the coefficient of \( x^{m-n} \) in
\[ \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \frac{1}{1-x^n} \]
\[ = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \frac{1}{1-x^n} \cdot (1+x^{n+1}+\cdots) \cdots. \]

Observe that \( m-n-(n+1) = m-2n-1 < 0 \).
Partition of Integers with Exact Number of Summands and
Largest Summand (continued)

- So $p(m-n)$ equals the coefficient of $x^{m-n}$ in
  $\frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \frac{1}{1-x^n}$.
- For (b), simply note that $p(2n,n) = p(n)$ by (a).

Partition of Integers into Distinct Summands Revisited

Let $q^\#(m,n)$ denote the number partitions of $m \in \mathbb{Z}^+$ into $n$
distinct summands.

**Lemma 70** $q^\#(m,n) = q(m - \binom{n}{2}, n)$.

- $x_1 + x_2 + \cdots + x_n = m$, where $0 < x_1 < x_2 < \cdots < x_n$,
  has $q^\#(m,n)$ integer solutions.

---

Partition of Integers with Exact Number of Summands and
Largest Summand (concluded)

**Lemma 69** $p(m+n,n) = p(m,1) + p(m,2) + \cdots + p(m,n)$.

- From Lemma 68a (p. 455), $p(m+n,n) = p_n(m)$.
- On the other hand,
  $p(m,1) + p(m,2) + \cdots + p(m,n) = p_n(m)$ by definition.

Partition of Integers into Distinct Summands Revisited

(continued)

- Adopt the following bijective transformation:
  $x_1 = w_1,$
  $x_2 = w_2 + 1,$
  $x_3 = w_3 + 2,$
  $\vdots$
  $x_n = w_n + (n-1).$
Partition of Integers into Distinct Summands Revisited (concluded)

• The equation becomes
\[ w_1 + w_2 + \cdots + w_n = m - \binom{n}{2}, \]
where \( 0 < w_1 \leq w_2 \leq w_3 \leq \cdots \leq w_n. \)
• This equation has \( q(m - \binom{n}{2}, n) \) integer solutions.

Exponential Generating Functions

• Let \( a_0, a_1, a_2, \ldots \) be a sequence of real numbers.
• The function
\[ f(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!} \]
is called the exponential generating function for the given sequence \( \{a_i\}_{i=0,1,\ldots} \).
• We may call the earlier generating function ordinary generating function to avoid ambiguity.

Some Properties of Exponential Generating Functions

• If \( f(x) \) is the exponential generating function for \( \{a_i\}_{i=0,1,\ldots} \), then \( xf'(x) \) is the generating function for \( \{ia_i\}_{i=0,1,\ldots}. \)
• If \( f(x) \) is the exponential generating function for \( \{a_i\}_{i=0,1,\ldots} \) and \( g(x) \) is the exponential generating function for \( \{b_i\}_{i=0,1,\ldots} \), then \( f(x)g(x) \) is the exponential generating function for \( \{h_i\}_{i=0,1,\ldots} \), where
\[ h_i = \sum_{k=0}^{i} a_k b_{i-k} \]
is the convolution of \( a_i \) and \( b_i. \)

Some Properties of Exponential Generating Functions (concluded)

Lemma 71 Let
\[ f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!} \]
be the exponential generating function for \( \{a_i\}_{i=0,1,\ldots} \). Then
\[ a_i = \frac{d^i f(x)}{dx^i} \bigg|_{x=0}, \quad i = 0, 1, 2, \ldots. \]
• By Taylor’s expansion (review p. 397).
Generating Function for Bell Numbers

- Let
  \[ p(x) = \sum_{n=0}^{\infty} \frac{P_n}{n!} x^n \]
  be the exponential generating function for the Bell numbers \( \{ P_n \}_{n=0,1,2,\ldots} \) defined on p. 228.

- From relation (28) on p. 231,
  \[
  p(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) P_k
  = 1 + \sum_{k=0}^{\infty} \frac{P_k}{k!} \sum_{n=k+1}^{\infty} \frac{x^n}{n!} \frac{1}{(n-k-1)!}.
  \]

The Proof (concluded)

- Differentiate to obtain
  \[
  p'(x) = \sum_{k=0}^{\infty} \frac{P_k}{k!} \sum_{n=k+1}^{\infty} \frac{x^{n-1}}{(n-k-1)!}
  = \sum_{k=0}^{\infty} \frac{P_k x^k}{k!} \sum_{r=0}^{\infty} \frac{x^r}{r!} = p(x) e^x.
  \]
  - In other words, \( (\ln p(x))' = p'(x)/p(x) = e^x. \)
  - Hence \( p(x) = e^{e^x+c} \) for some constant \( c. \)
  - As \( 1 = p(0) = e^{0+c} \), we conclude that \( c = -1 \) and
    \[ p(x) = e^{e^x-1}. \]

Recurrence Relations Arise Naturally

- When a problem has a recursive nature, recurrence relations often arise.
  - A problem can be solved by solving 2 subproblems of the same nature.
- When an algorithm is of the divide-and-conquer type, a recurrence relation describes its running time.
  - Sorting, fast Fourier transform, etc.
- Certain combinatorial objects are constructed recursively such as hypercubes (p. 572).
First-Order Linear Homogeneous Recurrence Relations
• Consider the recurrence relation
  \[ a_{n+1} = da_n, \]
  where \( n \geq 0 \) and \( d \) is a constant.
• The general solution is given by
  \[ a_n = Cd^n \]
  for any constant \( C \).
  – It satisfies the relation: \( Cd^{n+1} = dCd^n \).
• There are infinitely many solutions, one for each choice of \( C \).

--

First-Order Linear Nonhomogeneous Recurrence Relations
• Consider the recurrence relation
  \[ a_{n+1} + da_n = f(n). \]
  \( - \ n \geq 0. \)
  \( - \ d \) is a constant.
  \( - \ f(n) : \mathbb{N} \to \mathbb{N}. \)
• A general solution no longer exists.

--

kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients
• Consider the \( k \)-th order recurrence relation
  \[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0, \] \hspace{1cm} (63)
  where \( C_n, C_{n-1}, \ldots, C_{n-k} \in \mathbb{R}, C_n \neq 0 \), and \( C_{n-k} \neq 0 \).
• Add \( k \) initial conditions for \( a_0, a_1, \ldots, a_{k-1} \).
• Clearly, \( a_n \) is well-defined for each \( n = k, k+1, \ldots \).
**kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients (concluded)**

- A solution $y$ for $a_n$ is general if for any particular solution $y^*$, the undetermined coefficients of $y$ can be found so that $y$ is identical to $y^*$.
- Any general solution for $a_n$ that satisfies the $k$ initial conditions and Eq. (63) is a particular solution.
- In fact, it is the unique particular solution because any solution agreeing at $n = 0, 1, \ldots, k - 1$ must agree for all $n \geq 0$.

---

**Conditions for the General Solution**

**Theorem 72** Let $a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)}$ be $k$ particular solutions of Eq. (63). If

$$
\begin{vmatrix}
    a_1^{(1)} & a_2^{(1)} & \cdots & a_k^{(1)} \\
    a_1^{(2)} & a_2^{(2)} & \cdots & a_k^{(2)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_1^{(k)} & a_2^{(k)} & \cdots & a_k^{(k)}
\end{vmatrix} \neq 0,
$$

then $a_n = C_1 a_n^{(1)} + C_2 a_n^{(2)} + \cdots + C_k a_n^{(k)}$ is the general solution, where $C_1, C_2, \ldots, C_k$ are arbitrary constants.

---

---

**Fundamental Sets**

- The particular solutions of Eq. (63) on p. 472,

$$a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)},$$

that also satisfy inequality (64) in Theorem 72 (p. 475) are said to form a **fundamental set of solutions**.
- Solving a linear homogeneous recurrence equation thus reduces to finding a fundamental set!
\(k\)th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Distinct Roots

- Let \(r_1, r_2, \ldots, r_k\) be the (characteristic) roots of the characteristic equation

\[
C_n r^k + C_{n-1} r^{k-1} + \cdots + C_{n-k} = 0. \tag{65}
\]

- If \(r_1, r_2, \ldots, r_k\) are distinct, then the general solution has the form

\[
a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n,
\]

for constants \(c_1, c_2, \ldots, c_k\) determined by the initial conditions.

The Proof

- Assume \(a_n\) has the form \(cr^n\) for nonzero \(c\) and \(r\).
- After substitution into recurrence equation (63) on p. 472, \(r\) satisfies characteristic equation (65).
- Let \(r_1, r_2, \ldots, r_k\) be the \(k\) distinct (nonzero) roots.
- Hence \(a_n = r_i^n\) is a solution for \(1 \leq i \leq k\).
- Solutions \(r_i^n\) form a fundamental set because

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
r_1 & r_2 & \cdots & r_k \\
\vdots & \vdots & \ddots & \vdots \\
r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1}
\end{vmatrix} \neq 0.
\]

The \(k \times k\) matrix is called a Vandermonde matrix, which is nonsingular whenever \(r_1, r_2, \ldots, r_k\) are distinct.\(^a\)

\(^a\)This is a standard result in linear algebra.

The Proof (continued)

- Hence

\[
a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n
\]

is the general solution.
- The \(k\) coefficients \(c_1, c_2, \ldots, c_k\) are determined uniquely by the \(k\) initial conditions \(a_0, a_1, \ldots, a_{k-1}\):

\[
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{k-1}
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
r_1 & r_2 & \cdots & r_k \\
\vdots & \vdots & \ddots & \vdots \\
r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_k
\end{bmatrix}. \tag{66}
\]
The Fibonacci Relation

- Consider $a_{n+2} = a_{n+1} + a_n$.
- The initial conditions are $a_0 = 0$ and $a_1 = 1$.
- The characteristic equation is $r^2 - r - 1 = 0$, with two roots $(1 \pm \sqrt{5})/2$.
- The fundamental set is hence $\left\{ \left( \frac{1 + \sqrt{5}}{2} \right)^n, \left( \frac{1 - \sqrt{5}}{2} \right)^n \right\}$.

The Fibonacci Relation (continued)

- For example, $\left( \frac{1 + \sqrt{5}}{2} \right)^n$ satisfies the Fibonacci relation, as
  $\left( \frac{1 + \sqrt{5}}{2} \right)^n = \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \left( \frac{1 - \sqrt{5}}{2} \right)^n$.
- The general solution is hence
  $a_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n$. (67)

Don’t Believe It?

- Solve
  $0 = a_0 = c_1 + c_2$
  $1 = a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2}$

  for $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$.
- The solution is finally
  $a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$,

  known as the Binet formula.
Initial Conditions

- Different initial conditions give rise to different solutions.
- Suppose \( a_0 = 1 \) and \( a_1 = 2 \).
- Then solve

\[
\begin{align*}
1 &= a_0 = c_1 + c_2, \\
2 &= a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},
\end{align*}
\]

for \( c_1 = \frac{[(1 + \sqrt{5})/2]^2}{\sqrt{5}} \) and

\[
c_2 = -\frac{[(1 - \sqrt{5})/2]^2}{\sqrt{5}}
\]
to obtain

\[
a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2}.
\]

(68)

Generating Function for the Fibonacci Numbers

- From \( a_{n+2} = a_{n+1} + a_n \), we obtain

\[
\sum_{n=0}^{\infty} a_{n+2}x^{n+2} = \sum_{n=0}^{\infty} \left( a_{n+1}x^{n+2} + a_n x^{n+2} \right).
\]

- Let \( f(x) \) be the generating function for \( \{ a_n \}_{n=0,1,2,...} \).
- Then

\[
f(x) - a_0 - a_1 x = x [ f(x) - a_0 ] + x^2 f(x).
\]

- Hence

\[
f(x) = \frac{-a_0 x + a_0 + a_1 x}{1 - x - x^2}.
\]

(69)

A Formula for the Fibonacci Numbers

\[
a_n = \left( \begin{array}{c} n - 1 \\ 0 \end{array} \right) + \left( \begin{array}{c} n - 2 \\ 1 \end{array} \right) + \left( \begin{array}{c} n - 3 \\ 2 \end{array} \right) + \cdots + \left( \begin{array}{c} n - [n/2] \\ [n/2] - 1 \end{array} \right).
\]

- From Eq. (69) on p. 486, the generating function is

\[
\frac{-a_0 x + a_0 + a_1 x}{1 - x - x^2} = \frac{x}{1 - x(1+x)} = x + x^2(1+x) + x^3(1+x)^2 + \cdots + x^{n-1}(1+x)^{n-2} + x^n(1+x)^{n-1} + \cdots
\]

\[
= \cdots + \left( \begin{array}{c} n - [n/2] \\ [n/2] - 1 \end{array} \right) + \cdots + \left( \begin{array}{c} n - 2 \\ 1 \end{array} \right) + \left( \begin{array}{c} n - 1 \\ 0 \end{array} \right) x^n + \cdots.
\]

Number of Binary Sequences without Consecutive 0s

- Let \( a_n \) denote the number of binary sequences of length \( n \) without consecutive 0s.
- There are \( a_{n-1} \) valid sequences with the \( n \)th symbol being 1.
- There are \( a_{n-2} \) valid sequences with the \( n \)th symbol being 0 because any such sequence must end with 10.
- Hence \( a_n = a_{n-1} + a_{n-2} \), a Fibonacci sequence.
- Because \( a_1 = 2 \) and \( a_2 = 3 \), we must have \( a_0 = 1 \) to retrofit the Fibonacci sequence.
- The formula is Eq. (68) on p. 485 (contrast it with p. 70).
Number of Subsets without Consecutive Numbers

- A binary sequences $b_1b_2\cdots b_n$ of length $n$ can be interpreted as the set $\{i : b_i = 0\} \subseteq \{1, 2, \ldots, n\}$.
- Hence there are $a_n$ subsets of $\{1, 2, \ldots, n\}$ that contain no 2 consecutive integers.
  - $a_n$ is the Fibonacci number with $a_0 = 1$ and $a_1 = 2$ (formula on p. 485).
- How many subsets of $\{1, 2, \ldots, n\}$ contain no 2 consecutive integers when 1 and $n$ are considered consecutive?
  - Assume $n \geq 3$.

Number of Subsets without Consecutive Numbers (continued)

- There are $a_{n-1}$ acceptable subsets that do not contain $n$.
- If $n$ is included, an acceptable subset cannot contain 1.
- Hence there are $a_{n-2}$ such subsets.
- The total is therefore $L_n \equiv a_{n-1} + a_{n-2}$, the Lucas number.
- Surely, $L_n = L_{n-1} + L_{n-2}$ with $L_1 = 1$ and $L_2 = 3$.
- The general solution is hence

$$L_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

by Eq. (67) on p. 482.