Generalized Principle of Inclusion and Exclusion $E_m$

- Let $E_m$ denote the number of elements in $S$ that satisfy exactly $m$ of the $t$ conditions.
  - The principle of inclusion and exclusion corresponds to $E_0$.
- Then
  $$E_m = S_m - \binom{m+1}{1}S_{m+1} + \binom{m+2}{2}S_{m+2} - \ldots + (-1)^{t-m}\binom{t}{t-m}S_t$$
  $$= \sum_{k=m}^t (-1)^{k-m}\binom{k}{k-m}S_k. \quad (43)$$

The Proof

- If $x \in S$ satisfies fewer than $m$ conditions, then $x$ should contribute zero to $E_m$.
  - Indeed, it contributes zero to $S_m, S_{m+1}, \ldots, S_t$.
- If $x \in S$ satisfies exactly $m$ conditions, then $x$ should contribute one to $E_m$.
  - Indeed, it contributes one to $S_m$ and zero to $S_{m+1}, S_{m+2}, \ldots, S_t$.

The Proof (continued)

- If $x \in S$ satisfies $m < r \leq t$ of the conditions $c_i$, then $x$ should contribute zero to $E_m$.
- Indeed, it is counted $\binom{r}{m}$ times in $S_m$, $\binom{r}{m+1}$ times in $S_{m+1}$, \ldots, $\binom{r}{r}$ times in $S_r$, and zero times for all terms beyond $S_r$.
- The total count is
  $$\sum_{k=m}^r (-1)^{k-m}\binom{k}{k-m}\binom{r}{k} = \sum_{k=m}^r (-1)^{k-m}\binom{k}{m}\binom{r}{k}. \quad (44)$$

The Proof (concluded)

By Newton’s identity (p. 28),

$$\sum_{k=m}^r (-1)^{k-m}\binom{k}{m}\binom{r}{k} = \sum_{k=m}^r (-1)^{k-m}\binom{r}{m}\binom{r-m}{k-m} = \sum_{k=0}^{r-m} (-1)^{k}\binom{r}{m}\binom{r-m}{k}$$
$$= \binom{r}{m}\sum_{k=0}^{r-m} (-1)^{k}\binom{r-m}{k} = \binom{r}{m}(1-1)^{r-m} = 0.$$
Permutations with \( m \) Fixed Points

- Recall from p. 352 that a bijective function \( f \) on \( \{1, 2, \ldots, n\} \) has a fixed point at \( i \) if \( f(i) = i \).
- What is the number of permutations with \( m \) fixed points?
- Let \( c_i \) denote the condition that \( i \) is a fixed point.
- Then

\[
S_k = \binom{n}{k} (n-k)! = \frac{n!}{k!}, \quad (44)
\]

Generalized Principle of Inclusion and Exclusion \( L_m \)

- Let \( L_m \) denote the number of elements in \( S \) that satisfy at least \( m \) of the \( t \) conditions.
- Then

\[
L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots + (-1)^{t-m} \binom{t-1}{m-1} S_t
\]

\[
= \sum_{k=m}^{t} (-1)^{k-m} \binom{k-1}{m-1} S_k, \quad (45)
\]

The Proof (concluded)

- From Eq. (43) on p. 372,

\[
E_m = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{k-m} S_k
\]

\[
= \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{k-m} \frac{n!}{k!(n-k)!}
\]

\[
= \frac{n!}{m!} \sum_{k=m}^{n} (-1)^{k-m} \frac{1}{(k-m)!}
\]

- For example, \( E_{n-2} = n(n-1)/2 \).

The Proof

- By definition,

\[
L_m - L_{m+1} = E_m
\]

for \( m < t \) by definition.
- Now we prove the identity by induction on \( m \).
- First note that \( E_t = L_t = S_t \).
- Inductively, assume that

\[
L_{m+1} = \sum_{k=m+1}^{t} (-1)^{k-(m+1)} \binom{k-1}{m} S_k
\]

- Also \( E_m = \sum_{k=m}^{t} (-1)^{k-m} \binom{k}{m} S_k \) from (43) on p. 372.
**The Proof (concluded)**

- Finally, \( L_m \) equals

\[
L_{m+1} + E_m = \sum_{k=m+1}^{m} (-1)^{k-(m+1)} \binom{k-1}{m} S_k + \sum_{k=m}^{i} (-1)^{k-m} \binom{k}{m} S_k
\]

- Hence

\[
L_1 = \sum_{k=1}^{n} (-1)^{k-1} S_k = n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \approx n! \left(1 - \frac{1}{e}\right).
\]

**Permutations with Fixed Points**

- Consider permutations on \( \{1, 2, \ldots, n\} \).

- Let \( c_i \) stand for “\( i \) is a fixed point.”

- From Eq. (45) on p. 378, the number of permutations with at least one fixed point is

\[
L_1 = \sum_{k=1}^{i} (-1)^{k-1} S_k.
\]

**Checking for Consistency**

- The sum of the number of permutations without fixed points \( (E_0) \) and those with fixed points \( (L_1) \) should be \( n! \).

- Indeed, from Eq. (38) on p. 354 for \( E_0 \) and Eq. (46) on p. 382 for \( L_1 \),

\[
n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} = n!.
\]

- Note that \( E_0 \) is \( d_n \), the number of derangements.
Generating Functions

• Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers.
• The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

is called the generating function for the given sequence $\{a_i\}_{i=0,1,\ldots}$.

Convergence Issues

• We treat $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ as a formal power series.
• Convergence is not an issue unless
  - We substitute a numerical value for $x$ or
  - $a_0 + a_1 x + a_2 x^2 + \cdots$ represents some function rather than being an asymptotic series of some function.
• These issues will arise but will not lead to problems.\(^a\)


Examples

$(1 + x)^n$ is the generating function for sequence

$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, 0, 0, \ldots$

• $(1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$. 

$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$
Examples (continued)

\( (1 - x^{n+1})/(1 - x) \) is the generating function for the sequence
\[ 1, 1, \ldots, 1, 0, 0, \ldots \]
• \((1 - x^{n+1})/(1 - x) = 1 + x + x^2 + \cdots + x^n \).

Examples (continued)

\( -\ln(1 - x) \) is the generating function for
\[ 1, 1/2, 1/3, \ldots \]
• \(-\ln(1 - x) = 1 + (1/2) x + (1/3) x^2 + (1/4) x^3 + \cdots \).

Examples (continued)

1/(1 - x) is the generating function for
\[ 1, 1, 1, \ldots \]
• \(1/(1 - x) = 1 + x + x^2 + x^3 + \cdots \).

1/(1 - ax) is the generating function for
\[ 1, a, a^2, \ldots \]

Examples (continued)

Lemma 62

If \( f(x) \) is the generating function for
\[ a_0, a_1, a_2, \ldots, \]
then \( f'(x) \) is the generating function for
\[ a_1, 2a_2, 3a_3, \ldots \]
• \( f'(x) = \sum_{i=0}^{\infty} \frac{d}{dx} a_i x^i = \sum_{i=1}^{\infty} i a_i x^{i-1} \).

*Equivalently \( f'(x) \) is the generating function for \( \{ (i + 1) a_{i+1} \}_{i=0,1,\ldots} \) or \( xf'(x) \) is the generating function for \( \{ ia_i \}_{i=0,1,\ldots} \).
Examples (continued)

1/(1 − x)^2 is the generating function for
1, 2, 3, 4, . . . .

• 1/(1 − x) is the generating function for 1, 1, 1, . . .
  (p. 389).

• As 
  \[ \frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x}, \]
  1/(1 − x)^2 is the generating function for
  1 × 1, 2 × 1, 3 × 1, . . . = 1, 2, 3, . . ., as desired.

Examples (continued)

(x + 1)/(1 − x)^3 is the generating function for
1^2, 2^2, 3^2, 4^2, . . . .

Reason:

\[ \frac{x + 1}{(1-x)^3} = d \frac{d}{dx} \frac{x}{(1-x)^2} \]
\[ = d \frac{d}{dx} \sum_{i=1}^{\infty} ix^i \quad \text{from p. 392 and p. 393} \]
\[ = \sum_{i=1}^{\infty} i^2 x^{i-1}. \]

Examples (continued)

If \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) is the generating function for

\[ a_0, a_1, a_2, \ldots, \]

then \( xf(x) = \sum_{i=1}^{\infty} a_{i-1} x^i \) is the generating function for

\[ 0, a_0, a_1, a_2, \ldots. \]

• This operation is called the shift operator in signal
  processing or the operator \( E \) in difference equations.

Examples (continued)

If \( f(x) \) is the generating function for

\[ a_0, a_1, a_2, \ldots \]

and \( g(x) \) is the generating function for

\[ b_0, b_1, b_2, \ldots, \]

then \( f(x) + g(x) \) is the generating function for

\[ a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots. \]

• This is because \( f(x) + g(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i \).
Examples (concluded)

As an application,

\[ x \left[ \frac{1}{(1-x)^2} + \frac{x+1}{(1-x)^3} \right] \]

is the generating function for the sequence \( \{n + n^2\}_{n=0,1,2,...} \)
(see p. 392 and p. 394).

From Generating Function to Sequence

**Lemma 63** Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) be the generating function
for \( \{a_i\}_{i=0,1,...} \). Then

\[ a_i = \frac{1}{i!} \left. \frac{d^i f(x)}{dx^i} \right|_{x=0}, \quad i = 0, 1, 2, \ldots \]

- By the Taylor expansion.

Convolution and Generating Functions

- Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) be the generating function for
  \( a_0, a_1, a_2, \ldots \)
- Let \( g(x) = \sum_{i=0}^{\infty} b_i x^i \) be the generating function for
  \( b_0, b_1, b_2, \ldots \)
- Then \( f(x) g(x) \) is the generating function for
  \( h_0, h_1, h_2, \ldots \), where

\[ h_i = \sum_{k=0}^{i} a_k b_{i-k} \] (47)

called the **convolution** of \( a_i \)'s and \( b_i \)'s.

An Example

- Let \( f(x) = a_0 + a_1 x + a_2 x^2 \) and \( g(x) = b_0 + b_1 x + b_2 x^2 \).
- Then

\[
\begin{align*}
f(x) g(x) &= a_0 b_0 + (a_1 b_0 + a_0 b_1) x \\
&\quad + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 \\
&\quad + (a_2 b_1 + a_1 b_2) x^3 \\
&\quad + a_2 b_2 x^4.
\end{align*}
\]

- The convolution of two finite sequences can be computed using the **fast Fourier transform**.
Identity \( \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \) for \( n \in \mathbb{Z}^+ \) (Also p. 51)

\[
(1 + x)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i,
\]

\[
[(1 + x)^n]^2 = \left[ \sum_{i=0}^{n} \binom{n}{i} x^i \right]^2 = \sum_{i=0}^{2n} \left[ \sum_{k=0}^{i} \binom{n}{k} \binom{n}{i-k} \right] x^i.
\]

The coefficients for \( x^n \) are \( \binom{2n}{n} \) and \( \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2 \), respectively.

---

A Corollary of Vandermonde’s Convolution

**Corollary 64** \( \binom{r+n}{n} = \sum_{k=0}^{n} \binom{r}{k} \binom{n}{n-k}, \quad n \in \mathbb{N} \).

- Substitute \( n \) for \( s \) in Eq. (48) on p. 401 to obtain

\[
\binom{r+n}{n} = \sum_{k=0}^{n} \binom{r}{k} \binom{n}{n-k}.
\]

- Now note that \( \binom{n}{n-k} = \binom{n}{k} \).

---

Inverse of a Generating Function

**Theorem 65** Every generating function \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) with \( a_0 \neq 0 \) has a unique generating function \( g(x) \) such that \( f(x) g(x) = 1 \).

- Let

\[
g(x) = \sum_{i=0}^{\infty} b_i x^i,
\]

\[
h(x) = f(x) g(x) = \sum_{i=0}^{\infty} h_i x^i.
\]
The Proof (concluded)

- By Eq. (47) on p. 398, \( h_i = \sum_{k=0}^{i} a_k b_{i-k} \).
- By the requirements, \( h(x) = 1 \); hence \( h_0 = 1 \) and \( h_1 = h_2 = \cdots = 0 \).
- First, \( h_0 = 1 = a_0 b_0 \) implies that \( b_0 = 1/a_0 \).
- Inductively, \( h_{j+1} = 0 \) implies that
  \[
  b_{j+1} = -\frac{\sum_{k=1}^{j+1} a_k b_{(j+1)-k}}{a_0}.
  \]

\[1 + 2 + \cdots + n = \frac{n(n + 1)}{2} \text{ (See Also p. 33)}\]

- As \( 1/(1-x)^2 = \sum_{i=1}^{\infty} ix^{i-1} \) (p. 392), we have
  \( g(x) = x/(1-x)^2 = \sum_{i=1}^{\infty} ix^i \).
- By the summation operator, \( f(x) = x/(1-x)^3 \) is the generating function for \( \{1 + 2 + \cdots + n\}_{n=0,1,2,\ldots}. \)
- \( g'(x) = 1/(1-x)^2 + (2x)/(1-x)^3 = \sum_{i=1}^{\infty} ix^{i-1} + 2f(x) \).
- On the other hand, \( g'(x) = \sum_{i=1}^{\infty} i^2 x^{i-1} \).
- Combine the above two equations to obtain
  \[
  f(x) = \left( \sum_{i=1}^{\infty} i^2 x^{i-1} - \sum_{i=1}^{\infty} ix^{i-1} \right)/2 = \sum_{i=0}^{\infty} \frac{i(i+1)}{2} x^i.
  \]

The Summation Operator

- Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) be the generating function for \( a_0, a_1, a_2, \ldots \).
- Then \( f(x)/(1-x) \) is the generating function of sums \( a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots \).\n  - Let \( g(x) = (1-x)^{-1} = \sum_{i=0}^{\infty} x^i \) be the generating function of \( 1, 1, 1, \ldots \) (p. 389).
  - Then \( f(x)g(x) \) is the generating function of the said sum by Eq. (47) on p. 398 with \( b_1 = b_2 = \cdots = 1 \).

Binomial Coefficients Revisited

\[
\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.
\]

- \( (1 + x)^n \) is the generating function for \( \{\binom{n}{r}\}_{r=0,1,\ldots,n} \) (p. 387).
- \( (1 + x)^{n-1} \) is the generating function for \( \{\binom{n-1}{r}\}_{r=0,1,\ldots,n-1} \).
- \( x(1+x)^{n-1} \) is the generating function for \( 0, \binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-1} \).
- Finally, \( (1 + x)^n = (1 + x)^{n-1} + x(1 + x)^{n-1} \).
Generalized Binomial Coefficients

- For $n \in \mathbb{R}$, define
  \[
  \binom{n}{i} = \frac{n(n-1) \cdots (n-i+1)}{i!},
  \]  
  where $i \in \mathbb{Z}^+$. \hfill (49)

- For $n \in \mathbb{Z}^+$,
  \[
  \binom{-n}{i} = \frac{(-n)(-n-1) \cdots (-n-i+1)}{i!} = (-1)^i \binom{n+i-1}{i}.
  \] \hfill (50)

- The convention is that $\binom{n}{0} = 1$. 

The Generating Function for $\{\binom{n}{i}\}_{i=0,1,2,\ldots}$

For $n \in \mathbb{R}$, the Maclaurin series expansion for $(1+x)^n$ is
\[
(1+x)^n = 1 + nx + \frac{(n)(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots
\]
\[
= \sum_{i=0}^{\infty} \binom{n}{i} x^i,
\] \hfill (51)

where the last equality is by Eq. (49) on p. 408.

The Generating Function for $\{-\binom{n}{i}\}_{i=0,1,2,\ldots}$

For $n \in \mathbb{R}^+$, the Maclaurin series expansion for $(1+x)^{-n}$ is
\[
(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)x^2}{2!} + \frac{(-n)(-n-1)(-n-2)x^3}{3!} + \cdots
\]
\[
= \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} x^i
\] \hfill (52)

where the last equality is by Eq. (50) on p. 408.

The Generating Function for $\{-1\}^i(i+1)\}_{i=0,1,2,\ldots}$

- $(1+x)^{-2} = 1/(1+x)^2$ is the generating function for
  $\{-\binom{-2}{i}\}_{i=0,1,2,\ldots}$ by Eq. (52) on p. 410.

- By Eq. (50) on p. 408,
  \[
  \binom{-2}{i} = (-1)^i \binom{i+1}{i} = (-1)^i(i+1).
  \]
  \hfill (53)

- Alternatively, the same sequence is the convolution of
  $\{-1\}^i\}_{i=0,1,2,\ldots}$ and $\{-1\}^i\}_{i=0,1,2,\ldots}$.
The Generating Function for $\{\binom{n+i-1}{i}\}_{i=0,1,2,\ldots}$

For $n \in \mathbb{Z}^+$,

$$(1 - x)^{-n} = \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} (-x)^i$$

$$= \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$$

(53)

from the Maclaurin series expansion for $[1 + (-x)]^{-n}$ on p. 409 and Eq. (52) on p. 410.

---

An Example

- There are $\binom{11}{10} = 1001$ nonnegative integer solutions to
  $$x_1 + x_2 + x_3 + x_4 + x_5 = 10.$$  
- Now,
  $$(1 - x^{11})^5(1 - x)^{-5} = 1 + 5x + 15x^2 + 35x^3 + 70x^4 + 126x^5 + 210x^6 + 330x^7 + 495x^8 + 715x^9 + 1001x^{10} + \cdots.$$  
- The coefficient of $x^{10}$ is indeed 1001.

---

Integer Solutions of a Linear Equation Revisited

There are $\binom{n+r-1}{r}$ integer solutions to

$$x_1 + x_2 + \cdots + x_n = r,$$

where $x_i \geq 0$ (p. 65).

- The desired number is the coefficient of $x^r$ in
  $$f(x) = (1 + x + x^2 + \cdots + x^r)^n$$
  because
  $$f(x) = \sum_{0 \leq x_1, x_2, \ldots, x_n \leq r} x_1 x_2 \cdots x_n.$$  
- By Eq. (53) on p. 412,
  $$f(x) = (1 - x^{r+1})^n (1 - x)^{-n}$$
  $$= \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i + x^{r+1}(-\cdots).$$

---

A Simplified Proof

- For each variable $x_i$, the series $1 + x + x^2 + \cdots$ represents the possible value for that variable: $0, 1, 2, \ldots$.
- The desired number is the coefficient of $x^r$ in
  $$(1 + x + x^2 + \cdots)^n.$$  
- Now,
  $$(1 + x + x^2 + \cdots)^n = (1 - x)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$$
  by Eq. (53) on p. 412.
Integer Solutions of a Linear Equation with Upper Bounds

- The number of integer solutions of $x_1 + x_2 + \cdots + x_n = r$, where $0 \leq x_1, x_2, \ldots, x_n < b$, was solved on p. 368.
- The solution also appears as the coefficient of $x^r$ in
  $$(1 + x + x^2 + \cdots + x^{b-1})^n.$$  
  - What is the number of positive integers $x$, where $x \leq 999$, whose sum of the digits equals 20?
  - As $(1 + x + x^2 + \cdots + x^n)^3 = (\cdots) + 36x^{20} + (\cdots)$, the answer is 36.
  - The same problem was solved on p. 371.

Compositions of Positive Integers Revisited

- Recall that a composition for $m$ is a sum of positive integers whose order is relevant and which sum to $m$ (p. 75).
- Next we use the generating function to reprove the fact that the number of compositions for $m$ is $2^{m-1}$.
- First, $x/(1-x) = x + x^2 + x^3 + \cdots$.
- So the coefficient of $x^m$ in $[x/(1-x)]^i$ is the number of compositions with $i$ summands for $m$.

The Proof (continued)

- The generating function for the number of compositions is therefore
  $$\sum_{i=1}^{\infty} \left( \frac{x}{1-x} \right)^i.$$  
- Now,
  $$\sum_{i=1}^{\infty} \left( \frac{x}{1-x} \right)^i = \frac{x}{1-x} \sum_{i=0}^{\infty} \left( \frac{x}{1-x} \right)^i = \frac{x}{1-x} \frac{1}{1-\frac{x}{1-x}} = \frac{x}{1-2x}.$$  

The Proof (concluded)

- Observe that
  $$\frac{x}{1-2x} = x \left[ 1 + (2x) + (2x)^2 + \cdots \right] = \left[ x + 2x^2 + 2^2x^3 + \cdots \right] = \sum_{i=1}^{\infty} 2^{i-1}x^i.$$  
  - So the number of compositions for $m$ is indeed $2^{m-1}$.
Partition of Integers

- We ask for the number of partitions of \( m \in \mathbb{Z}^+ \) into positive integers where the order of summands is irrelevant.
  - Let \( p(m) \) denote the number of partitions of \( m \).
  - The number of partitions of \( m = 3 \) is \( p(3) = 3: 3, 2 + 1, 1 + 1 + 1 \).

- Contrast it with composition on p. 75 and p. 417.

- The number \( m \) can be a sum of a few 1s, a few 2s, a few 3s, . . . , and a few \( m \)'s.

The Proof (concluded)

- The desired number \( p(m) \) is the coefficient of \( x^m \) in
  \[
  \frac{1}{1-x}(1+x+x^2+x^3+\cdots)(1+x^2+x^4+\cdots)
  \]
  \[
  = \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \frac{1}{1-x^m}.
  \] \( (54) \)

- The generating function for \( \{ p(n) \}_{n=0,1,2,\ldots} \) is
  \[
  \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots.
  \] \( (55) \)

An Example

- Note that
  \[
  \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}
  \]
  \[
  = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \cdots.
  \]
  - So there are 7 ways to partition 5.
  - Indeed, the partitions are: 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1.

- No known closed-form formula.

Comments on Calculation

- We were asked to calculate the coefficient of \( x^n \) in
  \[
  \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \frac{1}{1-x^n}
  \]
  in Eq. (54) on p. 421.

- But it is a product of infinite power series!

- The trick is to calculate only
  \[
  [1 - x + x^2 - \cdots + (-1)^n x^n] [1 - x^2 + x^4 - \cdots + (-1)^n x^{2n}]
  \]
  \[
  [1 - x^3 + x^6 - \cdots + (-1)^n x^{3n}]
  \]
  \[
  \cdots [1 - x^n + x^{2n} - \cdots + (-1)^n x^{n^2}].
  \]

- We can even cut those terms beyond \( x^n \).
Application: Number of Groupings

- The number of ways for \( m \) distinct objects to form groups is a Bell number (p. 228).
- How many ways are there for \( m \) identical objects to form groups?
- Well, it is the same as the number of partitions of \( m \) into positive integers, i.e., \( p(m) \) (p. 420).
  - Each partition of \( m \), say \( m_1 + m_2 + \cdots \), incurs a groupings of sizes \( m_1, m_2, \ldots \), and vice versa.

No Summands Appear More Than Twice

- What is the number of partitions of \( m \in \mathbb{Z}^+ \) into positive integers where no summands appear more than twice?
- The desired number is the coefficient of \( x^m \) in
  \[
  (1 + x + x^2)(1 + x^2 + x^4)(1 + x^3 + x^6)(1 + x^4 + x^8) \cdots \\
  = \frac{1 - x^3}{1 - x} \frac{1 - x^6}{1 - x^2} \frac{1 - x^{12}}{1 - x^3} \cdots = \prod_{i=1}^{\infty} \frac{1 - x^{3i}}{1 - x^i} \\
  = \frac{1}{1 - x} \frac{1}{1 - x^2} \frac{1}{1 - x^4} \frac{1}{1 - x^6} \frac{1}{1 - x^7} \cdots
  
  \]
- Same as partitions into summands not divisible by 3.

Weighted Integer Solutions of a Linear Equation

- What is the number of integer solutions to
  \[
  x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n,
  \]
  where \( x_i \geq 0 \)?
- For example, the number of solutions for \( n = 5 \) is 7:
  \[
  (x_1, x_2, x_3, x_4, x_5) \in \{(5, 0, 0, 0, 0), (3, 1, 0, 0, 0), (2, 0, 1, 0, 0), (1, 2, 0, 0, 0), (1, 0, 0, 1, 0), (0, 1, 1, 0, 0), (0, 0, 0, 0, 1)\}.
  \]

The Proof (concluded)

- This problem is the partition-of-integers problem in disguise.
  - Every solution \( (x_1, x_2, \ldots, x_n) \) implies a partition of \( n \) in which there are \( x_i \)’s, and vice versa.
- The desired number is again \( p(n) \), the coefficient of \( x^n \) in
  \[
  \frac{1}{1 - x} \frac{1}{1 - x^2} \frac{1}{1 - x^3} \cdots \frac{1}{1 - x^n}
  \]
  from Eq. (54) on p. 421.
Partition of Integers into Distinct Summands

- We ask for the number of partitions of \( m \in \mathbb{Z}^+ \) into \textit{distinct} positive integers.
  - Let \( p^\#(m) \) denote the number of such partitions of \( m \).
  - The number of partitions of \( m = 3 \) is \( p^\#(3) = 2 : 3, 2 + 1 \).
- The desired number \( p^\#(m) \) is the coefficient of \( x^m \) in
  \[ (1 + x)(1 + x^2)(1 + x^3) \cdots (1 + x^m). \] \hfill (56)

An Example

- Note that
  \[
  (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6) \\
  = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \cdots.
  \]
- So there are 4 ways to partition 6.
- Indeed, the partitions are: 6, 5 + 1, 4 + 2, 3 + 2 + 1.

Partition of Integers into Distinct Summands (concluded)

- There are no known closed-form formulas for \( p^\#(m) \).
- Can we compute \( p^\#(m) \) in \( o(m^3) \) steps (in the word model)?
- What is a good upper bound on \( p^\#(m) \)?
- How fast can we compute all the actual solutions?

\^aThanks to a lively discussion on November 29, 2004.

Partition of Integers into Odd Summands\(^a\)

- What is the number of partitions of \( m \in \mathbb{Z}^+ \) into \textit{odd} positive integers?
  - The number of partitions of \( m = 3 \) is 2: 3, 1 + 1 + 1.
  
- The desired number is the coefficient of \( x^m \) in
  \[
  (1 + x + x^2 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^5 + x^{10} + \cdots) \cdots.
  \]

\(^a\)Euler (1748).
The Proof (concluded)

• But,

\[
\begin{align*}
(1 + x + x^2 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^5 + x^{10} + \cdots) \cdots \\
= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots = \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdots \\
= (1+x)(1+x^2)(1+x^3)(1+x^4) \cdots (1+x^m) \cdots.
\end{align*}
\]

• Hence the count is same as that of partitions into
distinct summands \( p^\#(m) \) by Eq. (56) on p. 428.

• This is called Euler's theorem.