Upper-Bounding \( v(n) \)

Theorem 55 (Erdős and Moser (1964))

\[ v(n) \leq \lceil 2 \log_2 n \rceil. \]

- We will show that there exists a tournament on \( n \) players with no transitive subtournaments on \( v \equiv 1 + \lceil 2 \log_2 n \rceil \) players.
- This establishes the theorem because then \( v(n) < v \).
- Note that \( (v - 1)/2 \geq \log_2 n \).
- There are \( 2^\binom{v}{2} \) possible tournaments on \( n \) players.

The Proof (continued)

- With a subtournament on \( v \) players fixed, there are \( 2^\binom{v}{2} - \binom{v}{2} \) possible tournaments on \( n \) players.
- The total number of tournaments on \( n \) players that contain a transitive subtournament on \( v \) players is at most

\[
\binom{n}{v} \cdot \frac{n!}{2^\binom{v}{2} (n-v)!} \cdot 2^\binom{v}{2} \leq \frac{n(n-1) \cdots (n-v+1)}{2^v \log_2 n} 2^\binom{v}{2} < 2^\binom{v}{2}.
\]

The Proof (concluded)

- Recall that \( 2^\binom{n}{2} \) is the total number of tournaments on \( n \) players.
- The total number of tournaments on \( n \) players that contain a transitive subtournament on \( v \) players is less than the total number of tournaments on \( n \) players.
- So there exists a tournament on \( n \) players without a transitive subtournament on \( v \) players.

Antisymmetric Relations

- \( R \) is antisymmetric if \( (x, y) \in R \land (y, x) \in R \Rightarrow x = y \) for all \( x, y \in A \).
  - “\( \leq \)” is antisymmetric.
  - “\( \leq \)” is antisymmetric.
- Alternatively, \( R \) is antisymmetric if for all \( x, y \in A \).
  \[ x \neq y \Rightarrow (x, y) \notin R \lor (y, x) \notin R. \] (29)
  - “\( < \)” is antisymmetric because
  \[ x \neq y \Rightarrow x < y \lor y < x. \]
Antisymmetric Relations (concluded)

- Antisymmetry is not synonymous with “symmetric.”
  - “⊆” is antisymmetric but not symmetric.
- Antisymmetry is not synonymous with “not symmetric.”
  - Take \( R \) as the relation that is an empty set.
  - So \( (x, y) \notin R \) for any \( x, y \).
  - Then \( R \) is antisymmetric and symmetric.

Number of Antisymmetric Relations

**Lemma 56** If \( |A| = m \), then there are
\[
2^m 3^{(m^2 - m)/2}
\]
antisymmetric relations on \( A \).

- The \( m \) decisions on \( (x, x) \in R \) are arbitrary.
- For each of the other \( \binom{m}{2} = (m^2 - m)/2 \) unordered pairs
  \( \{x, y\} (x \neq y) \), there are 3 choices suggested by Eq. (29):
  1. \( (x, y) \in R \) but \( (y, x) \notin R \).
  2. \( (x, y) \notin R \) but \( (y, x) \in R \).
  3. \( (x, y) \notin R \) and \( (y, x) \notin R \).

Inverse Relations

- Let \( R \subseteq A \times B \) be a relation.
- The inverse of \( R \), denoted \( R^{-1} \), is this relation from \( B \) to \( A \):
  \[
  R^{-1} = \{(b, a) : (a, b) \in R\}
  \]
  - The inverse of \( “\leq” \) is \( “\geq” \) (not \( “>” \)).
  - The inverse of \( “<” \) is \( “>” \) (not \( “\geq” \)).
- Note that inversehood and complement are not different concepts.

A Property of \( R^{-1} \)

**Lemma 57** If \( R \) is reflexive on \( A \), then \( R^{-1} \) is also reflexive.

- Let \( a \in A \).
- Then \( (a, a) \in R \).
- Hence \( (a, a) \in R^{-1} \).
- So \( R^{-1} \) is reflexive.
Composite Relations

- Let $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$ be two relations.
- The composite relation $R_1 \circ R_2$ is a relation from $A$ to $C$ defined by
  \[ \{(x, z) : x \in A, z \in C, \exists y \in B \ [(x, y) \in R_1 \land (y, z) \in R_2]\}. \]
- The associative law holds:
  \[ R_1 \circ (R_2 \circ R_3) = (R_1 \circ R_2) \circ R_3. \]
- $R^n = R \circ R \circ \cdots \circ R$ is called the power of $R$.

Matrices and Zero-One Matrices

- The $m \times n$ matrix $(a_{ij})_{m \times n}$ denotes the entry in the $i$th row and the $j$th column is $a_{ij}$.
- The transpose of $A = (a_{ij})_{m \times n}$, written as $A^t$, is the matrix $(b_{ij})_{n \times m}$, where $b_{ij} = a_{ji}$.
- $I_n$ is the $n \times n$ identity matrix.
- A zero-one matrix has entries of zeros and ones.
  - $- + \rightarrow \lor$.
  - $- \times \rightarrow \land$.

Matrix Precedence

- Let $E = (e_{ij})$ and $F = (f_{ij})$ be two $m \times n$ zero-one matrices.
- We say $E$ precedes (or is less than) $F$, written as $E \leq F$, if $e_{ij} \leq f_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.
- For example,
  \[
  \begin{bmatrix}
  0 & 1 & 1 \\
  0 & 0 & 1 \\
  0 & 0 & 0 \\
  \end{bmatrix}
  \leq
  \begin{bmatrix}
  0 & 1 & 1 \\
  0 & 1 & 1 \\
  0 & 0 & 0 \\
  \end{bmatrix}.
  \]
The Zero-One Matrix Representation of Relations

- Let $R$ be a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
- The relation matrix of $R$, $M(R)$, is the $m \times n$ zero-one matrix $(r_{ij})_{m \times n}$, where
  \[
  r_{ij} = \begin{cases} 
  1, & \text{if } (a_i, b_j) \in R \\
  0, & \text{if } (a_i, b_j) \notin R 
  \end{cases}
  \]
- It can be shown that
  \[M(R_1 \circ R_2) = M(R_1)M(R_2).\]  
  \hspace{1cm} (30)

An Example

- Consider the binary relation $<$ on $\{1, 2, 3, 4\}$.
- Here is the relation matrix:
  \[
  M(<) = \begin{bmatrix}
  1 & 0 & 1 & 1 \\
  2 & 0 & 0 & 1 \\
  3 & 0 & 0 & 0 \\
  4 & 0 & 0 & 0
  \end{bmatrix}.
  \]

An Example (continued)

- Now,
  \[
  M(<)M(<) = \begin{bmatrix}
  0 & 0 & 1 & 1 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
  \end{bmatrix}.
  \]
- The entry at $(1,4)$ is calculated as follows:
  \[
  [0, 1, 1, 1] \cdot [1, 1, 1, 0] = (0 \land 1) \lor (1 \land 1) \lor (1 \land 1) \lor (1 \land 0) = 1.
  \]

An Example (concluded)

- By Eq. (30) on p. 288, the above denotes the relation $\ll$ where $x \ll y$ if there exists a $z$ with $x < z$ and $z < y$.
- Sensibly, it says $1 \ll 3$, $1 \ll 4$, and $2 \ll 4$. 
Relation Matrices and Relations

- Let $\mathcal{R}$ be a relation on $A$ with $|A| = n$ and $M = M(\mathcal{R})$.
- $\mathcal{R}$ is reflexive if and only if $I_n \leq M$.
  - This means that $m_{ii} = 1$ in $M = (m_{ij})_{1 \leq i,j \leq n}$.
- $\mathcal{R}$ is symmetric if and only if $M = M^{tr}$.
- $\mathcal{R}$ is transitive if and only if $M^2 \leq M$.
  - Verify this inequality with the $M(<)M(<)$ on p. 290.
- $\mathcal{R}$ is antisymmetric if and only if $(M \land M^{tr}) \leq I_n$.
  - Verify this inequality with the $M(<)$ on p. 289.

Directed Graphs* and Relations

- A directed graph (or digraph) $G = (V, E)$ is made up of the node set $V$ and the edge set $E \subseteq V \times V$.
- If $(a, b) \in E$, there is an edge from node $a$ to node $b$.
  - $a$ is adjacent to $b$, whereas $b$ is adjacent from $a$.
  - $(a, a)$ is a loop (at $a$).
- Clearly, a digraph $(V, E)$ corresponds to a relation $\mathcal{R}$ on $V$, and vice versa.
  - $(x, y) \in \mathcal{R}$ if and only if $(x, y) \in E$.

---

*Euler (1736).
Adjacency Matrices

- The relation matrix for a digraph is called an adjacency matrix.
- The number of 1s is therefore the number of edges.
- The adjacency matrix for the digraph on p. 295 appeared on p. 289.

Operations on Adjacency Matrices

- $M^2$ (over normal $+$ and $\times$) is the number of paths of length 2 between any two nodes.
  - $M^2_{i,j} = M_{i1}M_{1j} + M_{i2}M_{2j} + \cdots + M_{in}M_{nj}$.
- Take the adjacency matrix $M(<)$ on p. 289 for the digraph on p. 295.
- Then
  \[
  M(<)M(<)M(<) = \begin{bmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
  \end{bmatrix}.
  \]

Operations on Adjacency Matrices (continued)

- In general, $M^k$ (over normal $+$ and $\times$) is the number of paths of length $k$ between any two nodes.
- Again, take the adjacency matrix $M(<)$ on p. 289 for the digraph on p. 295.
- Then
  \[
  M(<)M(<)M(<) = \begin{bmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
  \end{bmatrix}.
  \]

Operations on Adjacency Matrices (concluded)

- Similarly, $M^k$ (over $\lor$ and $\land$) represents if there exist paths of length $k$ between any two nodes.
- Therefore the matrix
  \[
  M^0 \lor M^1 \lor M^2 \lor \cdots \lor M^{|V| - 1}
  \]
  (over $\lor$ and $\land$) represents if there exist paths between any two nodes.
Partial Order

- A relation $R$ on $A$ is called a partial order or partial ordering relation if it is
  1. Reflexive.
  2. Antisymmetric.
  3. Transitive.
- “$\leq$” is a partial order.
- “$|$” (divisibility) is a partial order on $\mathbb{Z}^+$.

Total Order

- Let $(A, R)$ be a poset.
- $A$ is totally ordered if for all $x, y \in A$, either $(x, y) \in R$ or $(y, x) \in R$.
- This $R$ is called a total order.
  - $(\mathbb{R}, \leq)$ is a total order.
- Elements in a totally ordered set can be ranked.
- There are $m!$ relations on $A$ that are total orders, where $m = |A|$.

Partially Ordered Sets (Posets)

- Let $A$ be a set.
- $R$ is a relation on $A$.
- $(A, R)$ is a partially ordered set or poset if $R$ is a partial order.
  - Often $R$ is not mentioned explicitly.
  - $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all posets with “$\leq$”.
  - $(\mathbb{Z}^+, |)$ is a poset.

Relation Matrices of Total Orders

- Let $R$ be a total order on $A$ with $m = |A|$.
- Its relation matrix $M(R)$ has $m(m + 1)/2$ 1s.
  - After reordering the rows and columns by ranks, $M(R)(i, j) = 1$ if and only if $i \leq j$.
  - In other words, the matrix is an upper triangular matrix.
Hasse\textsuperscript{a} Diagrams

- Let \((A, \mathcal{R})\) be a poset.
- Draw an edge from \(x\) to \(y\) if \((x, y) \in \mathcal{R}\) and there is no other \(z \in A\) such that \((x, z) \in \mathcal{R}\) and \((z, y) \in \mathcal{R}\).
  - Edge \((x, y)\) cannot be inferred from other edges via transitivity considerations.
- The resulting graph is called a Hasse diagram.

\textsuperscript{a}Helmut Hasse (1898–1979).

A Poset

- \((\{1, 2, 3, 4, 5, 6, 7, 8\}, |)\).
- A loop at each node (not shown).
- Delete loops and “unnecessary” transitive edges to form the Hasse diagram.
  - Edge \((1, 6)\) is deleted because it can be inferred from edges \((1, 3)\) and \((3, 6)\).
  - Edge \((2, 6)\), on the other hand, cannot be inferred from other edges via transitivity.

Topological Sort

- Given a partial order \(\mathcal{R}\) represented as a Hasse diagram.
- The topological sorting algorithm produces a total order \(\mathcal{T}\) for which \(\mathcal{R} \subseteq \mathcal{T}\).
  - The total order needs only honor those \((x, y)\) in \(\mathcal{R}\).
  - The total order may not be unique.
  - The partial order on p. 306 gives rise to one total order
    \[1, 2, 4, 3, 8, 7, 6, 5.\] (31)
  - 1, 2, 4, 3, 8, 7, 5, 6 is another total order.
  - Both honor the relations in the Hasse diagram.
The Topological Sorting Algorithm

1: $H_1$ is the input Hasse diagram; $|A| = n.$
2: {Think of $\mathcal{R}$ as $\leq$ for convenience below.}
3: for $k = 1, 2, \ldots, n$ do
4: Pick $v_k \in H_k$ such that no edge in $H_k$ starts at $v_k$;
5: if $k = n$ then
6: return $v_n \leq v_{n-1} \leq \cdots \leq v_1$;
7: end if
8: Remove $v_k$ and all edges that terminate at $v_k$ to yield $H_{k+1}$;
9: end for

Maximal and Minimal Elements of Posets

- Let $(A, \mathcal{R})$ be a poset.
- $x \in A$ is a maximal element of $A$ if $(x, a) \notin \mathcal{R}$ for all $a \in A$ and $a \neq x$.
  - The poset $\{1, 2, 3, 4, 5, 6, 7, 8\}$ has 4 maximal elements: 5, 6, 7, 8 as none of them divides a number in $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
- $y \in A$ is a minimal element of $A$ if $(b, y) \notin \mathcal{R}$ for all $b \in A$ and $b \neq y$.
  - The poset $(\mathbb{Z}^+, \leq)$ has minimal element 1 but no maximal elements.

Maximal and Minimal Elements of Finite Posets

- Let $(A, \mathcal{R})$ be a finite poset.
- Every $y \in A$ must have a minimal element $x \in A$ such that $(x, y) \in \mathcal{R}$.
  - Suppose for all minimal elements $x$, $(x, y) \notin \mathcal{R}$.
  - In particular, $y$ is not a minimal element as $(y, y) \in \mathcal{R}$.
  - Thus for some $x$, $(x, y) \in \mathcal{R}$.
  - A “smallest” $x$ is a minimal element, a contradiction.
- Similarly, every $x \in A$ must have a maximal element $y \in A$ such that $(x, y) \in \mathcal{R}$.

Maximal and Minimal Elements of Finite Posets (concluded)

- We can reexpress our findings as follows.
- Let $X$ be the set of all minimal elements.
- Let $Y$ be the set of all maximal elements.
- Then
  \[
  \{ x \in A : (p, x) \in \mathcal{R} \text{ for some } p \in Y \} = \emptyset, \quad (32)
  \]
  \[
  \{ x \in A : (p, x) \in \mathcal{R} \text{ for some } p \in X \} = A, \quad (33)
  \]
  \[
  \{ x \in A : (x, q) \in \mathcal{R} \text{ for some } q \in X \} = \emptyset, \quad (34)
  \]
  \[
  \{ x \in A : (x, q) \in \mathcal{R} \text{ for some } q \in Y \} = A. \quad (35)
  \]
Existence of Maximal and Minimal Elements

- If $A$ is a finite poset, then $A$ has both a maximal and a minimal element.
  - The topological sorting algorithm returns a maximal element as $v_1$ and a minimal element as $v_n$.
  - For the list on p. 307, 5 is returned by the algorithm as a maximal element of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
- A poset may have more than one maximal elements and/or more than one minimal elements.
  - Recall that the poset $\{1, 2, 3, 4, 5, 6, 7, 8\}$ has 4 maximal elements and 1 minimal element.

Relations between Maximal and Minimal Elements

- If $(A, \mathcal{R})$ is a finite poset, then $A$ has a minimal element $x$ and a maximal element $y$ such that $(x, y) \in \mathcal{R}$.
  - Start with a minimal element $x$, which exists (p. 312).
  - Follow the “up edges” of $\mathcal{R}$ until cannot go further.
  - The element $y$ where we stop at will be a maximal element.
  - Furthermore, $(x, y) \in \mathcal{R}$ by transitivity.\(^a\)

Least and Greatest Elements of Posets

- Let $(A, \mathcal{R})$ be a poset.
- Let $x \in A$ be a least element if $(x, a) \in \mathcal{R}$ for all $a \in A$.
- Let $y \in A$ be a greatest element if $(b, y) \in \mathcal{R}$ for all $b \in A$.
- Least element and greatest element, if they exist, are unique.
  - Suppose $x, y$ are both greatest (least) elements.
  - Then $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$, which imply $x = y$ because of antisymmetry.

Maximal vs. Greatest Elements of Posets\(^a\)

- It is possible for a poset to have maximal elements but no greatest elements (p. 309).
- It is also possible for a poset to have multiple maximal elements (p. 309 again).
- But the greatest element, if it exists, must be a maximal element.
- In fact, the greatest element, if it exists, must be the only maximal element.

\(^a\)It is possible that $x = y$ if $x$ is isolated.

\(^a\)Contributed by Ms. Li-Yin Wu (B91902051) on October 20, 2003.
Lattices

- Let \((A, \mathcal{R})\) be a poset with \(B \subseteq A\).
- \(x \in A\) is a lower bound of \(B\) if \((x, b) \in \mathcal{R}\) for all \(b \in B\).
- \(y \in A\) is an upper bound of \(B\) if \((a, y) \in \mathcal{R}\) for all \(a \in B\).
- \(x' \in A\) is a greatest lower bound (\(\text{glb}\)) of \(B\) if it is a lower bound of \(B\) and if for all lower bounds \(x''\) of \(B\), \((x'', x') \in \mathcal{R}\).
- \(y' \in A\) is a least upper bound (\(\text{lub}\)) of \(B\) if it is an upper bound of \(B\) and if for all upper bounds \(y''\) of \(B\), \((y', y'') \in \mathcal{R}\).
- \((A, \mathcal{R})\) is called a lattice if for all \(x, y \in A\), the elements \(\text{lub}\{x, y\}\) and \(\text{glb}\{x, y\}\) both exist in \(A\).

\*Friedrich Schröder (1841–1902).

Examples of Lattices

- \((\mathbb{N}, \leq)\).
  - \(\text{lub}\{x, y\} = \max(x, y)\).
  - \(\text{glb}\{x, y\} = \min(x, y)\).
- \((2^A, \subseteq)\).
  - \(\text{lub}\{S, T\} = S \cup T\).
  - \(\text{glb}\{S, T\} = S \cap T\).

Partitions

- Let \(\emptyset \neq A_i \subseteq A\) for \(i \in I\).
- \(\{A_i\}_{i \in I}\) is a partition of \(A\) if
  - \(A = \bigcup_{i \in I} A_i\), and
  - \(A_i \cap A_j = \emptyset\) for \(i \neq j\).
A Partition

- Let
  \[ A_i = \{ x \in \mathbb{Z} : x \equiv i \mod n \} . \]
- Then
  \[ \{ A_0, A_1, \ldots, A_{n-1} \} \]
  is a partition of \( \mathbb{Z} \).

Equivalence Relations

- A relation \( R \) on \( A \) is called an equivalence relation if it is reflexive, symmetric, and transitive.
  - “\( = \)” is an equivalence relation.
  - “\( \equiv \mod m \)” is an equivalence relation.
  - “\( < \)” is not an equivalence relation.

Number of Partitions

- The number of ways to partition a set of size \( n \) into \( k \) blocks is \( S(n, k) \), a Stirling number.
  - See p. 215 or Eq. (25) on p. 218 and Eq. (26) on p. 220 for easy-to-use recurrence relations.
- The number of ways to partition a set of size \( n \) is \( P_n \), the \( n \)th Bell number (p. 228).

Equivalence Classes

- Let \( R \) be an equivalence relation on \( A \).
- For each \( x \in A \), the equivalence class of \( x \), denoted by \( [x] \), is defined by
  \[ [x] = \{ y \in A : (y, x) \in R \} . \]
  - Anything that is related to \( x \) is in \( [x] \).
- Consider the equivalence relation \( \equiv \mod n \) on \( \mathbb{Z} \) (same remainder after division by \( n \)).
- Then \( [i] = \{ x \in \mathbb{Z} : x \equiv i \mod n \} \).
- \( \{ [i] \}_{i=0,1,\ldots,n-1} \) is a partition of \( \mathbb{Z} \).
Equivalence Classes as Partitions

• Let $\mathcal{R}$ be an equivalence relation on $A$ and $x, y \in A$.
  • $x \in [x]$.
  • $(x, y) \in \mathcal{R}$ if and only if $[x] = [y]$.
    - Suppose $(x, y) \in \mathcal{R}$.
      * Pick any $w \in [x]$.
      * $(w, y) \in \mathcal{R}$ by transitivity and, hence, $w \in [y]$.
      * We conclude $[x] \subseteq [y]$.
    - Similarly, $[y] \subseteq [x]$.
  • $[x] = [y] \Rightarrow x \in [y] \Rightarrow (x, y) \in \mathcal{R}$.

The Proof (continued)

• From $x \in [x]$, we have $\bigcup_{x \in A} [x] = A$.
• Furthermore, distinct equivalence classes are disjoint.
• Therefore, equivalence classes partition $A$.

Equivalence Relations and Partitions

• Let $A$ be a set.
  • Any equivalence relation $\mathcal{R}$ on $A$ induces a natural partition of $A$: $\{[a] : a \in A\}$.
  • This partition, written as $A/\mathcal{R}$, is called the quotient.
• Conversely, any partition of $A$ gives rise to an equivalence relation on $A$.
  - Define $\mathcal{R}$ by $(x, y) \in \mathcal{R}$ if $x$ and $y$ are in the same block.
  - There is a one-to-one correspondence between the set of equivalence relations on $A$ and the set of partitions of $A$. 

\[
\begin{align*}
  &\exists v \in A \left[ v \in [x] \cap [y] \right] \\
  \Rightarrow &\exists v \in A \left[ (x, v) \in \mathcal{R} \land (v, y) \in \mathcal{R} \right] \\
  \Rightarrow & (x, y) \in \mathcal{R} \\
  \Rightarrow & [x] = [y],
\end{align*}
\] 

a contradiction.