Combinations

• Suppose there \( n \) distinct objects and \( r \) is an integer, 
  \( 1 \leq r \leq n \).

• The number of \textit{combinations} (selections \textit{without} reference to order) of \( r \) of these objects is

\[
C'(n, r) = \binom{n}{r} = \frac{n(n-1) \cdots (n-r+1)}{r(r-1) \cdots 1}.
\]

- \( C(n, r) = P(n, r)/r! \) because order is irrelevant.

• \( C(n, 0) = 1 \) for all \( n \geq 0 \).

• \( C(n, r) = C(n, n-r) \) for all \( n \geq 0 \).
Combinations (concluded)

• A finite sequence \((a_1, a_2, \ldots, a_n)\) of real numbers is **unimodal** if there exists a positive integer \(1 < j < n\) such that

\[
a_1 < a_2 < \cdots < a_{j-1} \leq a_j > a_{j+1} > \cdots > a_n.
\]

• \((C(n, 0), C(n, 1), \ldots, C(n, n))\) is unimodal.
  \(- C(n, r + 1)/C(n, r) = (n - r)/(r + 1).

• \(C(n, n/2)\) is *the* maximum element when \(n\) is even.

• \(C(n, (n - 1)/2)\) and \(C(n, (n + 1)/2)\) are *the* maximum elements when \(n\) is odd.
An Example

How many ways are there to arrange TALLAHASSEE with no adjacent As?

- Rearrange the characters as AAAEEHLLSST.
- TALLAHASSEE has 11 characters among which there are 3 As.
- There are \( \frac{8!}{2! \cdot 2! \cdot 1! \cdot 1!} = 5,040 \) ways to arrange the 8 non-A characters.
- For each such arrangement, there are 9 places to insert the 3 As.
- The desired number is hence \( 5,040 \times \binom{9}{3} = 423,360 \).
Pascal’s³ Identity

Lemma 8 \( \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \).

- Algebraic proof.
- Combinatorial proof.
- Generating-function proof (p. 530).

³Blaise Pascal (1623–1662).
Combinatorial Proof: Newton’s Identity

Lemma 9 \( \binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k} \).

- A university has \( n \) professors.
- The faculty assembly requires \( r \) professors.
- Among the members of the assembly, \( k \) serve the executive committee.
- Let us count the number of ways the executive committee can be formed.

\(^a\)Isaac Newton (1643–1727).
The Proof (concluded)

• We can first form the assembly in \( \binom{n}{r} \) ways.

• Then we pick the executive committee members from the assembly in \( \binom{r}{k} \) ways.

• The total count is \( \binom{n}{r} \binom{r}{k} \).

• Alternatively, we can pick the executive committee first, in \( \binom{n}{k} \) ways.

• Then we pick the remaining \( r - k \) members of the assembly in \( \binom{n-k}{r-k} \) ways.

• The total count is \( \binom{n}{k} \binom{n-k}{r-k} \).
Combinatorial Proof: Counting It Twice

Lemma 10  For \( m, n \geq 0 \), \( \sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1} \).

- We want to pick \( m + 1 \) tickets from a set of \( n + 1 \) tickets.
- There are \( \binom{n+1}{m+1} \) ways.
- Alternatively, number the tickets from 0 through \( n \).
- There are \( \binom{k}{m} \) ways to do this when the ticket with the largest number is \( k \) (\( m \leq k \leq n \)).
- Alternative proof: Apply Lemma 8 (p. 27) iteratively.
Combinatorial Proof: Counting It Twice (continued)

**Corollary 11** \( \sum_{k=1}^{m} k(k+1) = 2\binom{m+2}{3} \).

\[
\begin{align*}
\sum_{k=1}^{m} k(k+1) &= 2 \sum_{k=1}^{m} \binom{k+1}{2} \\
&= 2 \binom{m+2}{3}
\end{align*}
\]

by Lemma 10 (p. 30).
Combinatorial Proof: Counting It Twice (continued)

Corollary 12  For $m, n \geq 0$, $\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$.

\[
\sum_{k=0}^{m} \binom{n+k}{k} = \sum_{k=0}^{m} \binom{n+k}{n} = \sum_{k=n}^{n+m} \binom{k}{n} = \binom{n+m+1}{n+1} \quad \text{by Lemma 10 (p. 30)}
\]

\[
= \binom{n+m+1}{m}.
\]
Combinatorial Proof: Counting It Twice (continued)

Corollary 13 \(1 + 2 + \cdots + n = n(n + 1)/2\).

- Set \(m = 1\) in Lemma 10 (p. 30) to obtain
  \[
  \sum_{k=1}^{n} \binom{k}{1} = \binom{n+1}{2}.
  \]

- But this is equivalent to
  \[
  \sum_{k=1}^{n} k = n(n + 1)/2,
  \]
as desired.
Combinatorial Proof: Counting It Twice (continued)

We shall adopt the convention that

\[
\binom{n}{i} = 0
\]

for \( i < 0 \) or \( i > n \), where \( n \) is a positive integer.
Combinatorial Proof: Counting It Twice (continued)

Lemma 14 \[ \binom{m+n}{2} - \binom{m}{2} - \binom{n}{2} = mn. \]

- Consider \( m \) men and \( n \) women.
- The number of heterosexual marriages is \( mn \).
- On the other hand, there are \( \binom{m+n}{2} \) ways to choose 2 persons.
- Among them, \( \binom{m}{2} + \binom{n}{2} \) are same-sex.

Corollary 15 \[ \binom{2n}{2} = n^2 + 2\binom{n}{2}. \]
Combinatorial Proof: Counting It Twice (continued)

**Corollary 16** \(1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6.\)

- From Corollary 15 (p. 35),

\[
\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} \binom{2k}{2} - 2 \sum_{k=1}^{n} \binom{k}{2}
\]

\[
= \sum_{k=1}^{n} k(2k - 1) - 2 \binom{n+1}{3}
\]

by Lemma 10 (p. 30).
Combinatorial Proof: Counting It Twice (concluded)

• So
\[ \sum_{k=1}^{n} k^2 = 2 \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k - 2 \binom{n+1}{3}. \]

• We conclude that
\[
\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} k + 2 \binom{n+1}{3}
= \frac{n(n+1)}{2} + \frac{n(n+1)(n-1)}{3}
= \frac{n(n+1)(2n+1)}{6}.
\]
Binomial Random Walk

- A particle starting at the origin can move right (up) or left (down) in each step.

- It is a standard model for stock price movements called the **binomial option pricing model**.\(^a\)

\(^a\)Cox, Ross, and Rubinstein (1979).
Dynamics of the Binomial Random Walk

Lemma 17  The number of ways the particle can move to position $k$ in $n$ steps is

$$\binom{n}{\frac{n+k}{2}}.$$  \hspace{1cm} (1)

(Assume that $n + k$ is even.)

- To reach position $k$, the number of up moves must exceed the number of down moves by exactly $k$.
- UUDUUDUUD reaches position 3 in 9 steps.
The Proof (concluded)

• Now \((n + k)/2 - (n - k)/2 = k\).

• So the number of ways to reach position \(k\) in \(n\) steps is the number of ways

\[
\frac{(n+k)/2}{UU \cdots U} \frac{(n-k)/2}{DD \cdots D}
\]

can be permuted.

• The desired number is

\[
\frac{n!}{[(n + k)/2]![(n - k)/2]!} = \binom{n}{\frac{n+k}{2}}.
\]
Probability of Reaching a Position

- Suppose the binomial random walk has a probability of $p$ of going up and $1 - p$ of going down.
- The number of ways it is at position $k$ after $n$ steps is
  \[
  \binom{n}{\frac{n+k}{2}}
  \]
  by Eq. (1) on p. 39.
- The probability for this to happen is
  \[
  \left( \frac{n}{n+k} \right) p^{\frac{n+k}{2}} (1 - p)^{\frac{n-k}{2}}.
  \]
Probability of Reaching a Position (concluded)

- Suppose a position is the result of \(i\) up moves and \(n - i\) down moves.
  - Clearly, the position is \(i - (n - i) = 2i - n\).

- The number of ways of reaching it after \(n\) steps is
  \[
  \binom{n}{i}.
  \]

- The probability for this to happen is
  \[
  \binom{n}{i} p^i (1 - p)^{n-i}. \tag{2}
  \]
Vandermonde’s Convolution\textsuperscript{a}

\[
\binom{n}{i} = \sum_{l=0}^{k} \binom{k}{l} \binom{n-k}{i-l}.
\]

- Let state \((n, i)\) be the result of \(i\) up moves and \(n - i\) down moves.
- Suppose the walk starts at state \((0, 0)\) and ends at \((n, i)\).
- There are \(\binom{n}{i}\) such walks.
- State \((k, l)\) is on \(\binom{k}{l} \binom{n-k}{i-l}\) walks that reach \((n, i)\), where \(0 \leq k \leq n\) and \(0 \leq l \leq k\).

\textsuperscript{a}Alexandre-Théophile Vandermonde (1735–1796).
Vandermonde’s Convolution (concluded)

- Every walk that reaches \((n, i)\) must go through a state \((k, l)\) for some \(0 \leq l \leq k\).

- Add up those walks that go through state \((k, l)\) over \(0 \leq l \leq k\) to obtain\(^a\)

\[
\binom{n}{i} = \sum_{l=0}^{k} \binom{k}{l} \binom{n-k}{i-l}.
\]

- Applications in artificial neural networks.\(^b\)

---

\(^a\)Technically, the summation should be over \(0 \leq l \leq \min(k, i)\). But recall that \(\binom{n}{i} = 0\) for \(i < 0\) or \(i > n\), where \(n\) is a positive integer.

\(^b\)Baum and Lyuu (1991) and Lyuu and Rivin (1992).
The Binomial Theorem\textsuperscript{a}

Theorem 18

\[(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.\]

\begin{itemize}
  \item \((x + y)^n = (x + y)(x + y) \cdots (x + y)\).
  \item Each term must have the form \(x^i y^{n-i}\).
  \item There are \(\binom{n}{i}\) ways to pick \(i\) \(x\)'s and \(n - i\) \(y\)'s.
\end{itemize}

\textsuperscript{a}Attributed to Newton. First appeared in a book by Colin Maclaurin (1698–1746).
Corollaries of the Binomial Theorem

\[ 2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}. \tag{3} \]

- Set \( x = y = 1 \) in the binomial theorem.

For odd \( n \),

\[ 2^{n-1} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}} \]

\[ = \binom{n}{\frac{n+1}{2}} + \binom{n}{\frac{n+3}{2}} + \cdots + \binom{n}{n}. \tag{4} \]

- Because \( \binom{n}{r} = \binom{n}{n-r} \).
Corollaries of the Binomial Theorem (continued)

• Set $x = 1$ and $y = -1$ in the binomial theorem to obtain

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0.$$ 

• As a by-product,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}. \quad (5)$$
Corollaries of the Binomial Theorem (continued)

\[
\sum_{i=n+1}^{2n+1} \binom{2n+1}{i} = 2^{2n} \quad (6)
\]

because

\[
2^{2n+1} = \sum_{i=0}^{2n+1} \binom{2n+1}{i} = \sum_{i=0}^{n} \binom{2n+1}{i} + \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} = \sum_{i=0}^{n} \left( \frac{2n+1}{2n+1-i} \right) + \sum_{i=n+1}^{2n+1} \binom{2n+1}{i} = 2 \sum_{i=n+1}^{2n+1} \binom{2n+1}{i}.
\]
Corollaries of the Binomial Theorem (continued)

\[
\binom{2n}{n} = \sum_{i=0}^{n} \left( \binom{n}{i} \right)^2.
\]  

(7)

• Consider \( f(x) = (1 + x)^n (1 + x^{-1})^n \).

• \( \left( \binom{n}{i} \right)^2 \) is the number of ways to pick \( i \) \( x \)'s and \( i \) \( x^{-1} \)'s.

• \( \sum_{i=0}^{n} \left( \binom{n}{i} \right)^2 \) is the constant term in \( f(x) \).

• Rewrite \( f(x) \) as \( (1 + x)^n (1 + x)^n x^{-n} = x^{-n} (1 + x)^{2n} \).

• The constant term in \( f(x) \) is hence \( \binom{2n}{n} \).

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Alternative Proof for Eq. (7) on p. 51

• Consider a $2n$-step binomial random walk that ends at the origin.

• There are $\binom{2n}{n}$ such walks by Eq. (1) on p. 39.

• Consider a walk that reaches position $i$ at step $n$, where $n + i$ is even.

• There are $\left( \frac{n}{(n+i)/2} \right)^2$ such walks by Eq. (1) on p. 39.

• So

\[
\binom{2n}{n} = \sum_{i=-n,-n+2,...,n} \left( \frac{n}{(n+i)/2} \right)^2 = \sum_{i=0}^{n} \binom{n}{i}^2.
\]
Corollaries of the Binomial Theorem (continued)

\[
\binom{2n}{n} = \sum_{i=0}^{2n} (-1)^{n+i} \binom{2n}{i}^2.
\]

- Consider \( g(x) = (1 + x)^{2n}(1 - x^{-1})^{2n}. \)
- \( \binom{2n}{i}^2 \) is the number of ways to pick \( i \) \( x \)'s and \( i \) \( x^{-1} \)'s.
- \( \sum_{i=0}^{2n} (-1)^i \binom{2n}{i}^2 \) is the constant term in \( g(x) \).
- Rewrite \( g(x) \) as \( (x - x^{-1})^{2n} = x^{-2n}(x^2 - 1)^{2n}. \)
- The constant term in \( g(x) \) is hence \( (-1)^n \binom{2n}{n} \).
Corollaries of the Binomial Theorem (concluded)

\[ \sum_{i=1}^{n} i \binom{n}{i} = n2^{n-1}. \]

• Differentiate \((1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i\) to obtain

\[ n(1 + x)^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} x^{i-1}. \]

• Now set \(x = 1\).
Binary Strings with Even Weight

• Consider a binary string $x_1x_2\cdots x_n$.
  - The **weight** of $x_1x_2\cdots x_n$ is defined as $\sum_i x_i$.
• There are $2^n$ strings.
• Among them,
  \[ \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} = 2^{n-1} \]
  have even weight (see Eq. (5) on p. 49).
  - 1s occur in $2i$ positions.
  - $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 16$, and $\binom{7}{0} + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} = 64$. 
Majority Decision

• In a court with $2n + 1$ judges, in how many ways can a majority “yes” decision be handed down?
  - Any vote has a majority; hence we consider cases when the majority vote “yes.”

• There are $\binom{2n + 1}{i}$ ways such that $i$ judges vote “yes.”

• From Eq. (6) on p. 50, the desired answer is

$$\sum_{i=n+1}^{2n+1} \binom{2n + 1}{i} = 2^{2n}$$
Ways To Merge Sets

What is the number of ways to merge subsets of \{\{1\}, \{2\}, \ldots, \{n\}\} to form \{\{1, 2, \ldots, n\}\} in \(n - 1\) steps?

- For example, the number is 3 when \(n = 3\):

\[
\begin{align*}
\{\{1\}, \{2\}, \{3\}\} & \rightarrow \{\{1, 2\}, \{3\}\} \rightarrow \{\{1, 2, 3\}\}, \\
\{\{1\}, \{2\}, \{3\}\} & \rightarrow \{\{1, 3\}, \{2\}\} \rightarrow \{\{1, 2, 3\}\}, \\
\{\{1\}, \{2\}, \{3\}\} & \rightarrow \{\{2, 3\}, \{1\}\} \rightarrow \{\{1, 2, 3\}\}.
\end{align*}
\]
Ways To Merge Sets (concluded)

• Each merge involves two subsets.
• The \(i\)th step begins with \(n - i + 1\) subsets.
• There are \(\binom{n-i+1}{2}\) ways to pick the two subsets.
• The desired number is thus

\[
\prod_{i=1}^{n-1} \binom{n-i+1}{2} = \frac{n!(n-1)! \cdots 2!}{2^{n-1}(n-2)!(n-3)! \cdots 1!} = \frac{n!(n-1)!}{2^{n-1}}.
\]
The Multinomial Theorem

Theorem 19 \((x_1 + x_2 + \cdots + x_t)^n\) equals

\[
\sum_{0 \leq n_1, n_2, \ldots, n_t \leq n, \sum_i n_i = n} \frac{n!}{n_1! n_2! \cdots n_t!} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.
\]

- Each term in the expansion of \((x_1 + x_2 + \cdots + x_t)^n\) must have the form \(x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}\), where \(0 \leq n_1, n_2, \ldots, n_t \leq n\) and \(\sum_i n_i = n\).
- To pick \(n_1\) \(x_1\)'s, \(n_2\) \(x_2\)'s, and so on, there are

\[
\binom{n!}{n_1, n_2, \ldots, n_t} \equiv \frac{n!}{n_1! n_2! \cdots n_t!}
\]

ways.
Coefficient of $a^2b^3c^2d^5$ in $(a + 2b - 3c + 2d + 5)^{16}$

- The coefficient of $a^2(2b)^3(-3c)^2(2d)^55^4$ is

$$\binom{16}{2, 3, 2, 5, 4} = \frac{16!}{2! \cdot 3! \cdot 2! \cdot 5! \cdot 4!} = 302,702,400$$

by the multinomial theorem.

- The desired coefficient is then

$$302,702,400 \times 2^3 \times (-3)^2 \times 2^5 \times 5^4 = 435,891,456,000,000.$$
Distinct Objects into Identical Containers

**Corollary 20** There are \( \frac{(rn)!}{(r!)^n n!} \) ways to distribute \( rn \) distinct objects into \( n \) identical containers so that each container contains exactly \( r \) objects.

- By Theorem 19 (p. 59), there are

\[
\binom{rn}{r, r, \ldots, r} = \frac{(rn)!}{r! r! \cdots r!}
\]

ways to distribute \( rn \) distinct objects into \( n \) distinct containers.

- Divide the above count by \( n! \) to remove the identities of the containers.
Distinct Objects into Identical Containers (concluded)

Corollary 21 \( \frac{(rn)!}{(r!)^n n!} \) is an integer.

- Immediate from Corollary 20 (p. 61).

- It is interesting to note that this result generalizes Lemma 7 (p. 19), which says \( \frac{(rn)!}{(r!)^n} \) is an integer.
An Alternative Proof of Corollary 21 (p. 62)\textsuperscript{a}

\[
\frac{(rn)!}{(r!)^nn!} = \frac{1}{n!} \frac{(rn)!}{[r(n-1)]!} \frac{[r(n-2)]!}{r!} \cdots \frac{[r(1)]!}{[r(n-n)]!r!} \\
= \prod_{k=0}^{n-1} \left( \frac{r(n-k)}{r} \right) \\
= n! \prod_{k=0}^{n-1} \frac{r(n-k)}{n-k} \\
= \prod_{k=0}^{n-1} \left( \frac{r(n-k)-1}{r-1} \right).
\]

\textsuperscript{a}Contributed by Mr. Ansel Lin (B93902003) on September 20, 2004.