Network Flows with Applications

Transport Networks: Edge Form

- Consider a loop-free connected directed graph $G = (V, E)$.
- $G$ is called a network or transport network if:
  - There exists a unique node $a \in V$, called the source, with an in degree of 0.
  - There exists a unique node $z \in V$, called the sink, with an out degree of 0.
  - The weight on each edge $e$ is a nonnegative integer $c(e)$, called the capacity.

Weighted Graphs

- Let $G = (V, E)$ be a directed graph.
- Assign a real number to each edge $e$.
- This number is called the weight of $e$.
  - Weights can mean distances, capacities, costs, etc., depending on the applications.
  - Shortest path problem: efficiently solvable.\(^a\)
  - Traveling salesman problem: quintessential example of hard problems (NP-complete).\(^b\)
- The resulting graph is called a weighted graph.

\(^a\) Dijkstra (1959).
\(^b\) Karp (1972).
**Flows**

- Let $G = (V, E)$ be a transport network.
- A function $f : E \to \mathbb{N}$ is called a **flow** if:
  - $f(e) = 0$ for $e \not\in E$.
  - $f(e) \leq c(e)$ for each edge $e \in E$.
    * The capacities are not exceeded.
  - For each $v \in V - \{a, z\}$,
    $$\sum_{w \in V} f(w, v) = \sum_{w \in V} f(v, w),$$
    * Except the source and the sink, the flows are **conserved**.

**Saturation**

- Let $G = (V, E)$ be a transport network and $f$ be a flow.
- An edge $e$ is **saturated** if $f(e) = c(e)$.
- When $f(e) < c(e)$, the edge is **unsaturated**.
- If $a$ is the source, then
  $$\text{val}(f) = \sum_{v \in V} f(a, v)$$
  is the **value of the flow**.
- Determining the maximum flow value is our key goal.

**Values of Flows**

- The value of any flow cannot exceed $1 + 10$.
- Nor can it exceed $4 + 3$.
- These are easy upper bounds.

**Cuts**

- Consider a transport network $G = (V, E)$.
- Let $P \subseteq V$ contain source node $a$ but not sink node $z$.
- $(P, P)$ is a **cut** separating $a$ from $z$, where $P = V - P$.
- Every directed path in $G$ from $a$ to $z$ must pass through some edge from a node in $P$ to a node in $P$.
- The **capacity of a cut** is given by
  $$c(P, P) = \sum_{v \in P, w \in P, (v, w) \in E} c(v, w).$$
The Max-Flow Min-Cut Theorem

**Theorem 47** The value of any flow $f$ cannot exceed the capacity of any cut.

- Theorem 47 says that
  \[
  \max_{f} \text{val}(f) \leq \min_{(P, P')} c(P, P).
  \]
- In fact, they are equal.

**Theorem 48** (Ford and Fulkerson (1956)) In a transport network, the maximum flow value equals the minimum capacity over all cuts. Furthermore, the maximum flow can be calculated efficiently.

The Integral Theorem

- The max-flow min-cut theorem holds for nonnegative real capacities as well.
- Because the capacities are assumed to be integers here, an important corollary of the max-flow min-cut theorem results.

**Theorem 49 (Integrality theorem)** There is a maximum flow with integral entries.

Matching

- Let $G = (V, E) = (X, Y, E)$ be a bipartite graph.
- $E' \subseteq E$ is a matching if no two edges in $E'$ share a common node.
- A complete matching of $X$ into $Y$ is a matching such that every $x \in X$ is the endpoint of an edge.
  - It is necessary that $|X| \leq |Y|$.
  - It defines a one-to-one function from $X$ to $Y$. 
A Complete Matching

Jill                  Ray
Helen                Brendan
Kathy                Tristan
Isolde               Paris
Regina              Ludwig

Now you know why it is called a matching problem.

Existence of Complete Matchings

**Theorem 50 (Hall's theorem (1935))**  
Let $G = (V, E)$ be a bipartite graph with $V$ partitioned as $X \cup Y$. A complete matching from $X$ into $Y$ exists if and only if for every $A \subseteq X$,  

$$|A| \leq |R(A)|.$$  

Here, $R(A)$ consists of those nodes in $Y$ adjacent to some node in $A$.

A brute-force implementation of Hall's theorem is inefficient; efficient algorithms exist.\(^{a,b}\)

---

Illustration of Hall’s Theorem

Proof of Hall’s Theorem

- Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$.
- Add source node $s$ and sink node $t$ to graph $G$.
- Add edges $(a, x_1), (a, x_2), \ldots, (a, x_m)$ with capacity 1 each.
- Add edges $(y_1, z), (y_2, z), \ldots, (y_n, z)$ with capacity 1 each.
- Add direction to each edge in $E$ from $X$ to $Y$.
- Assign weight $M$ to each edge in $E$, where $M$ is any integer greater than $|X|$.

\(^{a}\)Philip Hall (1904-1982).
\(^{b}\)Edmonds (1965).
Proof of Hall’s Theorem (continued)

- A complete matching exists if and only if there is a maximum flow in the transport network that uses all edges out of $a$.
- Equivalently, if and only if there exists a flow with value equal to $m = |X|$, such a flow results in exactly $|X|$ edges from $X$ to $Y$ having flow 1.
  * Use the integrality theorem (p. 512).
  * This flow defines a complete matching.

Proof of Hall’s Theorem (continued)

Proof ($\Rightarrow$): If $|A| \leq |R(A)|$ for any $A \subseteq X$, then there is a flow with value equal to $|X|$.

- By the max-flow min-cut theorem (p. 511), it suffices to show that the capacity of every cut is at least $|X|$.
- Pick any cut $(P, \bar{P})$.
- $a \in P$ and $z \in \bar{P}$.
- Define $A = X \cap P$ and $B = Y \cap P$.
- Without loss of generality, $A = \{x_1, x_2, \ldots, x_i\}$ for some $0 \leq i \leq m$. 
Proof of Hall’s Theorem (continued)

- Now,
  \[ P = \{a\} \cup A \cup B \]
  \[ P = \{z\} \cup (X - A) \cup (Y - B) \]

- If there is an edge from \( A \) to \( Y - B \), then
  \[ c(P, P) \geq M > |X|, \]
  and we are done for this part.

- On the other hand suppose no such edges exist.

- Then \( R(A) \subseteq B \); hence
  \[ |R(A)| \leq |B|. \]
Proof of Hall’s Theorem (continued)

- Then,

\[
c(P, P) = \sum_{v \in X} c(a, v) + \sum_{y \in R(A)} c(y, z)
\]

\[
= |X - A| + |R(A)|
\]

\[
= |X| - |A| + |R(A)|
\]

\[
= |X| + (|R(A)| - |A|)
\]

< |X|.

- By Theorem 47 (p. 511), every flow must have value less than |X|.

System of Distinct Representatives

- Let \( A_1, A_2, \ldots, A_n \) be a collection of sets.
- Elements \( a_1, a_2, \ldots, a_n \) are called a system of distinct representatives if:
  - \( a_i \in A_i \) for \( 1 \leq i \leq n \),
  - \( a_1, a_2, \ldots, a_n \) are distinct.

Conditions for SDR\(^a\)

**Theorem 51** Sets \( A_1, A_2, \ldots, A_n \) have a system of distinct representatives if and only if for all \( 1 \leq i \leq n \), the union of any \( i \) of the sets contains at least \( i \) elements.

- Construct a bipartite graph from \( X = \{A_1, A_2, \ldots, A_n\} \) to \( Y = \{1, 2, \ldots, n\} \).
- \( A_i \in X \) is connected to \( j \in Y \) if \( A_i \) contains \( j \).
- But \( R(A) \) is a union of \( |A| \) sets for any \( A \subseteq X \).

\(^a\) Leon Mirsky (1918 1983).
The Proof (concluded)

- By Hall's theorem (p. 515), a complete matching exists if and only if \( |A| \leq |R(A)| \) for any \( A \subseteq X \).
- This is exactly what the theorem says.\(^a\)

\(^a\)Did you notice that the SDR theorem is simply Hall's theorem restated?

---

Menger's Theorem\(^a\) (Edge Form)

**Theorem 52 (Menger (1927))** Let \( G \) be a digraph. There are \( k \) edge-disjoint paths from \( x \) to \( y \) if and only if \( G \) is \( k \)-edge-connected between \( x \) and \( y \).

\[ \begin{array}{c}
  x \\
  \downarrow
  \begin{array}{c}
    x
  \end{array}
  \downarrow
  \begin{array}{c}
    y
  \end{array}
\end{array} \]

\(^a\)Karl Menger (1902 1985).

---

Edge Connectivity

- A digraph is \( k \)-edge-connected between \( x \) and \( y \) if there exists a path from \( x \) to \( y \) even after fewer than \( k \) edges are removed.
- A digraph is \( k \)-edge-connected \((k \geq 2)\) if it has at least 2 nodes and is \( k \)-edge-connected between any two nodes.
- There are also node versions of connectivity (p. 535).
- All the above problems are efficiently solvable.

---

Proof of Menger's Theorem

**Proof (\( \Rightarrow \))**: 
- Suppose \( G \) has \( k \) edge-disjoint paths from \( x \) to \( y \).
- Clearly, the removal of any \( k - 1 \) or fewer edges cannot prevent \( x \) from reaching \( y \).

**Proof (\( \Leftarrow \))**: 
- Suppose \( G \) is \( k \)-edge-connected.
- Make \( x \) the source node and \( y \) the sink node,
  - Remove \( x \)'s incoming edges and \( y \)'s outgoing edges.
Proof of Menger’s Theorem (concluded)

- Assign capacity one to each edge.
- Every cut \((P, \overline{P})\) must contain \(\geq k\) edges from \(P\) to \(P\).
- Hence every cut has capacity \(\geq k\).
- By Theorem 49 (p. 512), there is a flow with integral value \(k\) from \(x\) to \(y\).
- Each flow on an edge is either one or zero because the capacity is one.
- The edges with a flow of one are the \(k\) edge-disjoint paths needed.

Menger’s Theorem (Node Form)

**Theorem 53** Let \(G = (V, E)\) be a digraph and \(x, y \in V\) be nonadjacent. There are \(k\) node-disjoint paths from \(x\) to \(y\) if and only if \(G\) is \(k\) connected between \(x\) and \(y\).

Node Connectivity

- A digraph is \(k\)-connected between \(x\) and \(y\) if there exists a path from \(x\) to \(y\) even after fewer than \(k\) nodes are removed.
- A digraph is \(k\)-connected \((k \geq 2)\) if one of the following holds:
  - It is \(K_{k+1}\).
  - It has at least \(k + 2\) nodes and is \(k\)-connected between any two nodes.

The Equivalence Theorem

**Theorem 54** The following theorems are equivalent.

1. König’s theorem.
2. The SDR theorem.
3. Hall’s theorem.
4. Dilworth’s theorem.
5. The max-flow min-cut theorem.
Algebra: Groups

A Loose End in Item 47\textsuperscript{a}

- Can a “right” inverse be different from a “left” inverse?
- Suppose \( a \circ b = e \) and \( b' \circ a = e \).
  - \( b \) is a right inverse of \( a \).
  - \( b' \) is a left inverse of \( a \).
- Then \( b' = b' \circ e = b' \circ (a \circ b) = (b' \circ a) \circ b = e \circ b = b \).
- Hence there is no point in distinguishing left and right inverses.
\textsuperscript{a}Contributed by Mr. Hao (E0902039) on December 23, 2002.

Group Theory\textsuperscript{a}

- Let \( G \neq \emptyset \) be a set and \( \circ \) be a binary operation on \( G \).
- \((G, \circ)\) is called a group if it satisfies the following.
  1. For all \( a, b \in G \), \( a \circ b \in G \) (closure).
  2. For all \( a, b, c \in G \), \( a \circ (b \circ c) = (a \circ b) \circ c \) (associativity).
  3. There exists \( e \in G \) with \( a \circ e = e \circ a = a \) for all \( a \in G \) (identity).
  4. For each \( a \in G \), there is an element \( b \in G \) such that \( a \circ b = b \circ a = e \) (inverse).
- \( G \) is commutative or abelian if \( a \circ b = b \circ a \) for all \( a, b \in G \).
\textsuperscript{a}Niels Henrik Abel (1802–1829) and Evariste Galois (1811–1832).

Examples of Groups

- Under ordinary +, \((\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +)\) are groups.
  - The inverse of \( a \) is simply \(-a\), which exists.
- Under ordinary \( \times \), none of \((\mathbb{Z}, \times), (\mathbb{Q}, \times), (\mathbb{R}, \times), (\mathbb{C}, \times)\) are groups.
  - The number 0 has no inverses.
- Under ordinary \( \times \), \((\mathbb{Q}^*, \times), (\mathbb{R}^*, \times), (\mathbb{C}^*, \times)\) are groups.
  - \( A^* \) denotes the nonzero elements of \( A \).
- Under ordinary –, \((\mathbb{Z}, -), (\mathbb{Q}, -), (\mathbb{R}, -)\) are not groups.
  - The associative axiom fails: \( a - (b - c) \neq (a - b) - c \).
Properties of Groups\textsuperscript{a}

- The identity of $G$ is unique.
  - If $e_1, e_2$ are both identities, then $e_1 = e_1 \circ e_2 = e_2$ by the identity condition.
- The inverse of each element of $G$ is unique (it is $a^{-1}$ under $\times$ and $-a$ under $+$, e.g.).
  - Suppose $b, c$ are both inverses of $a \in G$.
  - Then $b = b \circ e = b \circ (a \circ c) = (b \circ a) \circ c = e \circ c = c$.

\textsuperscript{a}Properties must be proved using only the four axioms or their logical corollaries.

\vspace{1cm}

Inverses

- $(a^{-1})^{-1} = a$.
  - Both are inverses of $a^{-1}$ and inverses are unique.
- $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$.
  - $(b^{-1} \circ a^{-1}) \circ (a \circ b) = b^{-1} \circ (a^{-1} \circ a) \circ b = b^{-1} \circ b = e$.
- $(G, \circ)$ is abelian if and only if $(a \circ b)^{-1} = a^{-1} \circ b^{-1}$.
  - If $(G, \circ)$ is abelian, then $(a \circ b)^{-1} = (b \circ a)^{-1} = a^{-1} \circ b^{-1}$.
  - If $(a \circ b)^{-1} = a^{-1} \circ b^{-1}$, then $a \circ b = ((a \circ b)^{-1})^{-1} = (a^{-1} \circ b^{-1})^{-1} = (b^{-1})^{-1} \circ (a^{-1})^{-1} = b \circ a$.

\vspace{1cm}

The Cancellation Properties

The \textbf{left-cancellation property}: If $a, b, c \in G$ and $a \circ b = a \circ c$, then $b = c$.

- $b = (a^{-1} \circ a) \circ b = a^{-1} \circ (a \circ b) = a^{-1} \circ (a \circ c) = (a^{-1} \circ a) \circ c = c$.

The \textbf{right-cancellation property}: If $a, b, c \in G$ and $b \circ a = c \circ a$, then $b = c$.

\vspace{1cm}

Powers

- The associative property implies that $a_1 \circ a_2 \circ \ldots \circ a_n$ is well-defined.
- For $n > 0$, define $a^n = \underbrace{a \circ a \circ \ldots \circ a}_{n}$.
- For $n < 0$, define $a^n = \underbrace{a^{-1} \circ a^{-1} \circ \ldots \circ a^{-1}}_{n}$.
  - $a^{-n} = (a^{-1})^n$ because $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$.
  - Define $a^0 = e$. 
Operations on Powers

Lemma 55 \( a^n \circ a^m = a^{n+m} \) for \( n, m \in \mathbb{Z} \).

- For \( n, m \geq 0 \),
  \[
  a^n \circ a^m = \underbrace{a \circ \cdots \circ a}_{n} \circ \underbrace{a \circ \cdots \circ a}_{m} = a \circ \cdots \circ a = a^{n+m}.
  \]
- For \( n > 0, m < 0 \),
  \[
  a^n \circ a^m = \underbrace{a \circ \cdots \circ a}_{n} \circ \underbrace{a \circ \cdots \circ a}_{m} = a \circ \cdots \circ a = a^{n+m}.
  \]
- The other two cases are similar.

Criteria for Being a Subgroup

Only two axioms need to be checked.

Theorem 56 Let \( H \) be a nonempty subset of a group \((G, \circ)\). Then \( H \) is a subgroup of \( G \) if and only if (1) for all \( a, b \in H \), \( a \circ b \in H \), and (2) for all \( a \in H \), \( a^{-1} \in H \).

Proof (\( \Rightarrow \)):
- Assume that \( H \) is a subgroup of \( G \).
- Then \( H \) is a group.
- So \( H \) satisfies, among other things, the closure property (1) and the inverse property (2).

Subgroups

- Let \((G, \circ)\) be a group.
- Let \( \emptyset \neq H \subseteq G \).
- If \( H \) is a group under \( \circ \), we call it a subgroup of \( G \).
- For example, the set of even integers is a subgroup of \((\mathbb{Z}, +)\).
- \( H \) “inherits” \( \circ \) from \( G \) in that it is the same operation, producing the same result in both \( G \) and \( H \) wherever applicable.

The Proof (concluded)

Proof (\( \Leftarrow \)):
- Let \( H \neq \emptyset \) satisfy (1) and (2).
- We need to verify the associative property and the existence of identity,
  - **Associativity**: For all \( a, b, c \in H \),
    \[
    (a \circ b) \circ c = a \circ (b \circ c) \in G,
    \]
    hence in \( H \) by (1).
  - **Identity**: For any arbitrary \( a \in H \), \( a^{-1} \circ a \in H \) by (2) and is an identity.