Theorem 30 If $G = G_1 \times G_2$, then $d(G) = d(G_1) + d(G_2)$.

- Consider two nodes of $G$, $(x_1, y_1)$ and $(x_2, y_2)$.
- Take the shortest path of length at most $d(G_2)$ from $(x_1, y_1)$ to $(x_1, y_2)$ over nodes of $G_2$.
- Take the shortest path of length at most $d(G_1)$ from $(x_1, y_2)$ to $(x_2, y_2)$ over nodes of $G_1$.
- Since the two nodes are arbitrary, we have proved

$$d(G) \leq d(G_1) + d(G_2).$$

Cartesian Products and Diameters (continued)

- We proceed to prove that

$$d(G) \geq d(G_1) + d(G_2).$$

- Let $x_1, x_2 \in V_1$ be two nodes of $G_1$ with distance $d(G_1)$.
- Let $y_1, y_2 \in V_2$ be two nodes of $G_2$ with distance $d(G_2)$.
- Let $d$ be the distance between nodes

$$(x_1, y_1), (x_2, y_2) \in G.$$

Euler Circuits and Trails

- Let $G = (V, E)$ be an undirected graph or multigraph with no isolated nodes,
  - Isolated nodes are nodes without incident edges.
- $G$ is said to have an Euler circuit if there is a circuit in $G$ that traverses every edge of the graph exactly once.
- If there is an open trail from $x$ to $y$ in $G$ and this trail traverses every edge of the graph exactly once, the trail is called an Euler trail.
Characterization of Having Euler Circuits

**Theorem 31** Let \( G = (V, E) \) be an undirected graph or multigraph with no isolated nodes. Then \( G \) has an Euler circuit if and only if \( G \) is connected and every node in \( G \) has an even degree.

- Testing if a graph is Eulerian hence is trivial.
- The proof will be constructive.
- Testing for a graph-theoretical property and constructing a solution that satisfies it may have different complexities.
- Let \( n = |E| \).

\[
\text{The Proof } (\Rightarrow)
\]

- Clearly \( G \) is connected.
- Each time the Euler circuit enters a non-starting node \( v \), it must exit it before coming back again, if ever.
- This adds a count of 2 to \( \text{deg}(v) \).
- Because every edge is traversed, \( \text{deg}(v) \) must be even.
- The Euler circuit must start from the starting node \( s \) and end at the starting node.
- This adds 2 to \( \text{deg}(s) \).
- The other visits have the same property as visits to \( v \) and add an even number to \( \text{deg}(s) \).

\[
\text{The Proof } (\Leftarrow)
\]

- The \( n = 1 \) case is easy, by inspection.
- Assume the result is true when there are \( < n \) edges.
- If \( G \) has \( n \) edges, select a node \( s \in G \) as the starting and ending node.
- Construct a circuit \( C \) from \( s \).
  - Start from \( s \).
  - Traverse any hitherto untraversed edge, and so on.
  - We must eventually return to \( s \) because every node has an even degree.

\[
\text{The Proof } (\Leftarrow) \text{ (concluded)}
\]

- If \( C \) traverses every edge, we are done.
- Otherwise, remove the edges of \( C \) and isolated nodes as a result to yield a new graph \( K \).
- The degree of each node in \( K \) remains even.
**The Proof (⇐) (continued)**

- Suppose $K$ is connected.
- Construct an Euler circuit $c$ of $K$ (doable by the induction hypothesis).
- Node $s$ is on this Euler circuit because $K$ is connected.
- The desired Euler circuit: Start from $s$ and travel on $C$ until we end at $s$ and then traverse $c$ until we end at $s$ again.

---

**The Proof (⇐) (concluded)**

- Suppose $K$ is disconnected or $s$ is isolated.
- Construct an Euler circuit in each component of $K$ (doable by the induction hypothesis).
- Each Euler circuit $c_i$ must have a node $s_i$ on $C$ because originally $G$ is connected.
- The desired Euler circuit: Start from $s$ and travel on $C$ until we reach $s_1$, traverse $c_1$, return to $s_1$, continue on $C$ until we reach $s_2$, and so on.

---

**Constructing an Euler Circuit**

![Euler Circuit Diagram](image)

**Characterization of Having Euler Trails**

**Corollary 32** Let $G = (V, E)$ be an undirected graph or multigraph with no isolated nodes. Then $G$ has an Euler trail if and only if $G$ is connected and has exactly two nodes of odd degree.

- Let $x, y$ be the two nodes of odd degree.
- Add edge $(x, y)$ to $G$.
- Construct an Euler circuit, which exists by Theorem 31.
- Remove the edge $(x, y)$ from the circuit to arrive at an Euler trail.
In and Out Degrees

- Let $G$ be a directed graph.
- The **in degree** of $v \in V$ is the number of edges in $G$ that are incident into $v$.
- The **out degree** of $v \in V$ is the number of edges in $G$ that are incident from $v$.

Planar Graphs

- A graph or multigraph $G$ is called **planar** if it can be drawn in the plane with the edges intersecting only at nodes of $G$.
- Planarity can be tested efficiently.\(^a\)
- Such a drawing of $G$ is called an **embedding** of $G$ in the plane.

---

Euler’s Theorem

- Let $G = (V, E)$ be a connected planar graph or multigraph with $|V| = v$ and $|E| = e$.
- Let $r$ be the number of regions in the plane determined by a planar embedding of $G$.
- One of these regions has infinite area and is called the infinite region.
- Then

$$v - e + r = 2. \quad (40)$$

---

\(^a\)Hopcroft and Tarjan (1974).
The Proof When \( H \) is Connected

- \( H \) has \( v \) nodes, \( k \) edges, and \( r - 1 \) regions.
- A dotted edge is added to obtain a planar \( G \) (see p. 445 for illustration).
- The induction hypothesis applied to \( H \) says
  \[ v - k + (r - 1) = 2. \]
- Hence
  \[ v - (k + 1) + r = 2. \]
- The theorem is proved because \( G \) has \( v \) nodes, \( e = k + 1 \) edges, and \( r \) regions.

The Proof\(^a\)

- The theorem holds if \( e = 0, 1 \) (p. 511 of the textbook (4th ed.)).
- Assume the theorem holds for any connected planar graph with \( e \) edges, where \( 0 \leq e \leq k \).
- Let \( G = (V, E) \) be a graph with \( v \) nodes, \( r \) regions, and \( e = k + 1 \) edges.
- Let \( \{x, y\} \in E \).
- Delete \( \{x, y\} \) to obtain graph \( H: G = H + \{x, y\} \).

\(^a\)See Imre Lakatos (1976), Proofs and Refutations: The Logic of Mathematical Discovery (1989), for a most penetrating presentation.
The Proof When $H$ Is Not Connected

- $H$ has $v$ nodes, $k = e - 1$ edges, and $r$ regions.
- A dotted edge is added to obtain a planar $G$ (see p. 447 for illustration).
- $H$ has two components $H_1$ and $H_2$.\footnote{Thanks to a lively class discussion on December 1, 2003.}
- Let $H_i$ have $v_i$ nodes, $e_i$ edges, and $r_i$ regions.
- The induction hypothesis applied to $H_i$ says
  \[ v_i - e_i + r_i = 2. \]
- Therefore,
  \[ (v_1 + v_2) - (e_1 + e_2) + (r_1 + r_2) = 4. \] (41)

The Proof When $H$ Is Not Connected (concluded)

- Now,
  \[ v_1 + v_2 = v, \]
  \[ e_1 + e_2 = k = e - 1, \]
  \[ r_1 + r_2 = r + 1. \]
- Hence Eq. (41) on p. 446 becomes
  \[ v - (e - 1) + (r + 1) = 4. \]
- Hence again $v - e + r = 2$.

A Useful Corollary

**Corollary 33** Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e > 2$, and $r$ regions. Then
\[ e \leq 3v - 6. \]

- Each edge is shared by $\leq 2$ regions.
- The boundary of each region (including the infinite region) contains at least 3 edges ($G$ is not a multigraph).
- Hence $2e \geq \sum_{region R} |R's \ boundary| \geq 3r$.
- Euler’s theorem implies
  \[ 2 = v - e + r \leq v - e + (2/3)e = v - (1/3)e. \]
$K_5$ is Not Planar

- $K_5$ has $v = 5$ nodes and $e = 10$ edges.
- Suppose it is planar.
- By Corollary 33,
  
  $10 = e \leq 3v - 6 = 9,$
  
  a contradiction.

Bipartite Graphs

- A graph $G = (V, E)$ is called bipartite if:
  - $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, and
  - Every edge is of the form $\{x, y\}$ with $x \in V_1$ and $y \in V_2$.
- If each node in $V_1$ is joined with every node in $V_2$, we have a complete bipartite graph.
  - If $|V_1| = m$ and $|V_2| = n$, the complete bipartite graph is denoted by $K_{m,n}$.

$K_{3,3}$ is Not Planar

- $K_{3,3}$ has $v = 6$ nodes and $e = 9$ edges.
- Suppose it is planar.
- By Euler's formula (40) on p. 441, the number of regions is $r = 2 + e - v = 5$.
- But $K_{3,3}$ has no 3 nodes forming a complete subgraph.
- So the border of a region must contain at least 4 edges.
- The sum of those edges is at least $4r = 20$.
- By Eq. (39) on p. 420, $2e \geq 20$, a contradiction.
Kuratowski's Theorem

**Theorem 34** A graph is nonplanar if and only if it contains a subgraph that is "homeomorphic" to either $K_5$ or $K_{3,3}$.

*Kazimierz Kuratowski (1896-1980).

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Application: Tournaments
- Let $K_n^*$ be a directed graph with $n$ nodes.
- If for each distinct pair $x, y$ of nodes, either $(x, y) \in K_n^*$ or $(y, x) \in K_n^*$ but not both, then $K_n^*$ is called a tournament (recall p. 209).
- A tournament is not necessarily transitive.
- But the next theorem says that players can be ranked in at least one way.

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Hamiltonian Paths and Cycles
- Let $G = (V, E)$ be a graph with $|V| \geq 3$.
- A Hamiltonian cycle is a cycle in $G$ that contains every node in $V$.
- A Hamiltonian path is a path in $G$ that contains every node in $V$.
- Testing if $G$ has a Hamiltonian path or cycle is computationally hard; it is NP-complete.

*William Rowan Hamilton (1805-1865).

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Tournaments Are Hamiltonian

**Theorem 35 (Redei 1934)** A tournament always contains a directed Hamiltonian path.
- Let $p_m = (v_1, v_2, \ldots, v_m)$ be a path of maximum length.
- Assume $m < n$ and proceed to derive a contradiction.
- Let $v$ be a node not on $p_m$.
- $(v, v_1) \notin K_n^*$ for otherwise $p_m$ can be lengthened.
- Hence $(v_1, v) \in K_n^*$.
The Proof (continued)

- If there exists a $1 < j < m$ such that $\langle v_{j-1}, v \rangle \in K_n^*$ and $(v, v_j) \in K_n^*$, then the path $\langle v_1, \ldots, v_{j-1}, v, v_j, \ldots, v_m \rangle$ is longer than $p_m$, a contradiction.

- As $\langle v_1, v \rangle \in K_n^*$, we conclude that for each $1 < j < m$, $\langle v_{j-1}, v \rangle \in K_n^*$ and $(v, v_j) \notin K_n^*$ by induction.

\[ \begin{array}{c}
  \text{v}_1 \\
  \text{v}_2 \\
  \vdots \\
  \text{v}_n \\
  \text{v}_m 
\end{array} \]

The Proof (concluded)

- In particular, $\langle v, v_m \rangle \notin K_n^*$.

- So $\langle v_m, v \rangle \in K_n^*$.

- We can add $\langle v_m, v \rangle$ to $p_m$, a contradiction.

- Remark: Now that $K_n^*$ is Hamiltonian, how to find a Hamiltonian path efficiently?