Integer Solutions of a Linear Equation Revisited

There are \((n + r - 1)\) integer solutions to
\[ x_1 + x_2 + \cdots + x_n = r, \]
where \(x_i \geq 0\) (p. 38).

- The desired number is the coefficient of \(x^r\) in
  \[ f(x) = (1 + x + x^2 + \cdots + x^r)^n \]
  because
  \[ f(x) = \sum_{0 \leq x_1, x_2, \ldots, x_n \leq r} x_1 x_2 \cdots x_n. \]

- By Eq. (28) on p. 330,
  \[ f(x) = (1 - x^{r+1})(1 - x)^n \]
  \[ = \sum_{i=0}^{r} \binom{n+i-1}{i} x^i + x^{r+1}(\cdots). \]

A Simplified Proof

- For each variable \(x_i\), the series \(1 + x + x^2 + \cdots\)
  represents the possible value for that variable: 0, 1, 2, \ldots.

- The desired number is the coefficient of \(x^r\) in
  \[ (1 + x + x^2 + \cdots)^n. \]

- Now,
  \[ (1 + x + x^2 + \cdots)^n = (1 - x)^n \]
  \[ = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i \]
  by Eq. (28) on p. 330.

Partition of Integers

- We ask for the number of partitions of \(n \in \mathbb{Z}^+\) into
  positive integers where the order of summands is irrelevant.
  - The number of partitions of \(n = 3\) is 3: 3, 2 + 1,
    \[ 1 + 1 + 1. \]
  - Contrast it with composition on p. 47.
Partition of Integers (continued)

- The number $n$ can be a sum of a few 1s, a few 2s, a few 3s, \ldots, and a few $n$'s.
- The desired number is the coefficient of $x^n$ in
  \[
  \frac{1}{1-x} \left( \frac{1}{1-x^2} \right) \left( \frac{1}{1-x^4} \right) \left( \frac{1}{1-x^5} \right)
  = \frac{1}{1-x} + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \ldots.
  \]
  So there are 7 ways to partition 5.
- Indeed, the partitions are: 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.
- No known closed-form formula.

Comments on Calculation

- We were asked to calculate the coefficient of $x^n$ in
  \[
  \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \frac{1}{1-x^n}
  \]
  in Eq. (29) on p. 335.
- But it is a product of infinite power series!
- The trick is to calculate only
  \[
  [1 - x^2 + \ldots + (-1)^n x^n][1 - x^3 + x^6 - \ldots + (-1)^n x^{3n}]
  \]
  \[
  \cdots [1 - x^n + x^{2n} - \ldots + (-1)^n x^{n^2}].
  \]
- We can even cut those terms beyond $x^n$.

No Summands Appear More Than Twice

- What is the number of partitions of $m \in \mathbb{Z}^+$ into positive integers where the order of summands is irrelevant and no summands appear more than twice?
- The desired number is the coefficient of $x^m$ in
  \[
  \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \cdots \frac{1}{1-x^{m^2}} = \prod_{i=1}^{\infty} \frac{1}{1-x^{3i}}
  \]
  \[
  = \prod_{i=1}^{\infty} \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \frac{1}{1-x^4} \frac{1}{1-x^5} \frac{1}{1-x^6} \cdots.
  \]
- Same as partitions into summands not divisible by 3.
Weighted Integer Solutions of a Linear Equation

- What is the number of integer solutions to
  \[ x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n, \]
  where \( x_i \geq 0 \)?
- For example, the number of solutions for \( n = 5 \) is 7:
  \[(x_1, x_2, x_3, x_4, x_5) \in \{(5, 0, 0, 0, 0), (3, 1, 0, 0, 0), (2, 0, 1, 0, 0), (1, 2, 0, 0, 0), (1, 0, 0, 1, 0), (0, 1, 1, 0, 0), (0, 0, 0, 0, 1)\}.\]

Partition of Integers into Distinct Summands

- We ask for the number of partitions of \( m \in \mathbb{Z}^+ \) into distinct positive integers where the order of summands is irrelevant.
  - The number of partitions of \( m = 3 \) is 2: 3, 2 + 1.
- The desired number is the coefficient of \( x^m \) in
  \[(1 + x)(1 + x^2)(1 + x^3) \cdots (1 + x^m). \tag{29}\]
- No known closed-form formula.

Weighted Integer Solutions of a Linear Equation (concluded)

- This problem is the partition-of-integers problem in disguise.
  - Every solution \((x_1, x_2, \ldots, x_n)\) implies a partition of \( n \) in which there are \( x_i \)'s, and vice versa.
- The desired number is therefore the coefficient of \( x^n \) in
  \[
  \frac{1}{1 - x} \frac{1}{1 - x^2} \frac{1}{1 - x^3} \cdots \frac{1}{1 - x^n}.
  \]

Partition of Integers into Distinct Summands (concluded)

- Note that
  \[
  (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6) \\
  = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \cdots.
  \]
- So there are 4 ways to partition 6.
- Indeed, the partitions are: 6, 5 + 1, 4 + 2, 3 + 2 + 1.
Partition of Integers into Distinct Summands with Upper Bounds

- We ask for the number of partitions of \( m \in \mathbb{Z}^+ \) into distinct positive integers at most \( n \) where the order of summands is irrelevant.
- The desired number is the coefficient of \( x^m \) in
  \[
  (1 + x)(1 + x^2)(1 + x^3) \cdots (1 + x^n).
  \]
- No known closed-form formula, Applications in computational finance\(^a\).
- Can we compute all \( n(n+1)/2 \) coefficients in time \( o(n^3) \)?

\(^a\)Lynn (2002).

Partition of Integers into Odd Summands

- What is the number of partitions of \( m \in \mathbb{Z}^+ \) into odd positive integers where the order of summands is irrelevant?
  - The number of partitions of \( m = 3 \) is 2: 3, \( 1+1+1 \).
  - The desired number is the coefficient of \( x^m \) in
    \[
    
    \frac{1}{1-x} \frac{1}{1-x^3} \frac{1}{1-x^5} \cdots \frac{1}{1-x^m} = 
    \frac{1 - x^2 - x^4 - x^6 \cdots}{1 - x} \frac{1 - x^6 - x^{10} \cdots}{1 - x^3} \frac{1 - x^{10} - x^{12} \cdots}{1 - x^5} \cdots
    
    = (1+x)(1+x^3)(1+x^5)(1+x^7) \cdots (1+x^m) \ldots
    
    \]
  - This is the same as partitions into distinct summands (recall Eq. (29) on p. 341); Euler (1748).

Partition of Integers into Distinct Summands with Upper Bounds (concluded)

- Note that
  \[
  (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)
  = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 4x^7 + \cdots + x^{21}.
  \]
- So there are 4 ways to partition 7 into distinct positive integers at most 6.
  - Indeed, the partitions are: \( 6 + 1, 5 + 2, 4 + 3, 4 + 2 + 1 \).
- In fact, we solved 21 problems: The coefficient of \( x^i \), where \( 1 \leq i \leq 21 \), represents the number of ways to partition \( i \) into distinct positive integers at most 6.

Partition of Integers into Even Summands

- We ask for the number of partitions of \( m \in \mathbb{Z}^+ \) into positive even integers where the order of summands is irrelevant.
- The desired number is the coefficient of \( x^m \) in
  \[
  (1 + x^2 + x^4 + \cdots)(1 + x^4 + x^8 + \cdots)
  
  (1 + x^6 + x^{12} + \cdots)(1 + x^{|m/2|} + x^{|m/2|} + \cdots).
  \]
- We are more economical than in “partition of integers into odd summands” by removing higher-order terms.
Partition of Integers into Even Numbers of Each Summand

- We ask for the number of partitions of \( m \in \mathbb{Z}^+ \) into positive integer summands where each summand appears an even number of times and the order of summands is irrelevant.
- The desired number remains the coefficient of \( x^m \) in
  \[
  (1 + x^2 + x^4 + \cdots)(1 + x^4 + x^8 + \cdots) \\
  (1 + x^6 + x^{12} + \cdots)(1 + x^{2m/2} + x^{4m/2} + \cdots).
  \]
  - 1 appears an even number of times, 2 appears an even number of times, etc.

Partitions of Integers and Integer Solutions of Linear Equations

- These two issues are often related in subtle ways.
- To wit, what is the number of partitions of \( m \in \mathbb{N} \) into \( n \) nonnegative integers where the order of summands is relevant?
- Each nonnegative integer solution of
  \[
  x_1 + x_2 + \cdots + x_n = m
  \]
  corresponds to a valid partition.
- The answer is thus \( \binom{n+m-1}{m-1} \) from p. 331.

Partitions of Integers and Integer Solutions of Linear Equations (continued)

- What is the number of integer solutions of
  \[
  x_1 + x_2 + \cdots + x_n = n,
  \]
  where \( 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n? \)
- A solution corresponds to a partition of \( n \) into \emph{positive} integers where the order of summands is irrelevant.
  \(- (0, 0, 0, 1, 2, 3) \iff 6 = 1 + 2 + 3.\)
- From p. 335, the number equals the coefficient of \( x^n \) in
  \[
  \frac{1}{(1-x)(1-x^2)\cdots(1-x^n)}.
  \]

Partitions of Integers and Integer Solutions of Linear Equations (continued)

- In general, what is the number of integer solutions of
  \[
  x_1 + x_2 + \cdots + x_n = m,
  \]
  where \( 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n? \)
- Each solution corresponds to a partition of integer \( m \) into \( n \) nonnegative integers where the order of summands is irrelevant\(^a\)
  \(- (0, 0, 0, 1, 2, 3) \iff 6 = 0 + 0 + 0 + 1 + 2 + 3.\)

\(^a\)Professor Andrews, private communication, October 2001.
Partitions of Integers and Integer Solutions of Linear Equations (continued)

- Alternatively, each solution corresponds to a partition of $m$ into nonnegative integers at most $n$ where the order of summands is irrelevant.
  - See the Ferrers graph on the next slide.
- The desired number hence equals the coefficient of $x^m$ in
  \[
  \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^n)}.
  \]

Partitions of Integers and Integer Solutions of a Linear Equation (concluded)

- For instance,
  \[
  \frac{1}{(1-x)(1-x^2)(1-x^3)} = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + 10x^8 + 12x^9 + 14x^{10} + 16x^{11} + 19x^{12} + \cdots.
  \]
- There are 7 ways to partition 6 into 3 nonnegative integers:
  \[
  0 + 0 + 6, 0 + 1 + 5, \\
  0 + 2 + 4, 0 + 3 + 3, \\
  1 + 1 + 4, 1 + 2 + 3, 2 + 2 + 2.
  \]
Recurrence Relations Arise Naturally

- When a problem has a recursive nature, recurrence relations often arise.
  - A problem can be solved by solving 2 subproblems of the same nature.
- When an algorithm is of the divide-and-conquer type, a recurrence relation describes its running time.
  - Sorting, fast Fourier transform, etc.
- Certain combinatorial objects are constructed recursively such as hypercubes (p. 424).

First-Order Linear Homogeneous Recurrence Relations (concluded)

- Now suppose we impose the initial condition \( a_0 = A \).
- Then the (unique) particular solution is \( a_n = Ad^n \).
  - Because \( A = a_0 = Cd^0 = C \).
- Note that \( a_n = na_{n-1} \) is not a first-order linear homogeneous recurrence relation.
  - Its solution is \( n! \) when \( a_0 = 1 \).

First-Order Linear Homogeneous Recurrence Relations

- Consider the recurrence relation
  \[
  a_{n+1} = da_n,
  \]
  where \( n \geq 0 \) and \( d \) is a constant.
- The general solution is given by
  \[
  a_n = Cd^n
  \]
  for any constant \( C \).
  - It satisfies the relation: \( Cd^{n+1} = dCd^n \).
- There are infinitely many solutions, one for each choice of \( C \).

First-Order Linear Non-homogeneous Recurrence Relations

- Consider the recurrence relation
  \[
  a_{n+1} + da_n = f(n).
  \]
  - \( n \geq 0 \).
  - \( d \) is a constant.
  - \( f(n) : \mathbb{N} \to \mathbb{N} \).
- A general solution no longer exists.
**kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients**

- Consider the \( k \)-th order recurrence relation
  \[
  C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0, \quad (30)
  \]

  where \( C_n, C_{n-1}, \ldots, C_{n-k} \in \mathbb{R}, C_n \neq 0, \) and \( C_{n-k} \neq 0. \)

- Add \( k \) initial conditions for \( a_0, a_1, \ldots, a_k. \)

- Clearly, \( a_n \) is well-defined for each \( n = k, k+1, \ldots. \)

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**kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients (concluded)**

- A solution \( y \) for \( a_n \) is general if for any particular solution \( y' \), the undetermined coefficients of \( y \) can be found so that \( y \) is identical to \( y'. \)

- Any general solution for \( a_n \) that satisfies the \( k \) initial conditions and Eq. (30) is a particular solution.

- In fact, it is the unique particular solution because any solution agreeing at \( n = 0, 1, \ldots, k-1 \) must agree for all \( n \geq 0. \)

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**Conditions for the General Solution**

**Theorem 28** Let \( a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)} \) be \( k \) particular solutions of Eq. (30). If

\[
\begin{vmatrix}
  a_0^{(1)} & a_0^{(2)} & \cdots & a_0^{(k)} \\
  a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(k)} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k-1}^{(1)} & a_{k-1}^{(2)} & \cdots & a_{k-1}^{(k)}
\end{vmatrix} \neq 0, \tag{31}
\]

then \( a_n = C_1 a_n^{(1)} + C_2 a_n^{(2)} + \cdots + C_k a_n^{(k)} \) is the general solution, where \( C_1, C_2, \ldots, C_k \) are arbitrary constants.\(^a\)

Fundamental Sets

- The particular solutions of Eq. (30) on p. 359, 
  \[ a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)}, \]
  that also satisfy inequality (31) in Theorem 28 are said to form a fundamental set of solutions.
- Solving a linear homogeneous recurrence equation thus reduces to finding a fundamental set!

The Justification

- Assume \( a_n \) has the form \( cr^n \) for nonzero \( c \) and \( r \).
- After substitution into recurrence equation (30) on p. 359, \( r \) satisfies characteristic equation (32).
- Let \( r_1, r_2, \ldots, r_k \) be the \( k \) distinct (nonzero) roots.
- Hence \( a_n = r_i^n \) is a solution for \( 1 \leq i \leq k \).
- Solutions \( r_i^n \) form a fundamental set because

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**kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Distinct Roots**

- Let \( r_1, r_2, \ldots, r_k \) be the (characteristic) roots of the characteristic equation
  \[ C_n r^n + C_{n-1} r^{n-1} + \cdots + C_0 = 0. \quad (32) \]
- If \( r_1, r_2, \ldots, r_k \) are distinct, then the general solution has the form
  \[ a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n, \]
  for constants \( c_1, c_2, \ldots, c_k \) determined by the initial conditions.

**The Justification (continued)**

- The \( k \times k \) matrix is called a **Vandermonde matrix**, which is nonsingular whenever \( r_1, r_2, \ldots, r_k \) are distinct.\(^a\)

\(^a\)This is a standard result in linear algebra.
The Justification (concluded)

- Hence

\[ a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n \]

is the general solution.

- The $k$ coefficients $c_1, c_2, \ldots, c_k$ are determined uniquely by the $k$ initial conditions $a_0, a_1, \ldots, a_{k-1}$:

\[
\begin{pmatrix}
  a_0 \\
  a_1 \\
  \vdots \\
  a_{k-1}
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 & \cdots & 1 \\
  r_1 & r_2 & \cdots & r_k \\
  \vdots & \vdots & \ddots & \vdots \\
  r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1}
\end{pmatrix}
\begin{pmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_k
\end{pmatrix}.
\]

(33)

The Fibonacci Relation (concluded)

- Solve

\[
\begin{align*}
0 &= a_0 = c_1 + c_2 \\
1 &= a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2}
\end{align*}
\]

for $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = \frac{1}{\sqrt{5}}$.

- The solution is finally

\[
a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

The Fibonacci Relation

- Consider $a_{n+2} = a_{n+1} + a_n$.

- The initial conditions are $a_0 = 0$ and $a_1 = 1$.

- The characteristic equation is $r^2 - r - 1 = 0$, with two roots $(1 \pm \sqrt{5})/2$.

- The general solution is hence

\[
a_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

Don't Believe It?

\[
\begin{align*}
a_2 &= \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^2 - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^2 \\
&= \frac{1}{\sqrt{5}} \frac{1 + 2\sqrt{5} + 5}{4} - \frac{1}{\sqrt{5}} \frac{1 - 2\sqrt{5} + 5}{4} = 1, \\
a_3 &= \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^3 - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^3 \\
&= \frac{1}{\sqrt{5}} \frac{1 + 3\sqrt{5} + 15 + 5\sqrt{5}}{8} - \frac{1}{\sqrt{5}} \frac{1 - 3\sqrt{5} + 15 - 5\sqrt{5}}{8} = 2.
\end{align*}
\]
Initial Conditions

- Different initial conditions give rise to different solutions.
- Suppose $a_0 = 1$ and $a_1 = 2$.
- Then solve

$$1 = a_0 = c_1 + c_2$$

$$2 = a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2}$$

for $c_1 = \frac{[(1 + \sqrt{5})/2]^2}{\sqrt{5}}$ and $c_2 = \frac{[(1 \sqrt{5})/2]^2}{\sqrt{5}}$.

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Example: A Third-Order Relation

- Consider

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$$

with $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$.
- The characteristic equation $2r^3 - r^2 - 2r + 1 = 0$ has three distinct real roots: $1$, $-1$, and $0.5$.
- The general solution is

$$a_n = c_1 n^1 + c_2 n^(-1) + c_3 (1/2)^n$$

$$= c_1 + c_2 (1)^n + c_3 (1/2)^n.$$