Compositions of Positive Integers

- Let \( m \) be a positive integer.
- A composition for \( m \) is a sum of positive integers whose order is relevant and which sum to \( m \).
- For \( m = 3 \), the number of compositions is four: \( 3, 2 + 1, 1 + 2, 1 + 1 + 1 \).
- For \( m = 4 \), the number of compositions is eight: \( 4, 3 + 1, 2 + 2, 1 + 3, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 1 + 1 + 1 + 1 \).
- They suggest the number of compositions for \( m \) is \( 2^{m-1} \).

The Catalan Numbers (1838)

- A binomial random walk starts at the origin.
- What is the number of ways it can end at the origin in \( 2n \) steps without being in the negative territory?
- It is equivalent to the number of ways \( RR \cdots RLL \cdots L \) can be permuted so that no prefix has more \( Ls \) than \( Rs \).

\[ \text{The Catalan Numbers} \]

- For example,

\[
\begin{array}{c}
 0 \\
 1 \\
 2 \\
 1 \\
 0 \\
 1 \\
 0 \\
 R L R L R R L L.
\end{array}
\]

\[ \text{Eugène Charles Catalan (1814–1894).} \]
Formula for the Catalan Numbers

The number is

\[ b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}, n \geq 1. \]

with \( b_0 = 1. \)

- \( \overbrace{RR\cdots R}^{n} \overbrace{LL\cdots L}^{n} \) can be permuted in \( \binom{2n}{n} \) ways.
- Some of the permutations are illegal, such as \( RLLRR. \)
- We now prove that \( \binom{2n}{n} \) of the permutations are illegal.

Application: First Return to the Origin

- Let \( n \geq 1. \)
- What is the number of ways a binomial random walk first returns to the origin after \( 2n \) steps?
- The answer is \( 2b_n = \frac{1}{2n-1} \binom{2n}{n}. \)

The Proof (concluded)

- For every illegal permutation, we consider the first \( L \) move that makes the particle land at \(-1.\)
  - Such as \( RL[L]LRR. \)
- Swap \( L \) and \( R \) for this offending \( L \) and all earlier moves.
  - Such as \( [L][R][L]LRR. \)
- The result is some permutation of \( \overbrace{RR\cdots R}^{n+1} \overbrace{LL\cdots L}^{n}. \)
- As the correspondence is one-to-one, the offending walks are \( \binom{2n}{n} \) in number.

Application: Nonnegative Partial Sums

- Let \( n \geq 1. \)
- What is the number of ways we can arrange \( n \) \(+1\) and \( n \) \(-1\) such that all \( 2n \) partial sums (starting with the first summand) are nonnegative?
  - The six partial sums of \( (1, 1, -1, 1, -1, -1) \) are \( (1, 2, 1, 2, 1, 0). \)
  - The answer is \( b_n. \)
- The number remains \( b_n \) if we have only \( n-1 \) \(-1\).
  - In the original problem, the last number must be \(-1.\)
Application: A Variation
What is the number of ways
\[
\frac{n}{RR \ldots R} \frac{n}{LL \ldots L}
\]
can be permuted so that no prefix has more Rs than Ls?
- Apply the proof for \( b_n \) with \( L \) and \( R \) swapped.
- So it remains \( b_n \).

Application: Nonpositive Partial Sums
- Let \( n \geq 1 \).
- What is the number of ways we can arrange \( n \) “+1” and \( n \) “−1” such that all \( 2n \) partial sums (starting with the first summand) are nonpositive?
  - The six partial sums of \((-1, -1, 1, -1, 1, 1)\) are \((-1, -2, -1, -2, -1, 0)\).
  - The answer is \( b_n \).
- The number remains \( b_n \) if we have only \( n - 1 \) “+1”.
  - In the original problem, the last number must be 1.

Application: Another Variation
What is the number of ways a particle starting at the origin can end at the origin in \( 2n \) steps without touching \(-2\)?
- Pick up any sequence of \( 2(n + 1) \) moves without being in the negative territory.
- The first move must be a right move, and the last move must be a left move.
- Remove the first and the last moves.
- The remaining \( 2n \) moves will not touch \(-2\) (see next page).
- So the desired number is \( b_{n+1} \).
Application: Parenthesizing a Product

There are \( b_n \) ways to parenthesize \( x_1x_2 \cdots x_n \), where \( n \geq 1 \).

- There is a one-to-one correspondence between a legitimate parenthesization and an arrangement of \( n - 1 \) "+1" and \( n - 1 \) "−1" so that all the partial sums are nonnegative.
- The answer then follows from p. 54.

Two ways to parenthesize \( x_1x_2x_3 \)

\[
\begin{array}{c|c}
(\overline{x_1x_2}x_3) & (1, 1, -1, -1) \\
(x_1(\overline{x_2x_3})) & (1, -1, 1, -1)
\end{array}
\]

Application: Triangulations of a Convex \( n \)-Gon

- Similarly, a parenthesization of \( x_1x_2 \cdots x_{n-1} \) corresponds to a triangulation.
- So there are

\[
b_n = 1 - \binom{2n - 4}{n - 2}
\]

ways to triangulate the \( n \)-gon (p. 59).

Application: Triangulations of a Convex \( n \)-Gon

- A triangulation is a set of \( n - 3 \) diagonals no two of which intersect internally.
- A triangulation divides the \( n \)-gon into \( n - 2 \) triangles.
- Let \( v_1, v_2, \ldots, v_n \) be nodes of the \( n \)-gon.
- If \( v_i \) is joined to \( v_j \) by a diagonal in a triangulation, parenthesize the product \( x_1x_2 \cdots x_{n-1} \) with

\[
x_1 \cdots (x_i \cdots x_{j-1}) x_j \cdots x_{n-1}.
\]
- By so doing, we disallow \( v_i \) to be joined to \( v_{i+1} \).
- (Better proof, anybody?)

Combinatorics and "Higher" Mathematics
### Growth of Factorials

<table>
<thead>
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<th>n</th>
<th>n!</th>
<th>n</th>
<th>n!</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>8</td>
<td>40320</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>9</td>
<td>362880</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>10</td>
<td>3628800</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>11</td>
<td>39916800</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>12</td>
<td>479001600</td>
</tr>
<tr>
<td>5</td>
<td>120</td>
<td>13</td>
<td>6227020800</td>
</tr>
<tr>
<td>6</td>
<td>720</td>
<td>14</td>
<td>87178291200</td>
</tr>
<tr>
<td>7</td>
<td>5040</td>
<td>15</td>
<td>1307674368000</td>
</tr>
</tbody>
</table>

### A Useful Lower Bound for $n!$

**Lemma 7** $n! > (n/e)^n$.

**Proof:**

\[
\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n
\]

\[
> \sum_{k=1}^{n} \int_{k-1}^{k} \ln x \, dx \quad \text{as } \ln x \text{ is increasing}
\]

\[
= \int_{0}^{n} \ln x \, dx
\]

\[
= \left[ x \ln x - x \right]_{x=0}^{n}
\]

\[
= n \ln n - n.
\]
How Good Is the Bound?

Conclusion: good but not of the same order as $n!$.

A Useful Upper Bound for $C(n, m)$

**Lemma 9** $C(n, m) < (ne/m)^m$ for any $0 < m < n$.

Proof:

$$C(n, m) = \frac{n!}{(n-m)!m!} \leq \frac{n^m}{m!} < \frac{n^m}{(m/e)^m} \text{ by Lemma 7 (p. 65)}\]

\[= (ne/m)^m.\]

A Marginally Better Bound

**Lemma 8** $n! > e(n/e)^n$.

Proof:

$$\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \ldots + \ln n$$

$$> \sum_{k=2}^{n} \int_{k-1}^{k} \ln x \, dx$$

$$\geq \int_{1}^{n} \ln x \, dx \quad \text{as } \ln x \leq 0 \text{ when } x \leq 1$$

$$= \left[ x \ln x - x \right]_{x=1}^{n}$$

$$= n \ln n - n + 1.$$

Stirling’s Formula$^a$ (1730)

**Theorem 10** $n! \sim \sqrt{2\pi n} (n/e)^n$.

$^a$James Stirling (1692–1770); but actually due to Abraham de Moivre (1667–1754)!
Approximation of $C(n, m)$

- Stirling’s formula can be used to approximate $C(n, m)$ better than Lemma 9 (p. 69) under some conditions.
- For that purpose, a more refined Stirling’s formula is stated below without proof:

$$\sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}}.$$  \hspace{1cm} (3)

---

*Robbins (1955).*
Approximation of $C(n, m)$ (concluded)

- Now from inequalities (3) on p. 78,

\[
C(n, m) = \frac{n!}{(n - m)! m!} \frac{1}{\sqrt{2\pi (n - m)(n - m + 1)} e^{\frac{1}{24m}}} \geq \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^n \left( \frac{n}{m} \right)^{-m} \sqrt{\frac{n}{m(n - m)}} e^{\frac{1}{12m} - \frac{1}{144m^2}} - \frac{1}{8}\frac{1}{144m^2}.
\]

\[
1 < \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^n \left( \frac{n}{m - m} \right)^{-m} \sqrt{\frac{n}{m(n - m)}} e^{\frac{1}{12m} - \frac{1}{144m^2}} - \frac{1}{8}\frac{1}{144m^2}.
\]

Approximation of $C(n, m), 1 \leq m \leq n/2$ (concluded)

- Combining inequalities (4) on p. 79 and (5) on p. 80 under $1 \leq m \leq n/2$, we conclude that

\[
C(n, m) \sim \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n - m} \right)^{-m} \sqrt{\frac{n}{m(n - m)}}
\]

as $m \to \infty$.

Approximation of $C(n, m), 1 \leq m \leq n/2$

\[
\frac{C(n, m)}{C(n - m, m)} = \sqrt{\frac{2\pi(n - m)(n - m + 1)}{2\pi n(n - m + 1)}} e^{-\frac{1}{24m}} \geq \sqrt{\frac{n}{m(n - m)}} e^{-\frac{1}{12m} + \frac{1}{144m^2}} - \frac{1}{8}\frac{1}{144m^2}.
\]

\[
1 > \sqrt{\frac{n}{m(n - m)}} e^{-\frac{1}{12m} + \frac{1}{144m^2}} - \frac{1}{8}\frac{1}{144m^2}.
\]

\[
\frac{C(n, m)}{C(n - m, m)} > \sqrt{\frac{n}{m(n - m)}} e^{-\frac{1}{12m} + \frac{1}{144m^2}} - \frac{1}{8}\frac{1}{144m^2},
\]

An Upper Bound for $C(2n, n)$

Lemma 11 $C(2n, n) < 4^n$.

Proof: From inequality (4) on p. 79,

\[
C(2n, n) < \frac{1}{\sqrt{2\pi}} \left( \frac{2n}{n} \right)^n \left( \frac{2n}{2n - n} \right)^n \sqrt{\frac{2n}{n(2n - n)}} = \frac{1}{\sqrt{n\pi}} 4^n < 4^n.
\]

(Lemma 9 on p. 69 gives a looser upper bound of $(2e)^n$.)
Logic

Tautology\(^a\) and Contradiction

- A statement is a **tautology** if it is true for all truth value assignments for its component statements.
  - For example, \( p \lor \neg p \).
- A statement is a **contradiction** if it is false for all truth value assignments for its component statements.

\(^a\)Ludwig Wittgenstein (1889-1951), one of the most important philosophers of all time. “God has arrived,” Keynes said of him on January 18, 1928, “I met him on the 5:15 train.”

Propositional Logic:

- \( \neg p \): the negation of statement \( p \).
- \( p \land q \): the **conjunction** of statements \( p \) and \( q \).
  - “\( p \) and \( q \).”
- \( p \lor q \): the **disjunction** of statements \( p \) and \( q \).
  - “\( p \) or \( q \).”
- \( p \rightarrow q \): the **material implication** of \( q \) by \( p \).
  - “Hypothesis \( p \) implies conclusion \( q \).”
- \( p \leftrightarrow q \): the **biconditional** of \( p \) and \( q \).
  - “\( p \) if and only if \( q \).”

\(^a\)Attributed to Gottfried Wilhelm Leibniz (1646-1716).

Tautology and Contradiction (concluded)

- If

\[
(p_1 \land p_2 \land \cdots \land p_n) \rightarrow q
\]

is a tautology, then conclusion \( q \) follows **validly** from premises \( p_1, p_2, \ldots, p_n \).
- When \( p \rightarrow q \) is a tautology, we say \( p \) **logically implies** \( q \), written as \( p \Rightarrow q \).
Truth Tables

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p &amp; q</th>
<th>p \lor q</th>
<th>p \rightarrow q</th>
<th>p \leftrightarrow q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<tr>
<td>0</td>
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</tr>
</tbody>
</table>

- A truth table with \( n \) basic variables has \( 2^n \) entries.
- Verifying a tautology is a known computationally hard problem (\( \text{coNP-complete} \), to be exact).

Example of a Potentially Invalid Argument

"The premises he used were false, therefore his conclusions were false," said Dudley Sharp of \textit{Justice For All}.


Example of a Valid Argument

\( q \) follows validly from

\[ (p \rightarrow r) \land (\neg q \rightarrow p) \land \neg r. \]

- Write down the truth table for

\[ [(p \rightarrow r) \land (\neg q \rightarrow p) \land \neg r] \rightarrow q. \]

- Verify that it is a tautology.

Logical Equivalence

- Statements \( p \) and \( q \) are \textit{logically equivalent} (written as \( p \iff q \)) when \( p \) and \( q \) have the same truth value for all truth value assignments for the component statements.
- For example, \( p \rightarrow q \iff \neg p \lor q \).

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>\neg p \lor q</th>
<th>p \rightarrow q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
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<td>1</td>
</tr>
</tbody>
</table>
Some Easy Pieces

- $p \Rightarrow q$ if and only if $p \rightarrow q$ is a tautology.
- $p \Leftrightarrow q$ if and only if $p \Rightarrow q$ and $q \Rightarrow p$.
- $[p \land (p \rightarrow q)] \Rightarrow q$.
  - Modus ponens or rule of detachment.
- $[(p \rightarrow q) \land (q \rightarrow r)] \Rightarrow (p \rightarrow r)$.
  - Law of the syllogism.
- $[(p \rightarrow q) \land \neg q] \Rightarrow \neg p$.
  - Modus tollens.
- $(\neg p \rightarrow \text{false}) \Rightarrow p$.
  - Reductio ad absurdum or rule of contradiction.

Duality

- Let $s$ be a statement and contain no logical connectives other than $\lor$ and $\land$.
- The dual of $s$, $s^d$, is the statement obtained from $s$ by replacing each occurrence of
  - $\land$ with $\lor$;
  - $\lor$ with $\land$;
  - true with false, and
  - false with true,
  - Note that $\neg$ is unchanged.
- The principle of duality: If $s \iff t$, then $s^d \iff t^d$.

DeMorgan’s Laws

- $\neg(p \lor q) = \neg p \land \neg q$.
- $\neg(p \land q) = \neg p \lor \neg q$.
- Can be used to transform any boolean expression into an equivalent one where $\neg$ applies only to variables.
  - For example,
    
    $\neg(x_1 \lor (x_2 \land \neg x_3)) = (\neg x_1) \land (\neg x_2 \lor \neg x_3)$
    
    $= (\neg x_1) \land (\neg x_2 \lor \neg x_3)$
    
    $= (\neg x_1) \land (\neg x_2 \lor x_3)$.

Set Theory

- Let $A$ and $B$ be sets.
- $x \in A$ means $x$ is an element of $A$.
- $x \notin A$ means $x$ is not an element of $A$.
- $A \subseteq B$ means every element of $A$ is an element of $B$.
- $A = B$ means $A \subseteq B$ and $B \subseteq A$.
- $A \subset B$ means $A \subseteq B$ but $A \neq B$.
- $\emptyset$ is the empty set.
  - For any set $A$, $\emptyset \subseteq A$.

\[\text{Augustus De Morgan (1806 - 1871)},\]

\[\text{Founded by Georg Cantor (1845 - 1918)},\]
Set Operations

- $A \cup B$ is the **union** of $A$ and $B$.
- $A \cap B$ is the **intersection** of $A$ and $B$.
- $A \Delta B$ is the **symmetric difference** of $A$ and $B$, or $\{x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A)\}$.
- $A$ and $B$ are **disjoint** if $A \cap B = \emptyset$.
- $A$ is the **complement** of $A$.

DeMorgan’s Laws

\[
\bigcup_{i \in I} A_i = \bigcap_{i \in I} A_i^c \\
\bigcap_{i \in I} A_i = \bigcup_{i \in I} A_i^c
\]

More Combinatorics

Set Operations (concluded)

- $A - B = \{x : x \in A \land x \notin B\}$.
- $\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$.
- $\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}$. 
Common Sets

\[ N = \{ 0, 1, \ldots \} \]
\[ Z = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \]
\[ Z^+ = \{ 1, 2, \ldots \} \]
\[ \mathbb{R} = \text{set of real numbers} \]
\[ \mathbb{Q} = \text{set of rational numbers} \]

---

Mathematical Induction\(^a\)

- Let \( S(n) \) denote an (open) mathematical statement containing references to a positive integer \( n \) such that
  - \( S(1) \) is true and
  - \( S(k + 1) \) is true whenever \( S(k) \) is true for arbitrarily chosen \( k \in Z^+ \).
- Then \( S(n) \) is true for all \( n \in Z^+ \).

\(^a\)Richard Dedekind (1831–1916) and Giuseppe Peano (1858–1932).

---

The Well-Ordering Principle

**Theorem 12** Every nonempty subset of \( Z^+ \) contains a smallest element; \( Z^+ \) is well-ordered.

- Real numbers are not well-ordered.
  - \( \{ x \in \mathbb{R} : x > 1 \} \) does not contain a smallest element.
- Rational numbers are not well-ordered.
  - \( \{ x \in \mathbb{Q} : x > 1 \} \) does not contain a smallest element.

---

The Proof

- Let \( F = \{ t \in Z^+ : S(t) \text{ is false} \} \).
- Assume that \( F \neq \emptyset \).
- \( F \) has a least element \( \ell \) by the well-ordering principle.
- Clearly \( \ell > 1 \) and, hence, \( \ell - 1 \in Z^+ \).
- Because \( \ell - 1 \notin F \), \( S(\ell - 1) \) is true.
- It follows that \( S(\ell) \) is true, a contradiction.
- So \( F = \emptyset \).
Philosophical Issues

- Mathematical induction has nothing to do with induction in the physical and empirical sciences.
  - Sun rises on Monday, on Tuesday, etc., so it must rise everyday from now?
- Mathematical induction is merely a property of integers.