Problem 1 Show that if $L_1$ and $L_2$ are recursively enumerable languages, then so is $L_1 \cup L_2$.

Ans: Since $L_1$ and $L_2$ are RE languages, there must be a TM $M_1$ accepting $L_1$ and a TM $M_2$ accepting $L_2$. Now we construct another TM $M'$ that simulates $M_1$ and $M_2$ in an interleaving style. Make $M'$ accept if $M_1$ or $M_2$ accepts. Now,

- if $x \in L_1$, $M'(x) = M_1(x) = \text{"yes"}$,
- if $x \in L_2$, $M'(x) = M_2(x) = \text{"yes"}$,
- if $x \notin L_1 \cup L_2$, $M'(x) = \swarrow$.

So $M'$ accepts $L_1 \cup L_2$ and $L_1 \cup L_2$ is recursively enumerable.  

Problem 2 Show that the language

$$A = \{ (M; x) \mid M(x) = \text{"Yes"}} \}$$

is undecidable.
Ans: Suppose $A$ is recursive. Then there exists a TM $M_A$ that decides $A$ such that

- $M_A(M; x) = \text{"Yes"}$, if $(M; x) \in A$,
- $M_A(M; x) = \text{"No"}$, if $(M; x) \notin A$.

Consider the program $D(M)$ that calls $M_A$:

1: if $M_A(M; M) = \text{"Yes"}$ then
2: $D(M) = \text{"No"}$;
3: else
4: $D(M) = \text{"Yes"}$;
5: end if

Now, consider $D(D)$:

- $D(D) = \text{"No"}$, if $M_A(D; D) = \text{"Yes"}$.
- $D(D) = \text{"Yes"}$, if $M_A(D; D) = \text{"No"}.$

Note that, however, $M_A(D; D) = \text{"Yes"}$ implies $D(D) = \text{"Yes"}$ because $(D; D) \in A$. Similarly, $M_A(D; D) = \text{"No"}$ implies $D(D) \neq \text{"Yes"}$ because $(D; D) \notin A$. As contradiction follows in both cases, $A$ is undecidable. ■