Problem 1 (Chernoff Bound) Suppose $x_1, x_2, ..., x_n$ are independent random variables taking values 1 and 0 with probabilities $p$ and $1 - p$, respectively. Let $X = \sum_{i=1}^{n} X_i$. Then for $0 \leq \theta \leq 1$, $\Pr[X \leq (1 - \theta)n] \leq e^{-\frac{\theta^2 pn}{2}}$.

Proof: Let $t$ be any negative real number. By Markov inequality, $\Pr[X \leq (1 - \theta)n] = \Pr[e^{tX} \geq e^{t(1 - \theta)n}] \leq e^{-t(1 - \theta)n}E[e^{tX}]$. Since $X = \sum_{i=1}^{n} X_i$, $E[e^{tX}] = (1 + p(e^t - 1))^n$. Thus,

$$\Pr[X \leq (1 - \theta)n] \leq e^{-t(1 - \theta)n} (1 + p(e^t - 1))^n$$

(1)

Note that $(1 + a)^n \leq e^{an}$ for any $a > 0$. Let $t = \ln (1 - \theta)$. then

$$\Pr[X \leq (1 - \theta)n] \leq e^{-pn\ln(1 - \theta)}$$

(2)

The exponent expands to $-pn\left(\frac{\theta^2}{2} + \frac{\theta^3}{6} + \cdots\right)$ for $0 \leq \theta \leq 1$. Thus $\Pr[X \leq (1 - \theta)n] \leq e^{-\frac{\theta^2 pn}{2}}$.

Problem 2 Recall that $\text{EXP} = \text{TIME}(2^{n^k})$. Show that $\text{BPP} \subseteq \text{EXP}$.
**Proof:** It is known that PSPACE ⊆ EXP (p. 220 of the slides). Thus all we need to show is BPP ⊆ PSPACE. Let $L \in \text{BPP}$, and consider a precise polynomial-time NTM $N$ that decides $L$. Let $\epsilon \leq 1/4$ be the error probability, and $p(n)$ be the polynomial time complexity of $N$, where $n$ is the length of the input. Without loss of generality, assume $N$ has 2 options in each nondeterministic move. As in the textbook, in each run $N$ makes $p(n)$ nondeterministic moves. Thus $N$ has $2^{p(n)}$ possible computation paths each of length $p(n)$. Each computation path has the same probability of occurrence.

Construct a deterministic TM $M$ which simulates $N$ to generate all possible computation paths sequentially and reuses the space used by each previous path. $M$ counts the number $n_{\text{accept}}$ of the accepting paths. $M$ accepts the input if $\frac{n_{\text{accept}}}{2^{p(n)}} \geq 3/4$; otherwise, $M$ rejects. Thus $N$ runs in polynomial space. Clearly, BPP ⊆ PSPACE and BPP ⊆ EXP is proved.

■