Lecture 8: Noise and Error

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Roadmap

1. When Can Machines Learn?
2. Why Can Machines Learn?

**Lecture 7: The VC Dimension**

Learning happens if finite $d_{vc}$, large $N$, and low $E_{in}$

**Lecture 8: Noise and Error**

- Noise and Probabilistic Target
- Error Measure
- Algorithmic Error Measure
- Weighted Classification

3. How Can Machines Learn?
4. How Can Machines Learn Better?
Recap: The Learning Flow

unknown target function $f: \mathcal{X} \rightarrow \mathcal{Y}$

+ noise

training examples $\mathcal{D}: (x_1, y_1), \cdots, (x_N, y_N)$

learning algorithm $\mathcal{A}$

final hypothesis $g \approx f$

what if there is noise?
briefly introduced noise before pocket algorithm

<p>| | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>age</td>
<td>23 years</td>
</tr>
<tr>
<td>gender</td>
<td>female</td>
</tr>
<tr>
<td>annual salary</td>
<td>NTD 1,000,000</td>
</tr>
<tr>
<td>year in residence</td>
<td>1 year</td>
</tr>
<tr>
<td>year in job</td>
<td>0.5 year</td>
</tr>
<tr>
<td>current debt</td>
<td>200,000</td>
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credit? \{no(-1), yes(+1)\}

but more!

- **noise in y**: good customer, ‘mislabeled’ as bad?
- **noise in y**: same customers, different labels?
- **noise in x**: inaccurate customer information?

does VC bound work under noise?
Probabilistic Marbles

one key of VC bound: marbles!

‘deterministic’ marbles
- marble \( x \sim P(x) \)
- deterministic color \( [f(x) \neq h(x)] \)

‘probabilistic’ (noisy) marbles
- marble \( x \sim P(x) \)
- probabilistic color \( [y \neq h(x)] \) with \( y \sim P(y|x) \)

same nature: can estimate \( \mathbb{P}[\text{orange}] \) if \( i.i.d. \)

VC holds for \( (x,y)^{i.i.d.} P(x,y) \)

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Target Distribution $P(y|x)$

characterizes behavior of ‘mini-target’ on one $x$

- can be viewed as ‘ideal mini-target’ + noise, e.g.
  - $P(\circ|x) = 0.7$, $P(\times|x) = 0.3$
  - ideal mini-target $f(x) = \circ$
  - ‘flipping’ noise level = 0.3

- deterministic target $f$: special case of target distribution
  - $P(y|x) = 1$ for $y = f(x)$
  - $P(y|x) = 0$ for $y \neq f(x)$

Goal of learning:

predict ideal mini-target (w.r.t. $P(y|x)$) on often-seen inputs (w.r.t. $P(x)$)
The New Learning Flow

unknown target distribution $P(y|x)$ containing $f(x) + \text{noise}$

$y_1, y_2, \cdots, y_N$

training examples $D: (x_1, y_1), \cdots, (x_N, y_N)$

learning algorithm $A$

unknown $P$ on $X$

$x_1, x_2, \cdots, x_N$

$x, y$

final hypothesis $g \approx f$

hypothesis set $\mathcal{H}$

VC still works, pocket algorithm explained :-)

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Let's revisit PLA/pocket. Which of the following claim is true?

1. In practice, we should try to compute if $D$ is linear separable before deciding to use PLA.
2. If we know that $D$ is not linear separable, then the target function $f$ must not be a linear function.
3. If we know that $D$ is linear separable, then the target function $f$ must be a linear function.
4. None of the above

Reference Answer: 4

1. After computing if $D$ is linear separable, we shall know $w^*$ and then there is no need to use PLA. 2. What about noise? 3. What about ‘sampling luck’? :-)

Hsuan-Tien Lin (NTU CSIE)  Machine Learning Foundations  7/25
Error Measure

- how well? previously, considered out-of-sample measure
  \[ E_{\text{out}}(g) = \mathbb{E}_{x \sim P} [g(x) \neq f(x)] \]

- more generally, error measure \( E(g, f) \)
- naturally considered
  - out-of-sample: averaged over unknown \( x \)
  - pointwise: evaluated on one \( x \)
  - classification: \([\text{prediction} \neq \text{target}]\)

classification error \( [\ldots] \): often also called ‘0/1 error’
Pointwise Error Measure

can often express $E(g, f) =$ averaged $\text{err}(g(x), f(x))$, like

$$E_{out}(g) = \mathcal{E}_{x \sim P} \left[ g(x) \neq f(x) \right]$$

$\text{err}$: called pointwise error measure

---

**in-sample**

$$E_{in}(g) = \frac{1}{N} \sum_{n=1}^{N} \text{err}(g(x_n), f(x_n))$$

**out-of-sample**

$$E_{out}(g) = \mathcal{E}_{x \sim P} \text{err}(g(x), f(x))$$

will mainly consider pointwise $\text{err}$ for simplicity
Two Important Pointwise Error Measures

\[
\text{err} \begin{pmatrix}
g(x), f(x) \\
g(\tilde{y}), f(\tilde{y})
\end{pmatrix}
\]

0/1 error

\[
\text{err}(\tilde{y}, y) = \left[ \tilde{y} \neq y \right]
\]
- correct or incorrect?
- often for classification

Squared error

\[
\text{err}(\tilde{y}, y) = (\tilde{y} - y)^2
\]
- how far is \(\tilde{y}\) from \(y\)?
- often for regression

How does \(\text{err}\) ‘guide’ learning?
Ideal Mini-Target

interplay between noise and error:

\[ P(y|x) \text{ and } \text{err} \text{ define ideal mini-target } f(x) \]

\[
P(y = 1|x) = 0.2, \quad P(y = 2|x) = 0.7, \quad P(y = 3|x) = 0.1
\]

\[
\text{err}(\tilde{y}, y) = \left[ \tilde{y} \neq y \right]
\]

\[
\tilde{y} = \begin{cases} 
1 & \text{avg. err 0.8} \\
2 & \text{avg. err 0.3(\ast)} \\
3 & \text{avg. err 0.9} \\
1.9 & \text{avg. err 1.0 (really? :-))}
\end{cases}
\]

\[
f(x) = \arg \max_{y \in Y} P(y|x)
\]

\[
f(x) = \sum_{y \in Y} y \cdot P(y|x)
\]
Learning Flow with Error Measure

- **unknown target distribution** $P(y|x)$ containing $f(x) +$ noise
- **training examples** $D: (x_1, y_1), \cdots, (x_N, y_N)$
- **learning algorithm** $\mathcal{A}$
- **final hypothesis** $g \approx f$
- **hypothesis set** $\mathcal{H}$
- **unknown $P$ on $\mathcal{X}$**
- **error measure** $err$

Extended VC theory/‘philosophy’
works for most $\mathcal{H}$ and $err$
Consider the following $P(y|x)$ and $\text{err}(\tilde{y}, y) = |\tilde{y} - y|$. Which of the following is the ideal mini-target $f(x)$?

\[
P(y = 1|x) = 0.10, \ P(y = 2|x) = 0.35, \ P(y = 3|x) = 0.15, \ P(y = 4|x) = 0.40.
\]

1. $2 = \text{weighted median from } P(y|x)$
2. $2.5 = \text{average within } \mathcal{Y} = \{1, 2, 3, 4\}$
3. $2.85 = \text{weighted mean from } P(y|x)$
4. $4 = \text{argmax } P(y|x)$

Reference Answer: 1

For the ‘absolute error’, the weighted median provably results in the minimum average $\text{err}$. 
Choice of Error Measure

Fingerprint Verification

two types of error: **false accept** and **false reject**

<table>
<thead>
<tr>
<th>$f$</th>
<th>+1</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>no error</td>
<td>false reject</td>
</tr>
<tr>
<td>-1</td>
<td>false accept</td>
<td>no error</td>
</tr>
</tbody>
</table>

$0/1$ error penalizes both types **equally**
Fingerprint Verification for Supermarket

Two types of error: false accept and false reject

<table>
<thead>
<tr>
<th>$f$</th>
<th>+1</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>no error</td>
<td>false reject</td>
</tr>
<tr>
<td>+1</td>
<td>no error</td>
<td>false reject</td>
</tr>
<tr>
<td>-1</td>
<td>false accept</td>
<td>no error</td>
</tr>
</tbody>
</table>

- Supermarket: fingerprint for discount
- False reject: very unhappy customer, lose future business
- False accept: give away a minor discount, intruder left fingerprint :-)
two types of error: \textit{false accept} and \textit{false reject}

\begin{center}
\begin{tabular}{c|cc}
  & +1 & -1 \\
\hline
+1 & no error & false reject \\
-1 & false accept & no error \\
\end{tabular}
\end{center}

- CIA: fingerprint for entrance
- \textbf{false accept: very serious consequences!}
- \textbf{false reject: unhappy employee, but so what? :-)}
Take-home Message for Now

\( \text{err} \) is application/user-dependent

Algorithmic Error Measures \( \hat{\text{err}} \)

- true: just \( \text{err} \)
- plausible:
  - 0/1: minimum ‘flipping noise’—NP-hard to optimize, remember? :-)
  - squared: minimum Gaussian noise
- friendly: easy to optimize for \( \mathcal{A} \)
  - closed-form solution
  - convex objective function

\( \hat{\text{err}} \): more in next lectures
Learning Flow with Algorithmic Error Measure

unknown target distribution $P(y|x)$ containing $f(x) + \text{noise}$

$y_1, y_2, \ldots, y_N$

training examples $\mathcal{D}: (x_1, y_1), \ldots, (x_N, y_N)$

unknown $P$ on $\mathcal{X}$

$x_1, x_2, \ldots, x_N$

learning algorithm $\mathcal{A}$

final hypothesis $g \approx f$

error measure $\text{err}$

hypothesis set $\mathcal{H}$

$\hat{\text{err}}$: application goal;

$\hat{\text{err}}$: a key part of many $\mathcal{A}$
Consider \( \text{err} \) below for CIA. What is \( E_{\text{in}}(g) \) when using this \( \text{err} \)?

1. \[
\frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{y_n \neq g(x_n)}
\]

2. \[
\frac{1}{N} \left( \sum_{y_n = +1} \mathbb{1}_{y_n \neq g(x_n)} + 1000 \sum_{y_n = -1} \mathbb{1}_{y_n \neq g(x_n)} \right)
\]

3. \[
\frac{1}{N} \left( \sum_{y_n = +1} \mathbb{1}_{y_n \neq g(x_n)} - 1000 \sum_{y_n = -1} \mathbb{1}_{y_n \neq g(x_n)} \right)
\]

4. \[
\frac{1}{N} \left( 1000 \sum_{y_n = +1} \mathbb{1}_{y_n \neq g(x_n)} + \sum_{y_n = -1} \mathbb{1}_{y_n \neq g(x_n)} \right)
\]

Reference Answer: 2

When \( y_n = -1 \), the false positive made on such \( (x_n, y_n) \) is penalized 1000 times more!
Noise and Error

Weighted Classification

CIA Cost (Error, Loss, ...) Matrix

<table>
<thead>
<tr>
<th></th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y=+1$</td>
<td>0</td>
</tr>
<tr>
<td>$y=-1$</td>
<td>1000</td>
</tr>
</tbody>
</table>

out-of-sample

$$E_{\text{out}}(h) = \mathcal{E}_{(x,y) \sim P} \left\{ \begin{array}{ll} 1 & \text{if } y = +1 \\ 1000 & \text{if } y = -1 \end{array} \right\} \cdot [y \neq h(x)]$$

in-sample

$$E_{\text{in}}(h) = \frac{1}{N} \sum_{n=1}^{N} \left\{ \begin{array}{ll} 1 & \text{if } y_n = +1 \\ 1000 & \text{if } y_n = -1 \end{array} \right\} \cdot [y_n \neq h(x_n)]$$

weighted classification:

different ‘weight’ for different $(x, y)$
Minimizing $E_{in}$ for Weighted Classification

$$E^w_{in}(h) = \frac{1}{N} \sum_{n=1}^{N} \left\{ \begin{array}{l} 1 \quad \text{if } y_n = +1 \\ 1000 \quad \text{if } y_n = -1 \end{array} \right\} \cdot \| y_n \neq h(x_n) \|$$

Naïve Thoughts

- PLA: *doesn’t matter if linear separable. :-)*
- pocket: modify pocket-replacement rule
  —if $w_{t+1}$ reaches smaller $E^w_{in}$ than $\hat{w}$, replace $\hat{w}$ by $w_{t+1}$

pocket: some guarantee on $E^{0/1}_{in}$?
modified pocket: similar guarantee on $E^w_{in}$?
Noise and Error

Weighted Classification

Systematic Route: Connect $E_{in}^w$ and $E_{in}^{0/1}$

original problem

<table>
<thead>
<tr>
<th>$h(x)$</th>
<th>+1</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>+1</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>1000</td>
<td>0</td>
</tr>
</tbody>
</table>

$D$: 

$(x_1, +1)$

$(x_2, -1)$

$(x_3, -1)$

$\ldots$

$(x_{N-1}, +1)$

$(x_N, +1)$

equivalent problem

<table>
<thead>
<tr>
<th>$h(x)$</th>
<th>+1</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>+1</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$D$: 

$(x_1, +1)$

$(x_2, -1)$

$(x_2, -1)$, $(x_2, -1)$, $\ldots$, $(x_2, -1)$

$(x_3, -1)$

$(x_3, -1)$, $(x_3, -1)$, $\ldots$, $(x_3, -1)$

$\ldots$

$(x_{N-1}, +1)$

$(x_N, +1)$

after copying $-1$ examples 1000 times,

$E_{in}^w$ for LHS $\equiv E_{in}^{0/1}$ for RHS!
Weighted Pocket Algorithm

using ‘virtual copying’, weighted pocket algorithm include:

- weighted PLA:
  randomly check \(-1\) example mistakes with 1000 times more probability

- weighted pocket replacement:
  if \(\mathbf{w}_{t+1}\) reaches smaller \(E_{\text{in}}^w\) than \(\hat{\mathbf{w}}\), replace \(\hat{\mathbf{w}}\) by \(\mathbf{w}_{t+1}\)

systematic route (called ‘reduction’):
can be applied to many other algorithms!
Consider the CIA cost matrix. If there are 10 examples with $y_n = -1$ (intruder) and 999,990 examples with $y_n = +1$ (you). What would $E_{\text{in}}^w(h)$ be for a constant $h(x)$ that always returns $+1$?

<table>
<thead>
<tr>
<th>$y$</th>
<th>$+1$</th>
<th>$-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+1$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$-1$</td>
<td>1000</td>
<td>0</td>
</tr>
</tbody>
</table>

- 1. 0.001
- 2. 0.01
- 3. 0.1
- 4. 1

**Reference Answer:** 2

While the quiz is a simple evaluation, it is not uncommon that the data is very unbalanced for such an application. Properly ‘setting’ the weights can be used to avoid the lazy constant prediction.
Summary

1. When Can Machines Learn?
2. **Why** Can Machines Learn?

Lecture 7: The VC Dimension

Lecture 8: Noise and Error

- Noise and Probabilistic Target
  
  can replace $f(x)$ by $P(y|x)$

- Error Measure
  
  affect ‘ideal’ target

- Algorithmic Error Measure
  
  user-dependent $\implies$ plausible or friendly

- Weighted Classification
  
  easily done by virtual ‘example copying’

- next: more algorithms, please? :-)

3. How Can Machines Learn?
4. How Can Machines Learn Better?