Homework #2 Selected Solution

2.1 Asymptotic Complexity

In this problem, you can use any theorems in the textbook and any theorems on the class slides as the foundation of your proof. You cannot use any other theorems unless you prove them first.

(1) (10%) Do Exercise 1(b) on page 41 of the textbook (Subsec. 1.5.3).

**Proof.** Let \( n_0 = 0 \) and \( c = 1 \). For \( n \geq n_0 \), we see that
\[
\begin{align*}
n! &= n \cdot (n-1) \cdot (n-2) \cdots 1 \\
&\leq n \cdot n \cdot n \cdots n \\
&= c \cdot n^n
\end{align*}
\]
Thus, by the definition in the textbook, \( n! = O(n^n) \).

(2) (10%) Do Exercise 1(f) on page 41 of the textbook (Subsec. 1.5.3).

**Proof.** We will take \( c_1 = 1 \), \( c_2 = 2 \), \( n_0 = 3 \) and try to use the definition to prove the results.

For \( n \geq n_0 = 3 \), \( n \) must be \( \geq 1 \). Then, \( 2^n \geq 2 \geq 0 \). Thus,
\[
n^{2^n} + 6 \cdot 2^n \geq n^{2^n} = c_1 \cdot n^n.
\]
Now, for \( n \geq n_0 = 3 \), we shall prove that the property \( 2^n \geq 2n \) first. The proof can be done with a mathematical induction. When \( n = 3 \) we see that the property is true by \( 2^3 = 8 \geq 6 = 2n \).

Assume that the property is true for \( n = k \). Then, for \( n = k+1 \geq 3 \), we first know that \( k \geq 2 \geq 1 \). That is, \( 2^k \geq 2 \). Thus,
\[
2^n = 2^{k+1} = 2^k + 2^k \geq 2k + 2 = 2(k + 1).
\]
Note that we used the assumed \( 2^k \geq 2k \) in the derivation. By mathematical induction, \( 2^n \geq 2n \) is true for all \( n \geq n_0 = 3 \).

Then, since \( \log_2 n \geq \log_2 3 \geq 1 \) for \( n \geq n_0 = 3 \), and \( n \geq n_0 = 3 \geq \log_2 6 \),
\[
2^{n \log_2 2} \geq 2^n \geq 2n \geq n + \log_2 6.
\]
Thus,
\[
n^{2^n} \geq 6 \cdot 2^n.
\]
That is,
\[
n^{2^n} + 6 \cdot 2^n \leq 2 \cdot n^{2^n} = c_2 n^{2^n}.
\]

(3) (10%) Do Exercise 2(c) on page 41 of the textbook (Subsec. 1.5.3).

Assume that the statement is true, there exists \( c_2 \) and \( n_0 \) such that
\[
c_2 n^2 \leq n^2 / \log n
\]
for all \( n \geq n_0 \). That is,
\[
\log n \leq \frac{1}{c_2}
\]
Take \( n = \max(n_0, 10^{\frac{1}{c_2} + 1}) \). We see that \( n \geq n_0 \) but
\[
\log n \geq \frac{1}{c_2} + 1 > \frac{1}{c_2}
\]
which is a contradiction. Thus, the statement is not true.
(4) (10%) Do Exercise 2(e) on page 41 of the textbook (Subsec. 1.5.3).

Assume that the statement is true, there exists $c$ and $n_0$ such that

$$3^n \leq c \cdot 2^n$$

for all $n \geq n_0$. That is,

$$n (\log_2 3 - 1) \leq \log_2 c$$

Take $n = \max(n_0, \frac{\log_2 c}{\log_2 3 - 1} + 1)$. We see that $n \geq n_0$ but

$$n (\log_2 3 - 1) \geq \log_2 c + \log_2 3 - 1 > \log_2 c,$$

which is a contradiction. Thus, the statement is not true.