Data Structures and Algorithms (I)

close-book midterm exam

December 1, 2006

You may answer the questions in any order. Dishonest behaviors and attempts will be most seriously punished. When you are asked to justify, prove, or disprove your answers, you may directly use anything that we have shown in class in a “black-box” manner.

Problem 1 (20 points)
Prove or disprove the following statements

• (10 points) \(2^n = \Omega(3^n)\).

• (10 points) \((\log_{2}n)^{\log_{10}n} = \Theta\left(10^{(\log_{2}n)^{2}}\right)\).

Problem 2 (15 points)
State the master method.

Problem 3 (20 points)
Prove or disprove the following statements for an \(n\)-node red-black tree \(T\):

• (10 points) It takes \(\omega(1)\) rotations (in the worst case) to insert a new node into \(T\) in order to maintain the red-black properties of the resulting tree.

• (10 points) It takes \(\omega(\log \log n)\) rotations (in the worst case) to delete a node from \(T\) in order to maintain the red-black properties of the resulting tree.

Problem 4 (15 points)
Solve the following recurrence relations asymptotically. Justify your answers.

• (5 points) \(T(n) = 4 \cdot T(n/2) + n\).

• (5 points) \(T(n) = 4 \cdot T(n/2) + n^2\).

• (5 points) \(T(n) = 4 \cdot T(n/2) + n^3\).

Problem 5 (15 points)
Suppose that the input consists of \(n\) numbers. You are asked to give an algorithm to determine the minimum as well as the maximum of these \(n\) numbers using \(3n^2 + O(1)\) comparisons. Justify your answer.

Problem 6 (15 points)
An array \(A[1 \ldots n]\) is strongly heapified if

• \(A[i] \leq A[\lfloor i/2 \rfloor]\) holds for each \(i = 2, 3, \ldots, n\) and

• \(A[2j] \geq A[2j + 1]\) holds for each \(j = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor\).

What is the worst-case asymptotic time complexity of problem of reordering any arbitrary array \(A\) into a strongly heapified array. Namely, is it \(\Theta(n)\), \(\Theta(n \log n)\), \(\Theta(n^2)\) or something else? Justify your answer.