Lesson 7: CFG = PDA

Theme: The equivalence between context-free grammars (CFG) and push-down automata (PDA).

We are going to show that CFG and PDA define precisely the same class of languages. More precisely, we are going to show the following.

- For every CFG $G$, there is a PDA $A$ such that $L(A) = L(G)$.
- Vice versa, for every PDA $A$, there is a CFG $G$ such that $L(A) = L(G)$.

1 From CFG to PDA

Allowing the PDA to push a string of symbols. First, we note that we can modify the definition of CFG to one that allows it to push a string of symbols to its stack. That is, the transitions can be of the form:

$$(p, x, \text{pop}(y)) \rightarrow (q, \text{push}(z)),$$

where $z \in \Gamma^*$

Allowing such transition does not change the capability of a CFG. We can simply add “new” states $t_1, \ldots, t_m$, where $|z| = m$ and $a_1 \ldots a_m$, and each $t_i$ is used to push the symbol $a_i$ into the stack. More formally, the transition $(p, x, \text{pop}(y)) \rightarrow (q, \text{push})$ can be replaced with the following transitions:

$$(p, x, \text{pop}(y)) \rightarrow (t_1, \text{push}(a_1))$$
$$(t_1, \epsilon, \text{pop}(\epsilon)) \rightarrow (t_2, \text{push}(a_2))$$
$$(t_2, \epsilon, \text{pop}(\epsilon)) \rightarrow (t_3, \text{push}(a_3))$$

... ...

$$(t_{i-1}, \epsilon, \text{pop}(\epsilon)) \rightarrow (t_i, \text{push}(a_i))$$

... ...

$$(t_{m-1}, \epsilon, \text{pop}(\epsilon)) \rightarrow (t_m, \text{push}(a_m))$$
$$(t_m, \epsilon, \text{pop}(\epsilon)) \rightarrow (q, \text{push}(\epsilon))$$

Constructing a PDA from a CFG. Let $G = \langle \Sigma, V, R, S \rangle$ be a CFG. Consider the following PDA $A = \langle \Sigma, \Gamma, Q, q_0, F, \delta \rangle$ where each component is as follows.

- $\Gamma = \Sigma \cup V \cup \{\bot\}$.
- $Q = \{p, q, r\}$
- $p$ is the initial state.
- $F = \{r\}$.
- $\delta$ consists of the following transition:

$$(p, \epsilon, \text{pop}(\epsilon)) \rightarrow (q, \text{push}(\bot S))$$
$$(q, a, \text{pop}(a)) \rightarrow (q, \text{push}(\epsilon))$$
$$(q, \epsilon, \text{pop}(A)) \rightarrow (q, \text{push}(w)) \quad \text{for each rule } A \rightarrow w \in R$$
$$(q, \epsilon, \text{pop}(\bot)) \rightarrow (r, \text{push}(\epsilon))$$
Notice that in the first and third transitions, \( \mathcal{A} \) pushes a string of symbols to its stack.

We can show that \( L(\mathcal{A}) = L(\mathcal{G}) \).

## 2 From PDA to CFG

Let \( \mathcal{A} = (\Sigma, \Gamma, Q, q_0, F, \delta) \) be a PDA. Without loss of generality, we can assume the following.

- It has only one final state, say \( q_f \). That is, \( F = \{ q_f \} \).
- The stack is empty before accepting an input word. More precisely, on every word \( w \), if \( \mathcal{A} \) accepts \( w \), there is an accepting run of \( \mathcal{A} \) on \( w \) from the initial configuration \((q_0, \epsilon)\) to a final configuration \((q_f, \epsilon)\) where the content of the stack is empty.
- In each transition, \( \mathcal{A} \) either pushes a symbol into the stack or pops one from the stack, but it cannot do both. More precisely, every transition can only be of the forms:
  \[
  (p, x, \text{pop}(y)) \rightarrow (q, \text{push}(\epsilon)) \\
  (p, x, \text{push}(z)) \rightarrow (q, \text{push}(\epsilon))
  \]

Consider the following CFG \( \mathcal{G} = (\Sigma, V, R, S) \) where each component is as follows.

- \( V = \{ A_{p,q} \mid p, q \in Q \} \).
- \( A_{q_0,q_f} \) is the start variable.
- \( R \) consists of the following rules:
  - For every state \( p, q, r, s \in Q \) and every symbol \( z \in \Gamma \cup \{ \epsilon \} \) and every symbol \( a, b \in \Sigma \cup \{ \epsilon \} \), if the following transitions are in \( \delta \):
    \[
    (p, a, \text{pop}(\epsilon)) \rightarrow (r, \text{push}(z)) \\
    (s, b, \text{pop}(z)) \rightarrow (q, \text{push}(\epsilon))
    \]
    then the following rule is in \( R \):
    \[
    A_{p,q} \rightarrow a A_{r,s} b
    \]
  - For every state \( p, q, r \in Q \) and every symbol \( a \in \Sigma \), if the following transition is in \( \delta \):
    \[
    (p, a, \text{pop}(\epsilon)) \rightarrow (r, \text{push}(\epsilon))
    \]
    then the following rule is in \( R \):
    \[
    A_{p,q} \rightarrow a A_{r,q}
    \]
  - For every \( p, q, r \in Q \), we have the following rule in \( R \):
    \[
    A_{p,q} \rightarrow A_{p,r} A_{r,q}
    \]
  - For every \( p \in Q \), we have the following rule in \( R \):
    \[
    A_{p,p} \rightarrow \epsilon
    \]

We can show that \( L(\mathcal{A}) = L(\mathcal{G}) \).