Performance of Low-Density Parity-Check (LDPC) Coded OFDM Systems

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Abstract— **Orthogonal Frequency Division Multiplexing (OFDM) is a very attractive technique for high-bit-rate data transmission in multipath environments. Many error-correcting codes have been applied to OFDM, convolutional codes, Reed-Solomon codes, Turbo codes, and so on. Recently, low-density parity-check (LDPC) codes have attracted much attention particularly in the field of coding theory. LDPC codes were proposed by Gallager in 1962 and the performance is very close to the Shannon limit with practical decoding complexity like Turbo codes. We proposed the LDPC coded OFDM (LDPC-COFDM) systems with BPSK and showed that the LDPC codes are effective to improve the bit error rate (BER) of OFDM in multipath environments [1]. LDPC codes can be decoded by using a probability propagation algorithm known as the sum-product algorithm or belief propagation. When the LDPC codes are used for the OFDM systems, the properties of the iterative decoding, such as the distribution of the number of iterations where the decoding algorithm stops, have not been clarified. In mobile communications, a high bandwidth efficiency is required, and thus the multilevel modulation is preferred. However, it has not been clarified how we can apply LDPC codes to the OFDM systems with multilevel modulation. In this paper, first we investigate the distribution of the number of iterations where the decoding algorithm stops in the LDPC-COFDM systems. Moreover, we propose the decoding algorithm for the LDPC-COFDM systems with M-PSK. From the simulation, we show that the LDPC-COFDM systems achieve good error rate performance with a small number of iterations on both an AWGN and a frequency-selective fading channels. We confirm that the algorithm for the LDPC-COFDM systems with M-PSK work correctly.**

I. Introduction

In the future mobile communication systems the high-bit-rate transmission is required for high quality communications. Orthogonal Frequency Division Multiplexing (OFDM), which divides the wide signal bandwidth into many narrowband subchannels that are transmitted in parallel, is a very attractive technique for the high-bit-rate data transmission in a multipath environment that causes intersymbol interference (ISI). The ISI in OFDM can be eliminated by adding a guard interval. In a multipath environment, some subcarriers of OFDM may be completely lost because of the deep fades. Hence, even though most subcarriers may be detected without errors, the overall bit error rate (BER) will be largely dominated by a few subcarriers with small amplitudes. To avoid this domination by the weakest subcarriers, forward-error correction coding is essential. Many error-correcting codes have been applied to OFDM, convolutional codes, Reed-Solomon codes, Turbo codes [2], and so on.

Recently, low-density parity-check (LDPC) codes have attracted much attention particularly in the field of coding theory. LDPC codes were proposed by Gallager in 1962 [3][4] and the performance is very close to the Shannon limit with practical decoding complexity like Turbo codes. LDPC codes have been applied to BPSK and 8PSK, and their fundamental performance has been evaluated on an additive white Gaussian noise (AWGN) channel [5]. The performance of LDPC codes has been also evaluated on a block fading channel, and it has been shown that the LDPC codes achieve a large gain with respect to convolutional codes for large packet length [6]. We proposed the LDPC coded OFDM (LDPC-COFDM) systems with BPSK to improve the BER of OFDM in multipath environments [1]. We showed that LDPC codes are effective to improve the error performance of OFDM in multipath environments.

LDPC codes can be decoded by using a probability propagation algorithm known as the sum-product algorithm or belief propagation [4][7]. When the LDPC codes are used for OFDM systems, the properties of the iterative decoding, such as the distribution of the number of iterations where the decoding algorithm stops, have not been clarified. In mobile communications, a high bandwidth efficiency is required, and thus the multilevel modulation is preferred. However, it has not been clarified how we can apply LDPC codes to the OFDM systems with multilevel modulation.

In this paper, first we investigate the distribution of the number of iterations where the decoding algorithm stops in the LDPC-COFDM systems. Moreover, we propose the algorithm for the LDPC-COFDM systems with M-PSK. From the simulation, we show that the LDPC-COFDM systems achieve good error rate performance with a small number of iterations on both an AWGN and a frequency-selective channels. We confirm that the algorithm for the LDPC-COFDM systems with M-PSK work correctly.

II. LDPC Code

LDPC codes and their iterative decoding algorithm were proposed by Gallager in 1962 [3][4]. LDPC codes have been almost forgotten for about thirty years, in spite of their excellent properties. However, LDPC codes are now recognized as good error-correcting codes achieving near Shannon limit performance [7].

LDPC codes are defined as codes using a sparse parity-check matrix with the number of 1's per column (column weight) and the number of 1's per row (row weight), both of which are very small compared to the block length. LDPC codes are classified into two groups, regular LDPC codes and irregular LDPC codes. Regular LDPC codes have a uniform column weight and row weight, and irregular LDPC codes have a nonuniform col-

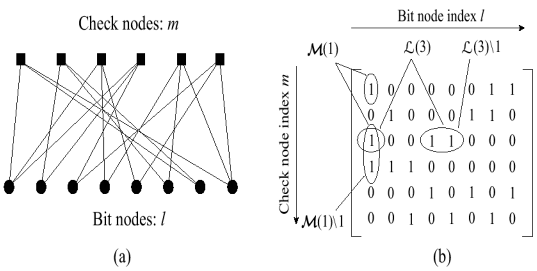

Fig. 1. (a) Factor graph and (b) notation of the sum-product algorithm

umn weight. We describe an LDPC code defined by $M \times N$ parity-check matrix $\mathbf{H}$ as (N,K) LDPC, where $K = N - M$ and the code rate is $R = K/N$. In the case that the $\mathbf{H}$ doesn't have full rank, $K > N - M$ and the error performance of an LDPC code becomes worse. Thus, when we construct the parity-check matrix $\mathbf{H}$, we ensure that all the rows of the matrix are linearly independent. LDPC codes can be decoded by using a probability propagation algorithm known as the sum-product or belief propagation algorithm [4][7]. LDPC codes have better block error performance than turbo codes, because the minimum distance of an LDPC code increases proportional to the code length with a high probability. Such a property is desirable for the high-bit-rate transmission that requires very low frame error probability.

LDPC codes can be represented by a Factor Graph that contains two types of nodes: the "bit nodes" and the "check nodes" [9]. Fig. 1 (a) shows an example of the Factor Graph. Each bit node corresponds to a column of a parity-check matrix, which also corresponds to a bit in the codeword. Each check node corresponds to a row of a parity-check matrix, which represents a parity-check equation. An edge between a bit node and a check node exists if and only if the bit participates in the parity-check equation represented by the check node.

III. Sum-product algorithm

First, we describe the notations of the sum-product algorithm in Fig. 1 (b). $\mathcal{M}(l)$ denotes the set of check nodes that are connected to the bit node l, i.e., positions of "1"s in the lth column of the parity-check matrix. $\mathcal{L}(m)$ denotes the set of bits that participates in the mth parity-check equation, i.e., the positions of "1"s in the mth row of the parity-check matrix. $\mathcal{L}(m)\backslash l$ represents the set $\mathcal{L}(m)$ with the lth bit excluded and $\mathcal{M}(l)\backslash m$ represents the set $\mathcal{M}(l)$ with the mth check excluded. $q^i_{l\to m}$, where $i = 0, 1$, denotes the probability information that the bit node l sends to the check node m, indicating $P(x_l = i)$. $r^i_{m\to l}$ denotes the probability information that the mth check node gathers for the lth bit being i. In other words, $r^i_{m\to l}$ is the likelihood information for $x_l = i$ from the mth parity-check equation, when the probabilities for other bits are designated by the $q^i_{l\to m}$. Therefore, $r^i_{m\to l}$ can be considered as the "extrinsic" information for the lth bit node from the mth check node. The $a\,posteriori$ probability for a bit is calculated by gathering all the extrinsic information from the check nodes that connect to it, which can be obtained by the following iterative belief propagation procedure.

For binary codes, the sum-product algorithm can be performed more efficiently in Log domain, where the probabilities are equivalently characterized by the log-likelihood ratios (LLRs): $L(r_{m\to l}) \triangleq \log \frac{r^1_{m\to l}}{r^0_{m\to l}}$, $L(q_{m\to l}) \triangleq \log \frac{q^1_{m\to l}}{q^0_{m\to l}}$, $L(p_l) \triangleq \log \frac{p^1_l}{p^0_l}$, $L(q_l) \triangleq \log \frac{q^1_l}{q^0_l}$. Note that p^i_l represents the likelihood that the lth bit is i.

Initialization

Each bit node l is assigned an $a\ priori$ LLR $L(p_l)$. In the case of equiprobable inputs on a memoryless AWGN channel with BPSK,

$$\begin{aligned} L(p_l) &= \log \frac{P(y_l \mid x_l = +1)}{P(y_l \mid x_l = -1)} \\ &= \frac{2}{\sigma^2} y_l \end{aligned}$$

where x, y represent the transmitted bit and received bit, respectively, and σ^2 is the noise variance. For every position (m, l) such that $H_{ml} = 1$, where H_{ml} represents the element of the mth row and the lth column in the parity-check matrix $\mathbf{H}$, $L(q_{l\to m})$ and $L(r_{m\to l})$ are initialized as:

$$\begin{aligned} L(q_{l\to m}) &= L(p_l) \\ L(r_{m\to l}) &= 0 \end{aligned}$$

L1. Checks to bits

Each check node m gathers all the incoming information $L(q_{l\to m})$'s, and updates the belief on the bit l based on the information from all other bits connected to the check node m.

$$L(r_{m\to l}) = 2\tanh^{-1}\left(\prod_{l'\in\mathcal{L}(m)\backslash l} \tanh(L(q_{l'\to m})/2)\right)$$

L2. Bits to checks

Each bit node l propagates its probability to all the check nodes that connect to it.

$$L(q_{l\to m}) = L(p_l) + \sum_{m'\in\mathcal{M}(l)\backslash m} L(r_{m'\to l})$$

L3. Check stop criterion

The decoder obtains the total $a\,posteriori$ probability for the bit l by summing the information from all the check nodes that connect to the bit l.

$$L(q_l) = L(p_l) + \sum_{m\in\mathcal{M}(l)} L(r_{m\to l})$$

Hard decision is made on the $L(q_l)$, and the resulting decoded input vector $\hat{\mathbf{x}}$ is checked against the parity-check matrix $\mathbf{H}$. If $\mathbf{H}\hat{\mathbf{x}} = \mathbf{0}$, the decoder stops and outputs $\hat{\mathbf{x}}$. Otherwise, it repeats the steps L1–L3. The sum-product algorithm sets the maximum number of iterations (max-iteration). If the number of iterations becomes the maximum number of iterations, the decoder stops and outputs $\hat{\mathbf{x}}$.

IV. LDPC Coded OFDM

A. Construction of LDPC Code

Fig. 2 shows the way to construct an LDPC code in this paper, which is depicted in [3]. A parity-check matrix is divided into three submatrices, each containing a single 1 in each column. The first of these submatrices contains 1's in descending order; i.e., the ith row contains 1's in the columns $(i-1)k+1$ to ik, where k is the row weight. The other submatrices are merely column permutations of the first submatrix. The permutations of the 2nd submatrix and the 3rd submatrix are independently selected.

B. System Model

In a multipath environment, some subcarriers of OFDM may be completely lost because of the deep fades. Hence, in this case, it is expected that lots of errors fix on continuous some subcarriers and the two dimensional errors in both time and frequency domains occur. That is why we apply LDPC codes, which can compensate for the two dimensional errors, to OFDM system.

Fig. 3 shows the model of the LDPC-COFDM system. At the transmitter, information bits are encoded at the LDPC encoder and modulated at the modulator. After the serial-to-parallel conversion, the OFDM sub-channel modulation is implemented by using an inverse fast Fourier transform (IFFT) and assigned to some OFDM symbols for the purpose of compensating two dimensional errors in the OFDM system. On a frequency-selective fading channel the guard interval is inserted for the purpose of eliminating the ISI. At the receiver, the guard interval is removed on a frequency-selective fading channel. After the serial-to-parallel conversion, the OFDM sub-channel demodulation is implemented by using a fast Fourier transform (FFT). The received OFDM symbols generated by the FFT are demodulated at the demodulator. The demodulated bits are decoded with each LDPC encoded block and data bits are restored.

C. Proposed Algorithm for M-PSK

The decoding algorithm of the proposed system is based on the sum-product algorithm. We initialize the first likelihood of the received signal as follows. We define the first likelihood corresponding to the tth bit of the sth received symbol as:

$$L(p_{s,t}^1) = \frac{P(\boldsymbol{y}_s \,|\, x_l = 1)}{P(\boldsymbol{y}_s \,|\, x_l = 0)} = \frac{\displaystyle\sum_{j \in \{\boldsymbol{J}_{t,1}\}} \exp\left[-\frac{(y_{s,I} - x_{I,j})^2 + (y_{s,Q} - x_{Q,j})^2}{2\sigma^2}\right]}{\displaystyle\sum_{j \in \{\boldsymbol{J}_{t,0}\}} \exp\left[-\frac{(y_{s,I} - x_{I,j})^2 + (y_{s,Q} - x_{Q,j})^2}{2\sigma^2}\right]}$$

where $x_l, \boldsymbol{y}_s$ represent the lth transmitted bit and the sth received symbol, respectively, and $\boldsymbol{J}_{t,i}$ represents the set of M-PSK symbols with the tth bit being i. Note that we use the

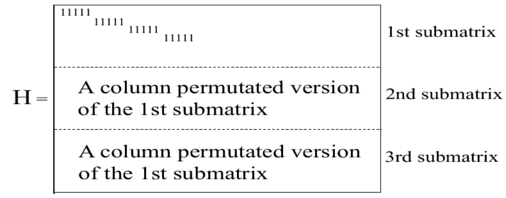

Fig. 2. Construction of an LDPC code

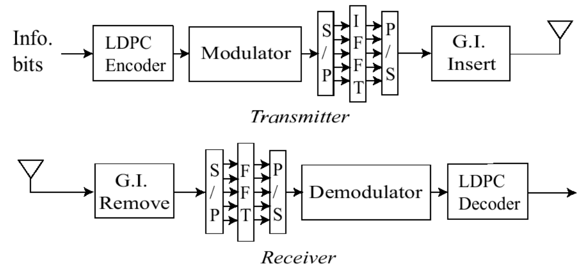

Fig. 3. LDPC-COFDM system model

M-PSK with Gray mapping. For the QPSK, we initialize the first likelihood of the received signal as:

$$L(p_l^1) = \begin{cases} L(p_{s,1}^1), & (l = 2s-1) \\ L(p_{s,2}^1), & (l = 2s) \end{cases}$$

For the 8PSK, we initialize the first likelihood of the received signal as:

$$L(p_l^1) = \begin{cases} L(p_{s,1}^1), & (l = 3s-2) \\ L(p_{s,2}^1), & (l = 3s-1) \\ L(p_{s,3}^1), & (l = 3s) \end{cases}$$

After definig the initialization like this, the decoding is done in the same procedure as for BPSK, L1 – L3.

V. Simulation Results

We present the results of our computer simulation. TABLE 1 shows the simulation parameters. The multipath condition used in our simulation is an equal power 2 path Rayleigh fading. The delay interval is 0.125 μsec (5 samples). Note that we use a (1080,525) LDPC code with column weight 3 and set the maximum number of iterations to 100.

Fig. 4 shows the probability distribution of the number of iterations at which the decoder stops for the (1080,525) LDPC coded OFDM system on an AWGN channel. We can see that as the E_b/N_0 becomes larger, the number of iterations where the probability takes its maximum other than 100-iterations becomes smaller and the peak value of the probability becomes higher. Note that when the E_b/N_0 is 2.1 or 3.1 dB, the probability doesn't take its maximum at 100-iterations, that is, the probability takes its maximum at 100-iterations only when E_b/N_0 is very small and LDPC codes cannot correct errors. We can also see that 20-iterations are enough for the LDPC-COFDM system on an AWGN channel. Thus, we can say that when the E_b/N_0 is not so small, the LDPC-COFDM systems achieve the good error performance with a small number of iterations on an

TABLE 1

SIMULATION PARAMETERS

Modulation	BPSK, QPSK, 8PSK
Amplifier	Linear
Number of subcarriers	64
Number of FFT points	512
Bandwidth	40 MHz
Guard interval	0.25 μsec
Channel models	AWGN Frequency-selective fading
Maximum doppler frequency	80 Hz
Multipath condition	Equal power 2 path Rayleigh fading
Delay interval	0.125 μsec (5 samples)

AWGN channel. We can also say that when the E_b/N_0 is very small, even if the maximum number of iterations increases, the error rate performance of the LDPC-COFDM systems hardly improve. Fig. 5 shows the probability distribution of the number of iterations at which the decoder stops for the (1080,525) LDPC coded OFDM system on a frequency-selective fading channel. Compared with the performance on an AWGN channel, we can see the similar trend: as the E_b/N_0 becomes larger, the number of iterations where the probability takes its maximum other than 100-iterations becomes smaller and the peak value of the probability becomes higher. We can also see that the number of iterations where the probability takes its maximum other than 100-iterations becomes a little bit larger than that on an AWGN channel. Thus, we can say that when the E_b/N_0 is large, the LDPC-COFDM systems achieve the good error rate performance with a small number of iterations on a frequency-selective fading channel.

Fig. 6 shows the BER of the LDPC-COFDM on an AWGN channel for various numbers of iterations in the decoding algorithm. We can see that as the number of iterations increases, the BER of the LDPC-COFDM is improved. We can also see that the BER of the LDPC-COFDM converges at 100-iterations. This is because the errors that cannot be corrected with 100-iterations would not be corrected even if we increase the number of iterations more. Fig. 7 shows the BER of the LDPC-COFDM on a frequency-selective fading channel for various numbers of iterations in the decoding algorithm. Compared with the performance on an AWGN channel, we can see the similar trend: as the number of iterations increases, the BER of the LDPC-COFDM is improved.

Fig. 8 shows the average number of the iterations in the decoding algorithm of the LDPC-COFDM systems on both an AWGN and a frequency-selective fading channels. Note that we set the maximum number of iterations to 100. We can see that as the E_b/N_0 becomes larger, the average number of iterations becomes smaller: The average numbers of iterations on an AWGN channel are 62 at $E_b/N_0 = 1.1$ dB and 5 at $E_b/N_0 = 3.1$ dB. The average numbers of iterations on a frequency-selective channel are 66 at $E_b/N_0 = 3.1$ dB and 7 at $E_b/N_0 = 4.5$ dB. Thus, when the E_b/N_0 is not small, the average numbers of iterations is small on both an AWGN and a frequency-selective fading channels.

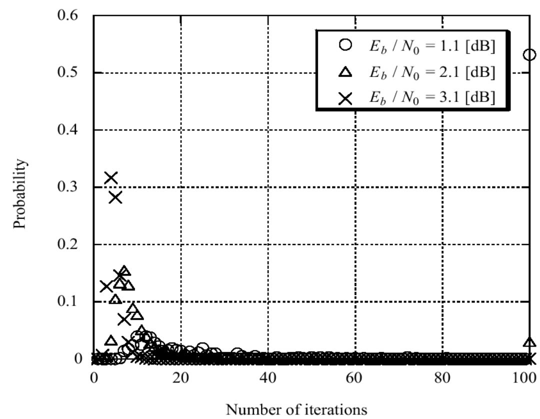

Fig. 4. Probability distribution of the number of iterations at which the decoder stops on an AWGN channel

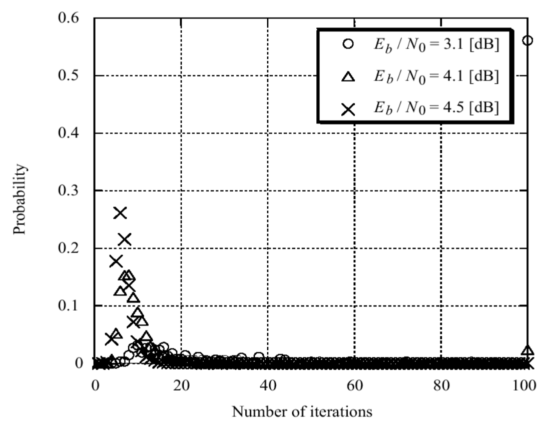

Fig. 5. Probability distribution of the number of iterations at which the decoder stops on a frequency-selective fading channel

Fig. 9 shows the BER of the LDPC-COFDM with BPSK, I-Q coded, QPSK, and 8PSK for the (1080,525) LDPC coded OFDM system on an AWGN channel. In the case of the I-Q coded, two LDPC encoders / decoders are employed. The LDPC encoded bits from each encoder are mapped to the I-channel and the Q-channel, respectively, and transmitted as the QPSK symbols. At the receiver the received bits of the demodulated QPSK symbols are decoded by the corresponding decoders, independently. Note that we set the maximum number of iterations to 100. We also show the BER of Turbo coded OFDM (TCOFDM) systems with BPSK on an AWGN channel. Note that we use a Turbo code of code rate $R = 1/2$ with the component interleaver of size $N_{int} = 512$. We use the Log-MAP algorithm as the decoding algorithm of the TCOFDM systems and set the number of iterations to 8. We can see that the BER of the LDPC-COFDM with I-Q coded is almost identical to that of the system with BPSK and the BER of the LDPC-COFDM with QPSK is better than that of the system with BPSK at the same value of E_b/N_0. We can also see that the BER of the LDPC-COFDM with 8PSK is about 1.3 dB worse than that of the system with BPSK. Note that the BER of the uncoded system with 8PSK is about 3 dB worse than that of the system with BPSK. Thus, we confirm that the algorithm

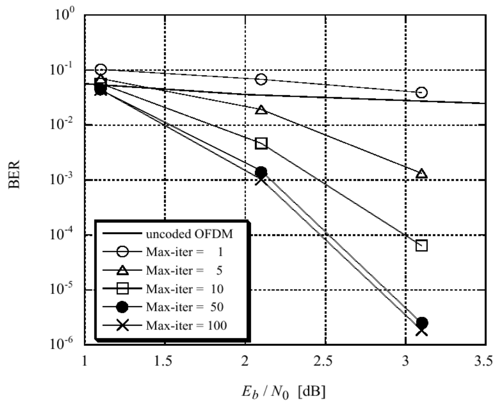

Fig. 6. BER on an AWGN channel for various number of iterations

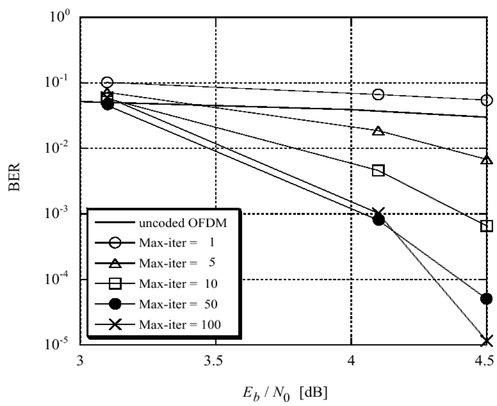

Fig. 7. BER on a frequency-selective fading channel for various number of iterations

for the LDPC-COFDM with M-PSK works correctly. Moreover, we can see that when $E_b/N_0 \leq 2.5$ dB, the BER of the LDPC-COFDM is worse than that of TCOFDM, while when $E_b/N_0 > 2.5$ dB, the BER of the LDPC-COFDM is better than that of the TCOFDM. Thus, when E_b/N_0 is not so small, the LDPC-COFDM achieves the better BER with a small number of iterations than the TCOFDM.

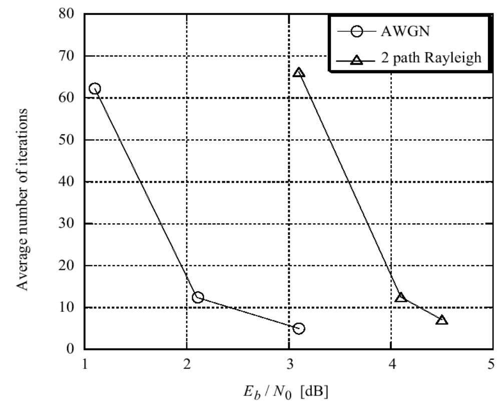

Fig. 8. Average number of iterations on both an AWGN and a frequency-selective fading channels

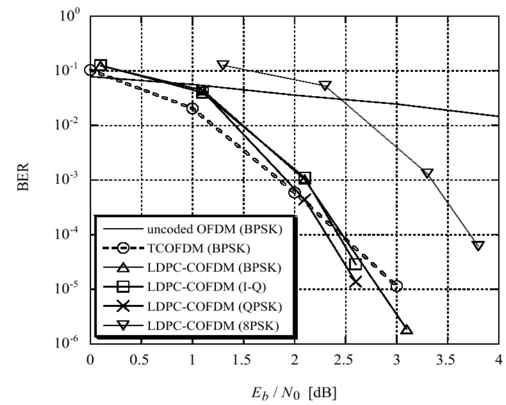

Fig. 9. BER of both the LDPC-COFDM with BPSK, I-Q coded, QPSK, 8PSK, and TCOFDM with BPSK on an AWGN channel

VI. Conclusions

In this paper, we evaluated the performance of the LDPC-COFDM systems. We showed that the LDPC-COFDM systems achieve the good error rate performance with a small number of iterations on both an AWGN and a frequency-selective fading channels. We also showed that when $E_b/N_0 > 2.5$ dB, the (1080,525) LDPC-COFDM with BPSK achieves the better BER than the TCOFDM with BPSK on an AWGN channel, while when $E_b/N_0 \leq 2.5$ dB, the LDPC-COFDM has the worse BER. We also showed that the BER of the LDPC-COFDM with QPSK is better than that of the system with BPSK at the same value of E_b/N_0. We confirmed that the decoding algorithm for the LDPC-COFDM systems with M-PSK works correctly.

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