Design of Low-Density Parity-Check Codes for Bandwidth Efficient Modulation

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Abstract— **We design low-density parity-check (LDPC) codes for bandwidth efficient modulation using a multi-level coding (MLC) technique. We develop a method to analyze the asymptotic performance of the LDPC codes using message-passing decoding at each level of the MLC scheme as the codeword length goes to infinity. Simulation of very large block size LDPC codes verifies the accuracy of the analytical results. We jointly optimize the code rates and code parameters of the LDPC codes at each level of the MLC scheme, and the asymptotic performance of the optimized irregular LDPC codes is very close to the channel capacity of the additive white Gaussian noise (AWGN) channel.**

I. INTRODUCTION

Multilevel coding (MLC) [1] is a coded modulation technique used for bandwidth-efficient transmission. The idea of MLC is to protect the individual bits at each level of the signal points by separate binary codes. It is shown in [1] that with a Gray mapping and independent decoding at each level, the information loss compared to the channel capacity is surprisingly small if optimal component codes are used. However, [1] only pointed out the advantage of this kind of parallel independent decoding (PID) strategy; it did not show a constructive way to design good codes to approach the capacity for the MLC/PID scheme. In [2], we used LDPC codes [3] at each level of the MLC/PID scheme with Gray-mapped 4-PAM modulation. The results there showed that the optimized LDPC codes at each level are able to achieve very low bit error probability at signal-to-noise ratios (SNR) very close to the capacity of the MLC/PID scheme. Here, we describe the code design method for a general case which applies to any higher-order constellations. Finally, we present the code optimization results for a Gray-mapped 8-PSK modulation system.

II. SYSTEM MODEL AND CAPACITY

The encoder structure of an MLC scheme is shown in Fig. 1. Each bit c_i , $i = 0, 1, ..., m-1$, is the encoder output of a different binary LDPC code C^i. The mapping device maps a binary vector $\mathbf{c}= (c_{m-1,...,}c_0)$ to a signal point $x \in A$, where A is the signal set and $|A| = 2^m$. We use a discrete equivalent AWGN channel model, and we denote by y the noisy channel output. The code rate R of the scheme is equal to the sum of the component code rates, i.e., $R = \sum_{i=0}^{m-1} R_i$.

With the PID strategy, the system is decomposed into an equivalent set of m parallel binary input channels, where the ith channel has input c_i , $i = 0, 1, ..., m-1$ and output y. Each equivalent binary input channel i is characterized by the following density function: $f(y|c_i = b) = E_{a\in A_b^i}\{f(y|x = a)\}$, where A_b^i denotes the subset of all the signal points of A whose labels have the value $b \in \{0, 1\}$ in position i.

Under the constraint of i.i.d. equiprobable inputs, the capacity of the channel with channel input[1] x and output y is given by [1]: $C = I(X;Y) = I(C_0, C_1, ..., C_{m-1}; Y)$. With the same input constraints, we can also calculate the "PID capacity" of the MLC/PID scheme as in [1] by: $\widehat{C} = \sum_{i=0}^{m-1} I(C_i;Y)$ where $I(C_i;Y)$ [2] represents the mutual information of the equivalent channel i with input c_i , $i = 0, 1, ..., m-1$, and output y. Since the inputs c_i , $i = 0, 1, .., m-1$, are independent of each other, it can be easily proved that $\sum_{i=0}^{m-1} I(C_i;Y) \leq I(C_0, C_1, ..., C_{m-1}; Y)$, implying that $\widehat{C} \leq C$. Surprisingly, if Gray mapping is used, it is pointed out in [1] that the gap between the channel capacity and the PID capacity is negligible if optimal component codes are used. We calculated both C and $\widehat{C}$ for the Gray mapped 8-PSK constellation (Fig. 2) on an AWGN channel. At a spectral efficiency of 2 bits/symbol, the minimum SNRs for reliable transmission corresponding to the channel capacity C and the PID capacity $\widehat{C}$ are 5.77 dB and 5.84 dB, respectively. Moreover, the equivalent channels with input c_1 and c_2 have equal PID capacities since the Gray labeling for c_1 and c_2 differs only by a rotation [1]. The PID capacity results also suggest the optimal individual code rates for each level: $C^0/C^1/C^2 = 0.510/0.745/0.745$.

III. LDPC CODES AND DENSITY EVOLUTION

In this section, we briefly review the terminology and techniques used in our code design.

A. LDPC Codes

An irregular LDPC code can be specified by a bit and check degree distribution pair (λ, ρ) with $\lambda(x) = \sum_{j=2}^{d_{v_{max}}} \lambda_j x^{j-1}$ and $\rho(x) = \sum_{j=2}^{d_{c_{max}}} \rho_j x^{j-1}$ [4]. A regular (d_v, d_c) LDPC code has $\lambda_{d_{v_{max}}} = 1$, $\rho_{d_{c_{max}}} = 1$, $d_v = d_{v_{max}}$ and $d_c = d_{c_{max}}$. We define an LDPC code ensemble $C^n(\lambda, \rho)$ as the set of all linear block codes of length n whose associated parity-check matrices can be specified by the bit and check degree distribution pair (λ, ρ).

For the MLC/PID scheme, we define an LDPC code ensemble set which consists of m LDPC code ensembles $\{C^n(\lambda^i, \rho^i), i = 0, 1, ..., m-1\}$ with each LDPC code ensemble corresponding to the LDPC code at one level of

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[1]We denote the random variables corresponding to the transmitted and received symbols by their capital letters.

the MLC scheme. If $\mathbf{v}^i = \left(v_1^i, v_2^i, ..., v_n^i\right)$, $i = 0, 1, ..., m-1$ is a codeword of an LDPC code at level i, we define an input sequence $\mathbf{s}$ of the MLC scheme as the time multiplex of the m codewords $\mathbf{v}^i$, $i = 0, 1, ..., m-1$, i.e., $\mathbf{s} = \left(v_1^0, v_1^1, ..., v_1^{m-1}, ..., v_n^0, v_n^1, ..., v_n^{m-1}\right)$.

B. Density Evolution

In [3], a numerical technique called density evolution is used to analyze the performance of message-passing decoders on a binary-input AWGN channel, enabling the determination of the noise thresholds [3] of LDPC code ensembles. The interpretation of the thresholds as predictors of actual decoder behavior and bounds on achievable performance relies upon a general "concentration theorem" - stating that, asymptotically in the blocklength, the decoder behavior for individual instances of a code, transmitted codeword, and channel noise realization concentrates around the average behavior - and a "convergence theorem" - stating that, again asymptotically in blocklength, the average behavior converges to that of a cycle-free graph, which can be computed using the density evolution algorithm. The application of density evolution to the optimization of the noise threshold and degree distribution over the ensemble of irregular LDPC codes is simplified by the symmetry of the channel (and decoder). Specifically, for any degree distribution, it is sufficient to carry out the density evolution and threshold determination for just the all-ones codeword.

Our objective is to develop a similar algorithmic approach to optimization of LDPC component codes for the MLC/PID scheme. Results pertaining to performance concentration and convergence to cycle-free graph behavior in this context will be presented elsewhere. Here, we simply describe the manner in which we apply density evolution to the MLC/PID system. Following [4], for a specified noise standard deviation σ^i at each level i of the MLC/PID scheme, $i = 0, 1, \ldots, m-1$, we use density evolution to track the fraction of incorrect messages $p_e^i(l)$ [3] after l decoding iterations on a cycle-free graph corresponding to a specified degree distribution pair (λ^i, ρ^i). We let f_0^i denote the initial density of the observed LAPPRs of the associated bit nodes and f_l^i denote the density of the messages passed from the bit nodes to the check nodes after l iterations. The density evolution can be described by $f_l^i = f_0^i \otimes \lambda^i\left(\Gamma^{-1}\left(\rho^i\left(\Gamma\left(f_{l-1}^i\right)\right)\right)\right)$, where $\lambda^i(f) = \sum \lambda_j^i f^{\otimes(j-1)}$, $\rho^i(f) = \sum \rho_j^i f^{\otimes(j-1)}$, $\otimes$ denotes convolution, and Γ and Γ^{-1} are operators defined in (5) and (6) of [4], respectively. For each level i, we define the corresponding noise threshold $(\sigma^*)^i$ to be the supremum of the σ^i for which $\lim_{l\to\infty} p_e^i(l) = 0$.

The application of density evolution to code design and threshold optimization for MLC/PID is complicated, at least in principle, by the fact that the equivalent binary-input component channels are not necessarily symmetric. Therefore, the decoding analysis of the all-ones codeword alone may not suffice to predict the average decoder behavior; in fact, for specific Gray-labeled constellations it is easy to see that this is the case. Instead, in our application of density evolution at the individual levels, we assume that the constellation symbols are equally likely to be transmitted, and the initial density functions f_0^i are determined accordingly.

We conjecture that the resulting thresholds represent the true average behavior of the MLC/PID system. Results of computer simulation described in the next section support this conjecture.

IV. MLC/PID CODE OPTIMIZATION

Fig. 3 shows the results of a bit-error-rate (BER) simulation for an MLC/PID scheme with Gray-mapped 8-PSK modulation on an AWGN channel. The overall code rate is 2 bits/symbol. Motivated by the PID capacity analysis, we used component code rates $R_0/R_1/R_2 = 0.51/0.745/0.745$. Specifically, for level 0, we designed a rate-0.51 quasi-regular (3,6/7) LDPC code and, for levels 1 and 2, we designed two rate-0.745 quasi-regular (3,11/12) LDPC codes. The component codewords $\mathbf{v}^i$, $i = 0, 1, 2$ all have length 10^6. The BER results are shown separately for the three component codes.

Also shown in Fig. 3 are performance thresholds, computed by density evolution and expressed in terms of equivalent (E_s/N_0) values, corresponding to the degree distributions used for the component codes on their equivalent binary channels. The two rate-0.745 (3,11/12) LDPC codes for levels 1 and 2 have the same threshold, 6.72dB. This follows from the fact that, under the Gray labeling, the assignments of c_1 and c_2 to the 8-PSK symbols differ only by a rotation, implying that $f_0^1 = f_0^2$. The threshold for the rate-0.51 (3,6/7) LDPC code for level 0 is 6.67dB. There is more than 0.8 dB between the thresholds of the quasi-regular LDPC codes and the PID capacity. (This gap will be substantially reduced when we consider irregular LDPC codes below.)

The simulated BER results show that the thresholds accurately predict the performance of the MLC/PID scheme with long LDPC component codes. In particular, the two rate-0.745 LDPC codes have virtually the same BER performance. These results support our conjecture and our interpretation of the computed thresholds.

We next consider the design of an MLC/PID scheme with irregular LPDC component codes. Since the optimal design of the MLC scheme requires that the component codes have (nearly) equal performance [1], we jointly optimize the component code rates and the degree distributions of the LDPC codes of the MLC/PID scheme so that all of the component code ensembles have the same thresholds. Specifically, for a specified overall rate R, we pick an initial combination of component rates R_i, $i = 0, 1, ..., m-1$ satisfying $R = \sum_{i=0}^{m-1} R_i$. At each level i, we then use density evolution in conjunction with differential evolution [4], a nonlinear optimization technique, to search for the degree distribution pair

(λ^i, ρ^i) with the largest noise threshold $(\sigma^*)^i$. We adjust the component code rates and repeat the optimization procedure at each level until the optimal degree distribution pairs (λ^i, ρ^i), $i = 0, 1, \ldots, m-1$ have equal noise thresholds. We use the resulting component code rates and the optimal degree distribution pairs to construct the LDPC codes for the MLC/PID scheme.

We carried out this optimization procedure for the MLC/PID scheme with Gray-mapped 8-PSK modulation when R =2 bits/symbol. As noted above, $f_0^1 = f_0^2$, so we only need to optimize the degree distribution pairs for the equivalent binary channels at levels 0 and 1. We found that the optimized component code rates were $R_0/R_1/R_2 = 0.51/0.745/0.745$, in agreement with the results predicted by the PID capacity calculation. Table I lists the optimized degree distribution pairs for the rate-0.51 LDPC code at level 0 and the rate-0.745 LDPC codes at level 1 and 2, subject to the constraint on the maximal bit degree $d_{v_{max}} = 50$. The table also shows the equivalent SNR thresholds $(E_s/N_0)^*$ (dB). The gap between the optimized thresholds and the minimum SNR for reliable transmission corresponding to the PID capacity $\widehat{C}$ is only 0.072 dB, which also suggests that, with sufficiently long blocklength, the MLC/PID scheme should achieve good performance within 0.142 dB of the channel capacity C.

V. CONCLUSIONS

We presented a bandwidth-efficient MLC/PID scheme with LDPC codes as component codes. We adapted the density evolution technique to determine a noise threshold corresponding to an LDPC degree distribution pair at any level, and we described a procedure for joint optimization of the code rates and degree distribution pairs of the component LDPC codes of the MLC/PID scheme. We presented results of code optimization and performance simulation for a Gray-mapped 8-PSK constellation on an AWGN channel. The thresholds of the optimized irregular LDPC component codes were found to be very close to the values corresponding to the PID capacity of the MLC scheme and the channel capacity.

REFERENCES

[1] U. Wachsmann, R. F. H. Fischer, J. B. Huber, "Multilevel codes: theoretical concepts and practical design rules," *IEEE Trans. Inform. Theory*, vol. 45, no. 5, pp. 1361-1391, July 1999.

[2] J. Hou, P. H. Siegel, L. B. Milstein, and H. D. Pfister, "Multilevel coding with low-density parity-check component codes," *submitted to IEEE Globecom'01*. (San Antonio, Texas, Nov. 25-29)

[3] T. J. Richardson and R. Urbanke, "The capacity of low-density parity-check codes under message-passing decoding," *IEEE Trans. Inform. Theory*, vol. 47, no. 2, pp. 599-618, February 2001.

[4] T. J. Richardson, A. Shokrollahi and R. Urbanke, "Design of capacity-approaching irregular low-density parity-check codes," *IEEE Trans. Inform. Theory*, vol. 47, no. 2, pp. 619-637, February 2001.

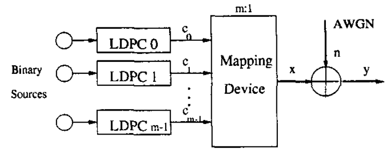

Fig. 1. MLC with LDPC codes as component codes.

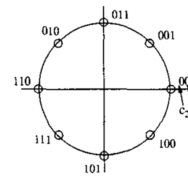

Fig. 2. The Gray-mapped 8-PSK constellation.

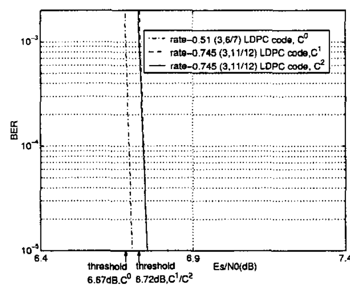

Fig. 3. Thresholds and simulation results of a Gray-mapped MLC/PID scheme with 8-PSK modulation on an AWGN channel. The component codeword lengths are 10^6.

i	0 (rate-0.51)	1/2(rate-0.745)
λ_2	0.15301	0.13542
λ_3	0.25900	0.20981
λ_4	0.00180	0.01720
λ_7	0.01464	0.03050
λ_8	0.00294	0.01748
λ_9	0.19801	0.05357
λ_{10}		0.16708
λ_{15}	0.10532	0.04867
λ_{30}	0.00525	0.00786
λ_{49}		0.09773
λ_{50}	0.26003	0.21468
ρ_9	0.01594	
ρ_{10}	0.75230	
ρ_{11}	0.23176	
ρ_{21}		0.57654
ρ_{22}		0.13921
ρ_{23}		0.28425
σ^*	0.358	0.358
$\left(\frac{E_s}{N_0}\right)^*$ (dB)	5.912	5.912

TABLE I

Optimal degree distributions for the MLC/PID scheme with Gray-mapped 8-PSK modulation on an AWGN channel.