Toward the Black-Scholes Formula

• As $n$ increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.

• Need to calibrate the BOPM’s parameters $u$, $d$, and $R$ to make it converge to the continuous-time model.

• We now skim through the proof.
Toward the Black-Scholes Formula (continued)

• Let $\tau$ denote the time to expiration of the option measured in years.

• Let $r$ be the continuously compounded annual rate.

• With $n$ periods during the option’s life, each period represents a time interval of $\tau/n$.

• Need to adjust the period-based $u$, $d$, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$. 
Toward the Black-Scholes Formula (continued)

• First, \( \hat{r} = r\tau/n \).
  – Each period is \( \tau/n \) years long.
  – The period gross return \( R = e^{\hat{r}} \).

• Let

\[
\ln \frac{S_{\tau}}{S}
\]

denote the continuously compounded rate of return of the stock.
Toward the Black-Scholes Formula (continued)

• Assume the stock’s true continuously compounded rate of return has mean $\mu \tau$ and variance $\sigma^2 \tau$.

• Call $\sigma$ the stock’s (annualized) volatility.

• We need one more condition to have a solution for $u, d, q$.

• Impose

$$ud = 1.$$  

  - It makes nodes at the same horizontal level of the tree have identical price.\(^a\)

\(^a\)Other choices are possible (see text).
Toward the Black-Scholes Formula (continued)

• Pick

\[ u = e^{\sigma \sqrt{\tau/n}}, \quad d = e^{-\sigma \sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\frac{\tau}{n}}. \]  

(12)

• With Eqs. (12), it can be checked that the mean \( \mu \tau \) is matched by the BOPM.

• Furthermore, the variance \( \sigma^2 \tau \) is asymptotically matched as well.
Toward the Black-Scholes Formula (continued)

- The choices (12) result in the CRR binomial model.\textsuperscript{a}

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (12) on p. 80 or Eq. (8) on p. 56.
  - If this happens, the probabilities lie outside $[0, 1]$.

- The problem disappears if $n$ is large enough.

\textsuperscript{a}Cox, Ross, and Rubinstein (1979).
Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- It approaches $N(\mu_\tau + \ln S, \sigma^2_\tau)$.
- Conclusion: $S_\tau$ has a lognormal distribution in the limit.
Toward the Black-Scholes Formula (continued)

• In the risk-neutral economy, pick

\[ q = \frac{R - d}{u - d}. \]

by Eq. (8) on p. 56.

Lemma 1 The continuously compounded rate of return
\[ \ln(S_\tau / S) \] approaches the normal distribution with mean
\( (r - \sigma^2/2) \tau \) and variance \( \sigma^2 \tau \) in a risk-neutral economy.\(^a\)

\(^a\)See Lemma 9.3.3 of the textbook.
Toward the Black-Scholes Formula (concluded)

Theorem 2 (The Black-Scholes Formula)

\[ C = SN(x) - X e^{-r \tau} N(x - \sigma \sqrt{\tau}), \]
\[ P = X e^{-r \tau} N(-x + \sigma \sqrt{\tau}) - SN(-x), \]

where

\[ x \equiv \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}. \]
BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: $S$, $X$, $\sigma$, $\tau$, and $r$.

- Binomial tree algorithms take 6 inputs: $S$, $X$, $u$, $d$, $\hat{r}$, and $n$.

- The connections are

  $u = e^{\sigma \sqrt{\tau/n}}$,
  $d = e^{-\sigma \sqrt{\tau/n}}$,
  $\hat{r} = r\tau/n$. 
\begin{itemize}
\item $S = 100$, $X = 100$ (left), and $X = 95$ (right).
\end{itemize}
BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.
- Oscillations can be dealt with by the judicious choices of $u$ and $d$.

---

\(^{a}\)Chang and Palmer (2007).
\(^{b}\)See Exercise 9.3.8 of the textbook.
Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market’s opinion of the volatility.
  - Solve for $\sigma$ given the option price, $S$, $X$, $\tau$, and $r$ with numerical methods.
- Implied volatility is the wrong number to put in the wrong formula to get the right price of plain-vanilla options.\(^a\)
- It is often preferred to historical volatility in practice.

\(^a\)Rebonato (2004).
Problems; the Smile

• Options written on the same underlying asset usually do not produce the same implied volatility.

• A typical pattern is a “smile” in relation to the strike price.
  – The implied volatility is lowest for at-the-money options.
  – It becomes higher the further the option is in- or out-of-the-money.

• Other patterns have also been observed.
Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
  \[ \max(0, X - S u^j d^{m-j}) \]
  and applies backward induction.
- At each intermediate node, it compares the payoff if exercised and the continuation value.
- It keeps the larger one.
Extensions of Options Theory
And the worst thing you can have is models and spreadsheets.

Barrier Options

• Their payoff depends on whether the underlying asset’s price reaches a certain price level $H$.

• A knock-out option is like an ordinary European option.

• But it ceases to exist if the barrier $H$ is reached by the price of its underlying asset.
Barrier Options (concluded)

- A knock-in option comes into existence if a certain barrier is reached.

- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$. 
A Formula for Down-and-In Calls\textsuperscript{a}

- Assume $X \geq H$.
- The value of a European down-and-in call on a stock paying a dividend yield of $q$ is

\[
Se^{-q\tau} \left( \frac{H}{S} \right)^{2\lambda} N(x) - Xe^{-r\tau} \left( \frac{H}{S} \right)^{2\lambda-2} N(x - \sqrt{\tau}),
\]

\[x \equiv \frac{\ln(H^2/(SX)) + (r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
\]

\[\lambda \equiv \frac{(r-q+\sigma^2/2)}{\sigma^2}.
\]

\textsuperscript{a}Merton (1973).
Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.

- Below is for the down-and-out option.

![Diagram of a binomial tree with a barrier at point H, starting from node 0.](image)
\[ S = 8, \ X = 6, \ H = 4, \ R = 1.25, \ u = 2, \text{ and } d = 0.5. \]

Backward-induction:
\[ C = \left(0.5 \times C_u + 0.5 \times C_d\right)/1.25. \]
Binomial Tree Algorithms (continued)

- But convergence is erratic because $H$ is not at a price level on the tree.\(^a\)
  - The barrier $H$ is moved to a node price.
  - This “effective barrier” changes as $n$ increases.

- In fact, the binomial tree is $O(1/\sqrt{n})$ convergent.\(^b\)

\(^a\)Boyle and Lau (1994).
\(^b\)Lin (R95221010) (2008).
Binomial Tree Algorithms (concluded)\textsuperscript{a}

Down-and-in call value

\textsuperscript{a}Lyuu (1998).
Path-Dependent Derivatives

- Let $S_0, S_1, \ldots, S_n$ denote the prices of the underlying asset over the life of the option.
- $S_0$ is the known price at time zero.
- $S_n$ is the price at expiration.
- The standard European call has a terminal value depending only on the last price, $\max(S_n - X, 0)$.
- Its value thus depends only on the underlying asset’s terminal price regardless of how it gets there.
Path-Dependent Derivatives (continued)

- Some derivatives are path-dependent in that their terminal payoff depends critically on the path.

- The (arithmetic) average-rate call has this terminal value:
  \[ \max \left( \frac{1}{n+1} \sum_{i=0}^{n} S_i - X, 0 \right) . \]

- The average-rate put’s terminal value is given by
  \[ \max \left( X - \frac{1}{n+1} \sum_{i=0}^{n} S_i, 0 \right) . \]
Path-Dependent Derivatives (concluded)

- Average-rate options are also called Asian options.\(^a\)

- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.

- The averaging clause is also common in convertible bonds and structured notes.

\(^a\)As of the late 1990s, the outstanding volume was in the range of 5–10 billion U.S. dollars (Nielsen & Sandmann, 2003).
Average-Rate Options

• Average-rate options are notoriously hard to price.

• The binomial tree for the averages does not combine (see next page).

• A naive algorithm enumerates the \(2^n\) paths for an \(n\)-period binomial tree and then averages the payoffs.\(^a\)

• But the complexity is exponential.

• The Monte Carlo method\(^b\) and approximation algorithms are some of the alternatives left.

\(^a\)Dai (B82506025, R86526008, D8852600) and Lyuu (2007) reduce it to \(2^{O(\sqrt{n})}\). Hsu (R7526001, D89922012) and Lyuu (2004) reduce it to \(O(n^2)\) given some regularity assumptions.

\(^b\)See pp. 142ff.
\[
\begin{align*}
\text{S} & \quad \text{S} \quad \text{S} \\
\text{Su} & \quad p & \quad \text{Suu} & \quad C_{\text{uu}} = \max \left( \frac{S + Su + Suu}{3} - X, 0 \right) \\
\text{Sd} & \quad 1-p & \quad \text{Sdd} \\
\text{Sd} & \quad p & \quad \text{Sdu} \\
\text{Su} & \quad p & \quad \text{Su} & \quad C_u = \frac{pC_{\text{uu}} + (1-p)C_{\text{ud}}}{e^r} \\
\text{Sd} & \quad 1-p & \quad \text{Sd} \\
\text{Su} & \quad p & \quad \text{Sd} & \quad C_{\text{ud}} = \max \left( \frac{S + Su + Sud}{3} - X, 0 \right) \\
\text{Sd} & \quad 1-p & \quad \text{Sd} \\
\text{Sd} & \quad p & \quad \text{Sdd} & \quad C_{\text{dd}} = \max \left( \frac{S + Sd + Sdd}{3} - X, 0 \right) \\
\end{align*}
\]
Continuous-Time Financial Mathematics
A proof is that which convinces a reasonable man; a rigorous proof is that which convinces an unreasonable man.

— Mark Kac (1914–1984)
Brownian Motion

• Brownian motion is a stochastic process \{X(t), t \geq 0\} with the following properties.

1. \(X(0) = 0\), unless stated otherwise.

2. for any \(0 \leq t_0 < t_1 < \cdots < t_n\), the random variables
   \[X(t_k) - X(t_{k-1})\]
   for \(1 \leq k \leq n\) are independent.\(^b\)

3. for \(0 \leq s < t\), \(X(t) - X(s)\) is normally distributed with mean \(\mu(t - s)\) and variance \(\sigma^2(t - s)\), where \(\mu\) and \(\sigma \neq 0\) are real numbers.

\(^a\)Robert Brown (1773–1858).

\(^b\)So \(X(t) - X(s)\) is independent of \(X(r)\) for \(r \leq s < t\).
Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener’s theorem.\(^a\)

- This process will be called a \((\mu, \sigma)\) Brownian motion with drift \(\mu\) and variance \(\sigma^2\).

- The \((0, 1)\) Brownian motion is called the Wiener process.

---

\(^a\)Norbert Wiener (1894–1964). He received his Ph.D. from Harvard in 1912.
Ito Process\(^a\)

- A shorthand\(^b\) is the following stochastic differential equation for the Ito differential \(dX_t\),

\[
dX_t = a(X_t, t) \, dt + b(X_t, t) \, dW_t.
\]

- Or simply

\[
dX_t = a_t \, dt + b_t \, dW_t.
\]

\(^a\)Ito (1944).
\(^b\)Paul Langevin (1872–1946) in 1904.
Ito Process (concluded)

- $dW$ is normally distributed with mean zero and variance $dt$.

- An equivalent form of Eq. (14) is

$$dX_t = a_t \ dt + b_t \sqrt{dt} \ \xi,$$  \hspace{1cm} (15)

where $\xi \sim N(0, 1)$. 
Modeling Stock Prices

- The most popular stochastic model for stock prices has been the geometric Brownian motion,
  \[ \frac{dS}{S} = \mu \, dt + \sigma \, dW. \]

- The continuously compounded rate of return \( X \equiv \ln S \) follows
  \[ dX = (\mu - \sigma^2/2) \, dt + \sigma \, dW \]

by Ito’s lemma.\(^a\)

\(^a\)Consistent with Lemma 1 (p. 84).
Local-Volatility Models

- The more general deterministic volatility model posits
  \[
  \frac{dS}{S} = (r_t - q_t) \, dt + \sigma(S, t) \, dW, 
  \]
  where instantaneous volatility \( \sigma(S, t) \) is called the local volatility function.\(^a\)

- One needs to recover \( \sigma(S, t) \) from the implied volatilities.

\(^a\)Derman and Kani (1994); Dupire (1994).
By Mr. Lok, U Hou (D99922028) on April 5, 2014.
Implied Trees

- The trees for the local volatility model are called implied trees.\textsuperscript{a}

- Their construction requires an implied volatility surface.

- An exponential-sized implied tree exists.\textsuperscript{b}

- How to construct a valid implied tree with efficiency has been open for a long time.\textsuperscript{c}

\textsuperscript{a}Derman & Kani (1994); Dupire (1994); Rubinstein (1994).

\textsuperscript{b}Charalambousa, Christofidesb, & Martzoukosa (2007).

\textsuperscript{c}Rubinstein (1994); Derman & Kani (1994); Derman, Kani, & Chriss (1996); Jackwerth & Rubinstein (1996); Jackwerth (1997); Coleman, Kim, Li, & Verma (2000); Li (2000/2001); Moriggia, Muzzioli, & Torricelli (2009).
Implied Trees (concluded)

• It is solved for separable local volatilities $\sigma$.\textsuperscript{a}
  
  – The local-volatility function $\sigma(S, V)$ is separable\textsuperscript{b} if
  \[
  \sigma(S, t) = \sigma_1(S) \sigma_2(t).
  \]

• A general solution is close.\textsuperscript{c}

\textsuperscript{a}Lok (D99922028) & Lyuu (2015, 2016).
\textsuperscript{b}Rebonato (2004); Brace, Gâtarek, & Musiela (1997).
\textsuperscript{c}Lok (D99922028) & Lyuu (2016).
The Hull-White Model

- Hull and White (1987) postulate the following model,

\[
\frac{dS}{S} = r \, dt + \sqrt{V} \, dW_1,
\]

\[
dV = \mu_v V \, dt + bV \, dW_2.
\]

- Above, $V$ is the instantaneous variance.

- They assume $\mu_v$ depends on $V$ and $t$ (but not $S$).
The SABR Model

- Hagan, Kumar, Lesniewski, and Woodward (2002) postulate the following model,

\[
\frac{dS}{S} = r \, dt + S^\theta V \, dW_1,
\]

\[
dV = bV \, dW_2,
\]

for \(0 \leq \theta \leq 1\).
The Hilliard-Schwartz Model

• Hilliard and Schwartz (1996) postulate the following general model,

\[
\frac{dS}{S} = r \, dt + f(S) V^a \, dW_1,
\]

\[
dV = \mu(V) \, dt + bV \, dW_2,
\]

for some well-behaved function \( f(S) \) and constant \( a \).
Heston’s Stochastic-Volatility Model

- Heston (1993) assumes the stock price follows

\[
\frac{dS}{S} = (\mu - q) \, dt + \sqrt{V} \, dW_1, \tag{16}
\]

\[
dV = \kappa (\theta - V) \, dt + \sigma \sqrt{V} \, dW_2. \tag{17}
\]

- \( V \) is the instantaneous variance, which follows a square-root process.
- \( dW_1 \) and \( dW_2 \) have correlation \( \rho \).
- The riskless rate \( r \) is constant.
Heston’s Stochastic-Volatility Model (concluded)

• It may be the most popular continuous-time stochastic-volatility model.\textsuperscript{a}

• For American options, we will need a tree for Heston’s model.\textsuperscript{b}

• They are all $O(n^3)$-sized.

\textsuperscript{a}Christoffersen, Heston, & Jacobs (2009).

\textsuperscript{b}Leisen (2010); Beliaeva & Nawalka (2010); Chou (R02723073) (2015).
Why Are Trees for Stochastic-Volatility Models Difficult?

- The CRR tree is 2-dimensional.\textsuperscript{a}

- The constant volatility makes the span from any node fixed.

- But a tree for a stochastic-volatility model must be 3-dimensional.
  - Every node is associated with a pair of stock price and a volatility.

\textsuperscript{a}Recall p. 82.
Why Are Trees for Stochastic-Volatility Models Difficult: Binomial Case?
Why Are Trees for Stochastic-Volatility Models Difficult: Trinomial Case?
Why Are Trees for Stochastic-Volatility Models Difficult? (concluded)

- Locally, the tree looks fine for one time step.
- But the volatility regulates the spans of the nodes on the stock-price plane.
- Unfortunately, those spans differ from node to node because the volatility varies.
- So two time steps from now, the branches will not combine!
Complexities of Stochastic-Volatility Models

- A few stochastic-volatility models suffer from subexponential ($c\sqrt{n}$) tree size.

- Examples include the Hull-White (1987), Hilliard-Schwartz (1996), and SABR (2002) models.\(^a\)

\(^a\)Chiu (R98723059) (2012).
Trees
I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)
Trinomial Tree

• Set up a trinomial approximation to the geometric Brownian motion\(^a\)

\[
\frac{dS}{S} = r\, dt + \sigma\, dW.
\]

• The three stock prices at time \(\Delta t\) are \(S\), \(Su\), and \(Sd\), where \(ud = 1\).

• Let the mean and variance of the stock price be \(SM\) and \(S^2V\), respectively.

\(^a\)Boyle (1988).
Trinomial Tree (continued)

• By Eqs. (5) on p. 24,

\[ M \equiv e^{r\Delta t}, \]
\[ V \equiv M^2(e^{\sigma^2\Delta t} - 1). \]

• Impose the matching of mean and that of variance:

\[ 1 = p_u + p_m + p_d, \]
\[ SM = (p_u u + p_m + (p_d/u)) S, \]
\[ S^2V = p_u (Su - SM)^2 + p_m (S - SM)^2 + p_d (Sd - SM)^2. \]
Trinomial Tree (concluded)

- Use linear algebra to verify that

\[
p_u = \frac{u(V + M^2 - M) - (M - 1)}{(u - 1)(u^2 - 1)},
\]

\[
p_d = \frac{u^2(V + M^2 - M) - u^3(M - 1)}{(u - 1)(u^2 - 1)}.
\]

- We must also make sure the probabilities lie between 0 and 1.
A Trinomial Tree

- Use $u = e^{\lambda \sigma \sqrt{\Delta t}}$, where $\lambda \geq 1$ is a tunable parameter.
- Then

$$p_u \rightarrow \frac{1}{2\lambda^2} + \frac{(r + \sigma^2) \sqrt{\Delta t}}{2\lambda \sigma},$$

$$p_d \rightarrow \frac{1}{2\lambda^2} - \frac{(r - 2\sigma^2) \sqrt{\Delta t}}{2\lambda \sigma}.$$
Barrier Options Priced by Trinomial Trees

Down-and-in call value

#Periods

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Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on $m$ assets has the terminal payoff

$$\max \left( \sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0 \right).$$
### Multivariate Contingent Claims (continued)\(^a\)

<table>
<thead>
<tr>
<th>Name</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exchange option</td>
<td>( \max(S_1(\tau) - S_2(\tau), 0) )</td>
</tr>
<tr>
<td>Better-off option</td>
<td>( \max(S_1(\tau), \ldots, S_k(\tau), 0) )</td>
</tr>
<tr>
<td>Worst-off option</td>
<td>( \min(S_1(\tau), \ldots, S_k(\tau), 0) )</td>
</tr>
<tr>
<td>Binary maximum option</td>
<td>( I{ \max(S_1(\tau), \ldots, S_k(\tau)) &gt; X } )</td>
</tr>
<tr>
<td>Maximum option</td>
<td>( \max(\max(S_1(\tau), \ldots, S_k(\tau)) - X, 0) )</td>
</tr>
<tr>
<td>Minimum option</td>
<td>( \max(\min(S_1(\tau), \ldots, S_k(\tau)) - X, 0) )</td>
</tr>
<tr>
<td>Spread option</td>
<td>( \max(S_1(\tau) - S_2(\tau) - X, 0) )</td>
</tr>
<tr>
<td>Basket average option</td>
<td>( \max((S_1(\tau) + \cdots + S_k(\tau))/k - X, 0) )</td>
</tr>
<tr>
<td>Multi-strike option</td>
<td>( \max(S_1(\tau) - X_1, \ldots, S_k(\tau) - X_k, 0) )</td>
</tr>
<tr>
<td>Pyramid rainbow option</td>
<td>( \max(</td>
</tr>
<tr>
<td>Madonna option</td>
<td>( \max(\sqrt{(S_1(\tau) - X_1)^2 + \cdots + (S_k(\tau) - X_k)^2} - X, 0) )</td>
</tr>
</tbody>
</table>

\(^a\)Lyuu & Teng (R91723054) (2011).
Multivariate Contingent Claims (concluded)

- Trees for multivariate contingent claims typically have size exponential in the number of assets.
- This is called the curse of dimensionality.
Numerical Methods
All science is dominated by the idea of approximation.

— Bertrand Russell
Monte Carlo Simulation\textsuperscript{a}

- Monte Carlo simulation is a sampling scheme.
- In many important applications within finance and without, Monte Carlo is one of the few feasible tools.

\textsuperscript{a}A top 10 algorithm according to Dongarra and Sullivan (2000).
Monte Carlo Option Pricing

- For the pricing of European options, we sample the stock prices.
- Then we average the payoffs.
- The variance of the estimator is now $1/N$ of that of the original random variable.
How about American Options?

- Standard Monte Carlo simulation is inappropriate for American options because of early exercise.

- It is difficult to determine the early-exercise point based on one single path.

- But Monte Carlo simulation can be modified to price American options with small biases.\(^a\)

- The LSM can be easily parallelized.\(^b\)

\(^a\)Longstaff and Schwartz (2001).

\(^b\)Huang (B96902079, R00922018) (2013); Chen (B97902046, R01922005) (2014); Chen (B97902046, R01922005), Huang (B96902079, R00922018) & Lyuu (2015).
Delta and Common Random Numbers

• In estimating delta $\partial f/\partial S$, it is natural to start with the finite-difference estimate

$$e^{-r\tau} \frac{E[P(S + \epsilon)] - E[P(S - \epsilon)]}{2\epsilon}.$$ 

– $P(x)$ is the terminal payoff of the derivative security when the underlying asset’s initial price equals $x$.

• Use simulation to estimate $E[P(S + \epsilon)]$ first.

• Use another simulation to estimate $E[P(S - \epsilon)]$.

• Finally, apply the formula to approximate the delta.
Delta and Common Random Numbers (concluded)

• This method is not recommended because of its high variance.

• A much better approach is to use common random numbers to lower the variance:

\[ e^{-r\tau} E \left[ \frac{P(S + \epsilon) - P(S - \epsilon)}{2\epsilon} \right]. \]

• Here, the same random numbers are used for \( P(S + \epsilon) \) and \( P(S - \epsilon) \).
Gamma

- The finite-difference formula for gamma $\partial^2 f / \partial S^2$ is
  $$e^{-r\tau} E \left[ \frac{P(S + \epsilon) - 2 \times P(S) + P(S - \epsilon)}{\epsilon^2} \right].$$

- Choosing an $\epsilon$ of the right magnitude can be challenging.
  - If $\epsilon$ is too large, inaccurate Greeks result.
  - If $\epsilon$ is too small, unstable Greeks result.

- This phenomenon is sometimes called the curse of differentiation.\(^a\)

\(^a\)Aït-Sahalia and Lo (1998); Bondarenko (2003).
Gamma (continued)

• In general, suppose

\[ \frac{\partial^i}{\partial \theta^i} e^{-r\tau} E[P(S)] = e^{-r\tau} E \left[ \frac{\partial^i P(S)}{\partial \theta^i} \right] \]

holds for all \( i > 0 \), where \( \theta \) is a parameter of interest.

  – A common requirement is Lipschitz continuity.\(^a\)

• Then formulas for the Greeks become integrals.

• As a result, we avoid \( \epsilon \), finite differences, and resimulation.

\(^a\) Broadie and Glasserman (1996).
Gamma (concluded)

- This is indeed possible for a broad class of payoff functions.\textsuperscript{a}

- In queueing networks, this is called infinitesimal perturbation analysis (IPA).\textsuperscript{b}

\textsuperscript{a}Teng (R91723054) (2004) and Lyuu and Teng (R91723054) (2011).

\textsuperscript{b}Cao (1985); Ho and Cao (1985).
Interest Rate Models
[Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.

Bond market terminology was designed less to convey meaning than to bewilder outsiders.
The Vasicek Model\textsuperscript{a}

- The short rate follows

\[ dr = \beta(\mu - r) \, dt + \sigma \, dW. \]

- The short rate is pulled to the long-term mean level \( \mu \) at rate \( \beta \).

- Superimposed on this “pull” is a normally distributed stochastic term \( \sigma \, dW \).

\textsuperscript{a}Vasicek (1977).
The Cox-Ingersoll-Ross Model\(^a\)

- It is the following square-root short rate model:

\[
dr = \beta(\mu - r) \, dt + \sigma \sqrt{r} \, dW. \tag{18}
\]

- The diffusion differs from the Vasicek model by a multiplicative factor \(\sqrt{r}\).

- The parameter \(\beta\) determines the speed of adjustment.

- The short rate can reach zero only if \(2\beta\mu < \sigma^2\).

\(^a\)Cox, Ingersoll, and Ross (1985).
The Ho-Lee Model\textsuperscript{a}

- The continuous-time limit of the Ho-Lee model is
  \[ dr = \theta(t) \, dt + \sigma \, dW. \]

- This is Vasicek’s model with the mean-reverting drift replaced by a deterministic, time-dependent drift.

- A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,
  \[ dr = \theta(t) \, dt + \sigma(t) \, dW. \]

\textsuperscript{a}Ho and Lee (1986). Thomas Lee is a “billionaire founder” of Thomas H. Lee Partners LP, according to \textit{Bloomberg} on May 26, 2012.
The Black-Derman-Toy Model\textsuperscript{a}

- The continuous-time limit of the BDT model is

\[ d \ln r = \left( \theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln r \right) dt + \sigma(t) dW. \]

- This model is extensively used by practitioners.

- The BDT short rate process is the lognormal binomial interest rate process.

- Lognormal models preclude negative short rates.

\textsuperscript{a}Black, Derman, and Toy (BDT) (1990), but essentially finished in 1986 according to Mehrling (2005).
The Black-Karasinski Model\textsuperscript{a}

- The BK model stipulates that the short rate follows
  \[ d \ln r = \kappa(t)(\theta(t) - \ln r) \, dt + \sigma(t) \, dW. \]

- This explicitly mean-reverting model depends on time through \( \kappa(\cdot), \theta(\cdot), \) and \( \sigma(\cdot). \)

- The BK model hence has one more degree of freedom than the BDT model.

- The speed of mean reversion \( \kappa(t) \) and the short rate volatility \( \sigma(t) \) are independent.

\textsuperscript{a}Black and Karasinski (1991).
The Extended Vasicek Model

- The extended Vasicek model adds time dependence to the original Vasicek model,

\[
    dr = (\theta(t) - a(t) r) \, dt + \sigma(t) \, dW.
\]

- Like the Ho-Lee model, this is a normal model.
- Many European-style securities can be evaluated analytically.
- Efficient numerical procedures can be developed for American-style securities.

\[^a\]Hull and White (1990).
The Hull-White Model

- The Hull-White model is the following special case,

\[ dr = (\theta(t) - ar) \, dt + \sigma \, dW. \]
The Extended CIR Model

• In the extended CIR model the short rate follows

\[ dr = (\theta(t) - a(t) r) \, dt + \sigma(t) \sqrt{r} \, dW. \]

• The functions \( \theta(t), a(t), \) and \( \sigma(t) \) are implied from market observables.

• With constant parameters, there exist analytical solutions to a small set of interest rate-sensitive securities.
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