Sample Term Structure

- We shall construct interest rate trees consistent with the sample term structure in the following table.
  - This was called calibration (the reverse of pricing).
- Assume the short rate volatility is such that
  \[ v \triangleq \frac{r_h}{r_\ell} = 1.5, \]
  independent of time.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate (%)</td>
<td>4</td>
<td>4.2</td>
<td>4.3</td>
</tr>
<tr>
<td>One-period forward rate (%)</td>
<td>4</td>
<td>4.4</td>
<td>4.5</td>
</tr>
<tr>
<td>Discount factor</td>
<td>0.96154</td>
<td>0.92101</td>
<td>0.88135</td>
</tr>
</tbody>
</table>
An Approximate Calibration Scheme

- Start with the implied one-period forward rates.
- Then equate the expected short rate with the forward rate (see Exercise 5.6.6 in text).
- For the first period, the forward rate is today’s one-period spot rate.
- In general, let $f_j$ denote the forward rate in period $j$.
- This forward rate can be derived from the market discount function via

$$f_j = \frac{d(j)}{d(j + 1)} - 1$$

(see Exercise 5.6.3 in text).
An Approximate Calibration Scheme (continued)

- Since the \( i \)th short rate \( r_j v_j^{i-1} \), \( 1 \leq i \leq j \), occurs with probability \( 2^{-(j-1)} \binom{j-1}{i-1} \), this means

\[
\sum_{i=1}^{j} 2^{-(j-1)} \binom{j-1}{i-1} r_j v_j^{i-1} = f_j.
\]

- Thus

\[
r_j = \left( \frac{2}{1 + v_j} \right)^{j-1} f_j. \tag{125}
\]

- This binomial interest rate tree is trivial to set up, in \( O(n) \) time.
An Approximate Calibration Scheme (continued)

• The ensuing tree for the sample term structure appears in figure next page.

• For example, the price of the zero-coupon bond paying $1 at the end of the third period is

\[
\frac{1}{4} \times \frac{1}{1.04} \times \left( \frac{1}{1.0352} \times \left( \frac{1}{1.0288} + \frac{1}{1.0432} \right) + \frac{1}{1.0528} \times \left( \frac{1}{1.0432} + \frac{1}{1.0648} \right) \right)
\]

or 0.88155, which exceeds discount factor 0.88135.

• The tree is thus \textit{not} calibrated.
Baseline rates

- A: 4.0%
- B: 3.52%
- C: 2.88%
- D: 5.28%
- E: 4.32%
- F: 6.48%

Implied forward rates:

- Period 1: 4.0%
- Period 2: 4.4%
- Period 3: 4.5%
An Approximate Calibration Scheme (concluded)

- Indeed, this bias is inherent: The tree *overprices* the bonds.\(^a\)
- Suppose we replace the baseline rates \(r_j\) by \(r_jv_j\).
- Then the resulting tree *underprices* the bonds.\(^b\)
- The true baseline rates are thus bounded between \(r_j\) and \(r_jv_j\).

\(^a\)See Exercise 23.2.4 in text.
Issues in Calibration

- The model prices generated by the binomial interest rate tree should match the observed market prices.
- Perhaps the most crucial aspect of model building.
- Treat the backward induction for the model price of the $m$-period zero-coupon bond as computing some function $f(r_m)$ of the unknown baseline rate $r_m$ for period $m$.
- A root-finding method is applied to solve $f(r_m) = P$ for $r_m$ given the zero’s price $P$ and $r_1, r_2, \ldots, r_{m-1}$.
- This procedure is carried out for $m = 1, 2, \ldots, n$.
- It runs in $O(n^3)$ time.
Binomial Interest Rate Tree Calibration

- Calibration can be accomplished in $O(n^2)$ time by the use of forward induction.\(^a\)

- The scheme records how much $1 at a node contributes to the model price.

- This number is called the state price, the Arrow-Debreu price, or Green’s function.
  - It is the price of a state contingent claim that pays $1 at that particular node (state) and 0 elsewhere.

- The column of state prices will be established by moving forward from time 0 to time $n$.

Binomial Interest Rate Tree Calibration (continued)

- Suppose we are at time $j$ and there are $j + 1$ nodes.
  - The unknown baseline rate for period $j$ is $r^\Delta = r_j$.
  - The multiplicative ratio is $v^\Delta = v_j$.
  - $P_1, P_2, \ldots, P_j$ are the known state prices at earlier time $j - 1$.
  - They correspond to rates $r, rv, \ldots, rv^{j-1}$ for period $j$ (recall p. 949).

- By definition, $\sum_{i=1}^{j} P_i$ is the price of the $(j - 1)$-period zero-coupon bond.

- We want to find $r$ based on $P_1, P_2, \ldots, P_j$ and the price of the $j$-period zero-coupon bond.
Binomial Interest Rate Tree Calibration (continued)

- One dollar at time $j$ has a known market value of $\frac{1}{[1 + S(j)]^j}$, where $S(j)$ is the $j$-period spot rate.

- Alternatively, this dollar has a present value of

$$g(r) \triangleq \frac{P_1}{1 + r} + \frac{P_2}{(1 + rv)} + \frac{P_3}{(1 + rv^2)} + \cdots + \frac{P_j}{(1 + rv^{j-1})}$$

(see next plot).

- So we solve

$$g(r) = \frac{1}{[1 + S(j)]^j} \quad (126)$$

for $r$. 
\[ P_i \rightarrow rv^{i-1} \]
Binomial Interest Rate Tree Calibration (continued)

- Given a decreasing market discount function, a unique positive solution for $r$ is guaranteed.
- The state prices at time $j$ can now be calculated (see panel (a) next page).
- We call a tree with these state prices a binomial state price tree (see panel (b) next page).
- The calibrated tree is depicted on p. 970.
Binomial Interest Rate Tree Calibration (concluded)

• The Newton-Raphson method can be used to solve for the \( r \) in Eq. (126) on p. 966 as \( g'(r) \) is easy to evaluate.

• The monotonicity and the convexity of \( g(r) \) also facilitate root finding.

• The total running time is \( O(n^2) \), as each root-finding routine consumes \( O(j) \) time.

• With a good initial guess,\(^a\) the Newton-Raphson method converges in only a few steps.\(^b\)

\(^a\)Such as the \( r_j = (\frac{2}{1+v_j})^{j-1} f_j \) on p. 959.

\(^b\)Lyuu (1999).
A Numerical Example

- One dollar at the end of the second period should have a present value of 0.92101 by the sample term structure.

- The baseline rate for the second period, $r_2$, satisfies

\[
\frac{0.480769}{1 + r_2} + \frac{0.480769}{1 + 1.5 \times r_2} = 0.92101.
\]

- The result is $r_2 = 3.526\%$.

- This is used to derive the next column of state prices shown in panel (b) on p. 969 as 0.232197, 0.460505, and 0.228308.

- Their sum gives the correct market discount factor 0.92101.
A Numerical Example (concluded)

• The baseline rate for the third period, $r_3$, satisfies

$$\frac{0.232197}{1 + r_3} + \frac{0.460505}{1 + 1.5 \times r_3} + \frac{0.228308}{1 + (1.5)^2 \times r_3} = 0.88135.$$ 

• The result is $r_3 = 2.895\%$.

• Now, redo the calculation on p. 960 using the new rates:

$$\frac{1}{4} \times \frac{1}{1.04} \times \left[ \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) + \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) \right],$$

which equals 0.88135, an exact match.

• The tree on p. 970 prices without bias the benchmark securities.
Spread of Nonbenchmark Bonds

- Model prices calculated by the calibrated tree as a rule do not match market prices of nonbenchmark bonds.
- The incremental return over the benchmark bonds is called spread.
- If we add the spread uniformly over the short rates in the tree, the model price will equal the market price.
- We will apply the spread concept to option-free bonds next.
Spread of Nonbenchmark Bonds (continued)

• We illustrate the idea with an example.

• Start with the tree on p. 976.

• Consider a security with cash flow $C_i$ at time $i$ for $i = 1, 2, 3$.

• Its model price is $p(s)$, which is equal to

$$
\frac{1}{1.04 + s} \times \left[ C_1 + \frac{1}{2} \times \frac{1}{1.03526 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.02895 + s} + \frac{C_3}{1.04343 + s} \right) \right) + \right.
\frac{1}{2} \times \frac{1}{1.05289 + s} \times \left( C_2 + \frac{1}{2} \left( \frac{C_3}{1.04343 + s} + \frac{C_3}{1.06514 + s} \right) \right) \right].
$$

• Given a market price of $P$, the spread is the $s$ that solves $P = p(s)$.
Implied forward rates: 4.0% 4.4% 4.5%

<table>
<thead>
<tr>
<th>Period</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.0%</td>
</tr>
<tr>
<td>2</td>
<td>4.4%</td>
</tr>
<tr>
<td>3</td>
<td>4.5%</td>
</tr>
</tbody>
</table>
Spread of Nonbenchmark Bonds (continued)

- The model price $p(s)$ is a monotonically decreasing, convex function of $s$.

- We will employ the Newton-Raphson root-finding method to solve
  \[ p(s) - P = 0 \]
  for $s$.

- But a quick look at the equation for $p(s)$ reveals that evaluating $p'(s)$ directly is infeasible.

- Fortunately, the tree can be used to evaluate both $p(s)$ and $p'(s)$ during backward induction.
Spread of Nonbenchmark Bonds (continued)

- Consider an arbitrary node A in the tree associated with the short rate \( r \).

- In the process of computing the model price \( p(s) \), a price \( p_A(s) \) is computed at A.

- Prices computed at A’s two successor nodes B and C are discounted by \( r + s \) to obtain \( p_A(s) \) as follows,

\[
p_A(s) = c + \frac{p_B(s) + p_C(s)}{2(1 + r + s)},
\]

where \( c \) denotes the cash flow at A.
Spread of Nonbenchmark Bonds (continued)

- To compute \( p'_A(s) \) as well, node A calculates

\[
p'_A(s) = \frac{p'_B(s) + p'_C(s)}{2(1 + r + s)} - \frac{p_B(s) + p_C(s)}{2(1 + r + s)^2}.
\]  

(127)

- This is easy if \( p'_B(s) \) and \( p'_C(s) \) are also computed at nodes B and C.

- When A is a terminal node, simply use the payoff function for \( p_A(s) \).\(^a\)

\(^a\)Contributed by Mr. Chou, Ming-Hsin (R02723073) on May 28, 2014.
\[ p_a(s) = c + \frac{p_b(s) + p_c(s)}{2(1+r+s)} \]

\[ p'_a(s) = \frac{p'_b(s) + p'_c(s)}{2(1+r+s)} - \frac{p_b(s) + p_c(s)}{2(1+r+s)^2} \]
Spread of Nonbenchmark Bonds (continued)

• Apply the above procedure inductively to yield $p(s)$ and $p'(s)$ at the root (p. 980).

• This is called the differential tree method.\textsuperscript{a}
  
  – Similar ideas can be found in automatic differentiation (AD)\textsuperscript{b} and backpropagation\textsuperscript{c} in artificial neural networks.

• The total running time is $O(n^2)$.

• The memory requirement is $O(n)$.

\textsuperscript{a}Lyuu (1999).
\textsuperscript{b}Rall (1981).
\textsuperscript{c}Werbos (1974); Rumelhart, Hinton, & Williams (1986).
## Spread of Nonbenchmark Bonds (continued)

<table>
<thead>
<tr>
<th>Number of partitions $n$</th>
<th>Running time (s)</th>
<th>Number of iterations</th>
<th>Number of partitions</th>
<th>Running time (s)</th>
<th>Number of iterations</th>
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</thead>
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<tr>
<td>9500</td>
<td>2834.170</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**75MHz Sun SPARCstation 20.**
Spread of Nonbenchmark Bonds (concluded)

• Consider a three-year, 5% bond with a market price of 100.569.

• Assume the bond pays annual interest.

• The spread can be shown to be 50 basis points over the tree (p. 984).

• Note that the idea of spread does not assume parallel shifts in the term structure.

• It also differs from the yield spread (p. 124) and static spread (p. 125) of the nonbenchmark bond over an otherwise identical benchmark bond.
Cash flows: 

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
<td>5</td>
<td>105</td>
</tr>
</tbody>
</table>

A 4.50% 100.569
B 5.789% 103.436
C 7.014% 103.118
D

B 4.026% 106.754
C 4.843% 105.150
D

C 3.395% 106.552
D

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More Applications of the Differential Tree: Calculating Implied Volatility (in seconds)\textsuperscript{a}

<table>
<thead>
<tr>
<th>Number of partitions</th>
<th>Running time</th>
<th>Number of iterations</th>
<th>Number of partitions</th>
<th>Running time</th>
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</tr>
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<tbody>
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<td>100</td>
<td>0.013845</td>
<td>3</td>
</tr>
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<td>200</td>
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<td>200</td>
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</tr>
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</tr>
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<td>0.201850</td>
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<tr>
<td>600</td>
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<td>0.323260</td>
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<td>800</td>
<td>0.569605</td>
<td>2</td>
</tr>
</tbody>
</table>

Intel 166MHz Pentium, running on Microsoft Windows 95.

\textsuperscript{a}Lyuu (1999).
Fixed-Income Options

• Consider a 2-year 99 European call on the 3-year, 5% Treasury.

• Assume the Treasury pays annual interest.

• From p. 987 the 3-year Treasury’s price minus the $5 interest at year 2 could be $102.046, $100.630, or $98.579 two years from now.
  – The accrued interest is not included as it belongs to the original bondholder.

• Now compare the strike price against the bond prices.

• The call is in the money in the first two scenarios out of the money in the third.
Fixed-Income Options (continued)

- The option value is calculated to be $1.458 on p. 987(a).
- European interest rate puts can be valued similarly.
- Consider a two-year 99 European put on the same security.
- At expiration, the put is in the money only when the Treasury is worth $98.579 without the accrued interest.
- The option value is computed to be $0.096 on p. 987(b).
Fixed-Income Options (concluded)

- The present value of the strike price is
  \[ PV(X) = 99 \times 0.92101 = 91.18. \]

- The Treasury is worth \( B = 101.955 \).

- The present value of the interest payments during the life of the options is
  \[ PV(I) = 5 \times 0.96154 + 5 \times 0.92101 = 9.41275. \]

- The call and the put are worth \( C = 1.458 \) and \( P = 0.096 \), respectively.

- Hence the put-call parity is preserved:
  \[ C = P + B - PV(I) - PV(X). \]
Delta or Hedge Ratio

- How much does the option price change in response to changes in the price of the underlying bond?
- This relation is called delta (or hedge ratio) defined as

\[
\frac{O_h - O_\ell}{P_h - P_\ell}.
\]

- In the above \( P_h \) and \( P_\ell \) denote the bond prices if the short rate moves up and down, respectively.
- Similarly, \( O_h \) and \( O_\ell \) denote the option values if the short rate moves up and down, respectively.
Delta or Hedge Ratio (concluded)

- Delta measures the sensitivity of the option value to changes in the underlying bond price.
- So it shows how to hedge one with the other.
- Take the call and put on p. 987 as examples.
- Their deltas are
  \[
  \frac{0.774 - 2.258}{99.350 - 102.716} = 0.441, \\
  \frac{0.200 - 0.000}{99.350 - 102.716} = -0.059, \\
  \]
  respectively.
Volatility Term Structures

• The binomial interest rate tree can be used to calculate the yield volatility of zero-coupon bonds.

• Consider an \( n \)-period zero-coupon bond.

• First find its yield to maturity \( y_h \) (\( y_\ell \), respectively) at the end of the initial period if the short rate rises (declines, respectively).

• The yield volatility for our model is defined as

\[
\frac{1}{2} \ln \left( \frac{y_h}{y_\ell} \right). \tag{128}
\]
Volatility Term Structures (continued)

• For example, based on the tree on p. 970, the two-year zero’s yield at the end of the first period is 5.289% if the rate rises and 3.526% if the rate declines.

• Its yield volatility is therefore

$$\frac{1}{2} \ln \left( \frac{0.05289}{0.03526} \right) = 20.273\%.$$
Volatility Term Structures (continued)

- Consider the three-year zero-coupon bond.

- If the short rate rises, the price of the zero one year from now will be
  \[
  \frac{1}{2} \times \frac{1}{1.05289} \times \left( \frac{1}{1.04343} + \frac{1}{1.06514} \right) = 0.90096.
  \]

- Thus its yield is \[\sqrt{\frac{1}{0.90096}} - 1 = 0.053531.\]

- If the short rate declines, the price of the zero one year from now will be
  \[
  \frac{1}{2} \times \frac{1}{1.03526} \times \left( \frac{1}{1.02895} + \frac{1}{1.04343} \right) = 0.93225.
  \]
Volatility Term Structures (continued)

• Thus its yield is \( \sqrt{\frac{1}{0.93225}} - 1 = 0.0357 \).

• The yield volatility is hence

\[
\frac{1}{2} \ln \left( \frac{0.053531}{0.0357} \right) = 20.256\%,
\]

slightly less than the one-year yield volatility.

• This is consistent with the reality that longer-term bonds typically have lower yield volatilities than shorter-term bonds.

• The procedure can be repeated for longer-term zeros to obtain their yield volatilities.

\[\text{\textsuperscript{a}The relation is reversed for price volatilities (duration).}\]
Spot rate volatility

Short rate volatility given flat %10 volatility term structure.
Volatility Term Structures (concluded)

- We started with $v_i$ and then derived the volatility term structure.
- In practice, the steps are reversed.
- The volatility term structure is supplied by the user along with the term structure.
- The $v_i$—hence the short rate volatilities via Eq. (123) on p. 948—and the $r_i$ are then simultaneously determined.
- The result is the Black-Derman-Toy model of Goldman Sachs.\(^a\)

\(^a\)Black, Derman, & Toy (1990).
Foundations of Term Structure Modeling
[Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.

[The] fixed-income traders I knew seemed smarter than the equity trader […] there’s no competitive edge to being smart in the equities business[.]
— Emanuel Derman,
_My Life as a Quant_ (2004)

Bond market terminology was designed less to convey meaning than to bewilder outsiders.
— Michael Lewis, _The Big Short_ (2011)
Terminology

• A period denotes a unit of elapsed time.
  - Viewed at time $t$, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.

• Bonds will be assumed to have a par value of one — unless stated otherwise.

• The time unit for continuous-time models will usually be measured by the year.
Standard Notations

The following notation will be used throughout.

$\mathbf{t}$: a point in time.

$r(t)$: the one-period riskless rate prevailing at time $t$ for repayment one period later.\textsuperscript{a}

$P(t, T)$: the present value at time $t$ of one dollar at time $T$.

\textsuperscript{a}Alternatively, the instantaneous spot rate, or short rate, at time $t$. 
Standard Notations (continued)

$r(t, T)$: the $(T - t)$-period interest rate prevailing at time $t$ stated on a per-period basis and compounded once per period.$^a$

$F(t, T, M)$: the forward price at time $t$ of a forward contract that delivers at time $T$ a zero-coupon bond maturing at time $M \geq T$.

$^a$In other words, the $(T - t)$-period spot rate at time $t$. 
Standard Notations (concluded)

\( f(t, T, L) \): the \( L \)-period forward rate at time \( T \) implied at time \( t \) stated on a per-period basis and compounded once per period.

\( f(t, T) \): the one-period or instantaneous forward rate at time \( T \) as seen at time \( t \) stated on a per period basis and compounded once per period.

- It is \( f(t, T, 1) \) in the discrete-time model and \( f(t, T, dt) \) in the continuous-time model.
- Note that \( f(t, t) \) equals the short rate \( r(t) \).
Fundamental Relations

• The price of a zero-coupon bond equals

\[ P(t, T) = \begin{cases} 
(1 + r(t, T))^{-(T-t)}, & \text{in discrete time}, \\
e^{-r(t,T)(T-t)}, & \text{in continuous time}.
\end{cases} \] (129)

• \( r(t, T) \) as a function of \( T \) defines the spot rate curve at time \( t \).

• By definition,

\[ f(t, t) = \begin{cases} 
r(t, t + 1), & \text{in discrete time}, \\
r(t, t), & \text{in continuous time}.
\end{cases} \]
Fundamental Relations (continued)

- Forward prices and zero-coupon bond prices are related:

\[ F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \quad (130) \]

- The forward price equals the future value at time \( T \) of the underlying asset.\(^a\)

- Equation (130) holds whether the model is discrete-time or continuous-time.

\(^a\)See Exercise 24.2.1 of the textbook for proof.
Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

\[ f(t, T, L) = \left( \frac{1}{F(t, T, T + L)} \right)^{1/L} - 1 = \left( \frac{P(t, T)}{P(t, T + L)} \right)^{1/L} - 1 \] (131)

in discrete time.

- The analog to Eq. (131) under simple compounding is

\[ f(t, T, L) = \frac{1}{L} \left( \frac{P(t, T)}{P(t, T + L)} - 1 \right). \]
Fundamental Relations (continued)

- In continuous time,

\[ f(t, T, L) = -\frac{\ln F(t, T, T + L)}{L} = \frac{\ln(P(t, T)/P(t, T + L))}{L} \]

by Eq. (130) on p. 1006.

- Furthermore,

\[ f(t, T, \Delta t) = \frac{\ln(P(t, T)/P(t, T + \Delta t))}{\Delta t} \rightarrow -\frac{\partial \ln P(t, T)}{\partial T} \]

\[ = -\frac{\partial P(t, T)/\partial T}{P(t, T)}. \]
Fundamental Relations (continued)

- So

\[
 f(t, T) \equiv \lim_{\Delta t \to 0} f(t, T, \Delta t) = -\frac{\partial P(t, T)/\partial T}{P(t, T)}, \quad t \leq T.
\] 

(133)

- Because Eq. (133) is equivalent to

\[
 P(t, T) = e^{-\int_{t}^{T} f(t, s) \, ds},
\]

(134)

the spot rate curve is

\[
 r(t, T) = \frac{\int_{t}^{T} f(t, s) \, ds}{T - t}.
\]
Fundamental Relations (concluded)

- The discrete analog to Eq. (134) is

\[ P(t, T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1)) \cdots (1 + f(t, T - 1))}. \]

- The short rate and the market discount function are related by

\[ r(t) = -\frac{\partial P(t, T)}{\partial T} \bigg|_{T=t}. \]
Risk-Neutral Pricing

- Assume the local expectations theory.
- The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  - For all $t + 1 < T$,
    \[
    \frac{E_t[ P(t + 1, T) ]}{P(t, T)} = 1 + r(t). \quad (135)
    \]
  - Relation (135) in fact follows from the risk-neutral valuation principle.\(^a\)

\(^{a}\)Theorem 18 on p. 521.
Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability \( \pi \).

- Rewrite Eq. (135) as

\[
E_t^\pi \left[ \frac{P(t+1, T)}{1 + r(t)} \right] = P(t, T).
\]

- It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.
Risk-Neutral Pricing (continued)

- Apply the above equality iteratively to obtain

\[
P(t, T) = E_t^\pi \left[ \frac{P(t + 1, T)}{1 + r(t)} \right] = E_t^\pi \left[ \frac{E_{t+1}^\pi \left[ P(t + 2, T) \right]}{(1 + r(t))(1 + r(t + 1))} \right] = \ldots
\]

\[
= E_t^\pi \left[ \frac{1}{(1 + r(t))(1 + r(t + 1)) \cdots (1 + r(T - 1))} \right]. \tag{136}
\]
Risk-Neutral Pricing (concluded)

- Equation (135) on p. 1011 can also be expressed as

\[ E_t[P(t+1,T)] = F(t,t+1,T). \]

- Verify that with, e.g., Eq. (130) on p. 1006.

- Hence the forward price for the next period is an unbiased estimator of the expected bond price.\(^a\)

\(^a\)But the forward rate is not an unbiased estimator of the expected future short rate (p. 962).
Continuous-Time Risk-Neutral Pricing

• In continuous time, the local expectations theory implies

\[ P(t, T) = E_t \left[ e^{-\int_t^T r(s) \, ds} \right], \quad t < T. \tag{137} \]

• Note that \( e^{\int_t^T r(s) \, ds} \) is the bank account process, which denotes the rolled-over money market account.
Interest Rate Swaps

- Consider an interest rate swap made at time $t$ (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.
- The fixed rate is $c$ per annum.
- The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.
- For simplicity, assume $t_{i+1} - t_i$ is a fixed constant $\Delta t$ for all $i$, and the notional principal is one dollar.
- If $t < t_0$, we have a forward interest rate swap.
- The ordinary swap corresponds to $t = t_0$. 
Interest Rate Swaps (continued)

• The amount to be paid out at time $t_{i+1}$ is $(f_i - c) \Delta t$ for the floating-rate payer.

• Simple rates are adopted here.

• Hence $f_i$ satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$
Interest Rate Swaps (continued)

- The value of the swap at time $t$ is thus

$$
\sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t}^{t_i} r(s) \, ds} (f_{i-1} - c) \Delta t \right]
$$

$$
= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t}^{t_i} r(s) \, ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right]
$$

$$
= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t}^{t_i} r(s) \, ds} \left( e^{\int_{t_{i-1}}^{t_i} r(s) \, ds} - (1 + c\Delta t) \right) \right]
$$

$$
= \sum_{i=1}^{n} \left[ P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \right]
$$

$$
= P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
$$
Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.
Swap Rate

- The swap rate, which gives the swap zero value, equals

\[ S_n(t) \triangleq \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i) \Delta t}. \]  

(138)

- The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.

- For an ordinary swap, \( P(t, t_0) = 1 \).
The Term Structure Equation\textsuperscript{a}

- Let us start with the zero-coupon bonds and the money market account.
- Let the zero-coupon bond price $P(r, t, T)$ follow
  \[
  \frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.
  \]
- At time $t$, short one unit of a bond maturing at time $s_1$ and buy $\alpha$ units of a bond maturing at time $s_2$.

\textsuperscript{a}Vasicek (1977).
The Term Structure Equation (continued)

- The net wealth change follows

\[
-dP(r, t, s_1) + \alpha dP(r, t, s_2)
\]

\[
= (-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)) \, dt \\
+ (-P(r, t, s_1) \sigma_p(r, t, s_1) + \alpha P(r, t, s_2) \sigma_p(r, t, s_2)) \, dW.
\]

- Pick

\[
\alpha \triangleq \frac{P(r, t, s_1) \sigma_p(r, t, s_1)}{P(r, t, s_2) \sigma_p(r, t, s_2)}.
\]
The Term Structure Equation (continued)

• Then the net wealth has no volatility and must earn the riskless return:

\[
\frac{-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)}{-P(r, t, s_1) + \alpha P(r, t, s_2)} = r.
\]

• Simplify the above to obtain

\[
\frac{\sigma_p(r, t, s_1) \mu_p(r, t, s_2) - \sigma_p(r, t, s_2) \mu_p(r, t, s_1)}{\sigma_p(r, t, s_1) - \sigma_p(r, t, s_2)} = r.
\]

• This becomes

\[
\frac{\mu_p(r, t, s_2) - r}{\sigma_p(r, t, s_2)} = \frac{\mu_p(r, t, s_1) - r}{\sigma_p(r, t, s_1)}
\]

after rearrangement.
The Term Structure Equation (continued)

- Since the above equality holds for any \( s_1 \) and \( s_2 \),

\[
\frac{\mu_p(r, t, s) - r}{\sigma_p(r, t, s)} \triangleq \lambda(r, t) \tag{139}
\]

for some \( \lambda \) independent of the bond maturity \( s \).

- As \( \mu_p = r + \lambda \sigma_p \), all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset’s volatility.

- The term \( \lambda(r, t) \) is called the market price of risk.

- The market price of risk must be the same for all bonds to preclude arbitrage opportunities.
The Term Structure Equation (continued)

• Assume a Markovian short rate model,

\[ dr = \mu(r, t)\, dt + \sigma(r, t)\, dW. \]

• Then the bond price process is also Markovian.

• By Eq. (14.15) on p. 202 of the textbook,

\[ \mu_p = \left( -\frac{\partial P}{\partial T} + \mu(r, t)\frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2}\frac{\partial^2 P}{\partial r^2} \right)/P, \]

(140)

\[ \sigma_p = \left( \sigma(r, t)\frac{\partial P}{\partial r} \right)/P, \]

(140')

subject to \( P(\cdot, T, T) = 1. \)
The Term Structure Equation (concluded)

- Substitute $\mu_p$ and $\sigma_p$ into Eq. (139) on p. 1024 to obtain

$$- \frac{\partial P}{\partial T} + \left[ \mu(r, t) - \lambda(r, t) \sigma(r, t) \right] \frac{\partial P}{\partial r} + \frac{1}{2} \sigma(r, t)^2 \frac{\partial^2 P}{\partial r^2} = rP. \quad (141)$$

- This is called the term structure equation.

- It applies to all interest rate derivatives: The differences are the terminal and boundary conditions.

- Once $P$ is available, the spot rate curve emerges via

$$r(t, T) = -\frac{\ln P(t, T)}{T - t}.$$
Numerical Examples

• Assume this spot rate curve:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate</td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

• Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:
Numerical Examples (continued)

• *No* real-world probabilities are specified.

• The prices of one- and two-year zero-coupon bonds are, respectively,

\[
\frac{100}{1.04} = 96.154, \\
\frac{100}{(1.05)^2} = 90.703.
\]

• They follow the binomial processes on p. 1029.
Numerical Examples (continued)

90.703 \leftrightarrow 92.593 \ (= 100/1.08)
   \downarrow 
   \downarrow 
98.039 \ (= 100/1.02) \leftrightarrow 96.154 \leftrightarrow 100 \leftrightarrow 100

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.
Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.

- Suppose all securities have the same expected one-period rate of return, the riskless rate.

- Then

\[(1 - p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\% ,\]

where \( p \) denotes the risk-neutral probability of a down move in rates.
Numerical Examples (concluded)

- Solving the equation leads to \( p = 0.319 \).

- Interest rate contingent claims can be priced under this probability.
Numerical Examples: Fixed-Income Options

• A one-year European call on the two-year zero with a $95 strike price has the payoffs,
  \[
  C \begin{cases} 
  0.000 \\
  3.039 
  \end{cases}
  \]

• To solve for the option value $C$, we replicate the call by a portfolio of $x$ one-year and $y$ two-year zeros.
Numerical Examples: Fixed-Income Options (continued)

- This leads to the simultaneous equations,

\[ x \times 100 + y \times 92.593 = 0.000, \]
\[ x \times 100 + y \times 98.039 = 3.039. \]

- They give \( x = -0.5167 \) and \( y = 0.5580 \).

- Consequently,

\[ C = x \times 96.154 + y \times 90.703 \approx 0.93 \]

to prevent arbitrage.
Numerical Examples: Fixed-Income Options (continued)

• This price is derived without assuming any version of an expectations theory.
• Instead, the arbitrage-free price is derived by replication.
• The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
• The dependence holds only indirectly via the current bond prices.
Numerical Examples: Fixed-Income Options (concluded)

• An equivalent method is to utilize risk-neutral pricing.
• The above call option is worth

\[
C = \frac{(1 - p) \times 0 + p \times 3.039}{1.04} \approx 0.93,
\]

the same as before.
• This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.
Numerical Examples: Futures and Forward Prices

• A one-year futures contract on the one-year rate has a payoff of 100 – r, where r is the one-year rate at maturity:

\[ F = 92 \ (= 100 - 8) \]
\[ F = 98 \ (= 100 - 2) \]

• As the futures price \( F \) is the expected future payoff,\(^a\)

\[
F = (1 - p) \times 92 + p \times 98 = 93.914.
\]

\(^a\)See Exercise 13.2.11 of the textbook or p. 522.
Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is\(^a\)

\[
90.703/96.154 = 94.331\%.
\]

- The forward price exceeds the futures price.\(^b\)

\(^a\)By Eq. (130) on p. 1006.
\(^b\)Recall p. 466.