Pricing Discrete Barrier Options

• Barrier options whose barrier is monitored only at discrete times are called discrete barrier options.

• They are more common than the continuously monitored versions.

• The main difficulty with pricing discrete barrier options lies in matching the monitored times.

• Here is why.
Pricing Discrete Barrier Options (continued)

• Suppose each period has a duration of $\Delta t$ and the $\ell > 1$ monitored times are

$$t_0 = 0, t_1, t_2, \ldots, t_\ell = T.$$ 

• It is unlikely that all monitored times coincide with the end of a period on the tree, or $\Delta t$ divides $t_i$ for all $i$.

• The binomial-trinomial tree can handle discrete options with ease, however.

• Simply build a binomial-trinomial tree from time 0 to time $t_1$, followed by one from time $t_1$ to time $t_2$, and so on until time $t_\ell$. 
\[ 2\sigma \sqrt{\Delta t_1} \]

\[ 2\sigma \sqrt{\Delta t_2} \]

\[ \Delta t'_1 \quad \Delta t_1 \quad \Delta t_1 \quad \Delta t'_2 \]

\[ \Delta t'_1 \quad \Delta t_1 \quad \Delta t_1 \quad \Delta t'_2 \]

\[ t_0 \quad t_1 \]
Pricing Discrete Barrier Options (concluded)

- This procedure works even if each $t_i$ is associated with a distinct barrier or if each window $[t_i, t_{i+1})$ has its own continuously monitored barrier or double barriers.
Options on a Stock That Pays Known Dividends

- Many ad hoc assumptions have been postulated for option pricing with known dividends.a
  1. The one we saw earlier (p. 305) models the stock price minus the present value of the anticipated dividends as following geometric Brownian motion.
  2. One can also model the stock price plus the forward values of the dividends as following geometric Brownian motion.

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aFrishling (2002).
Options on a Stock That Pays Known Dividends (continued)

- Realistic models assume:
  - The stock price decreases by the amount of the dividend paid at the ex-dividend date.
  - The dividend is part cash and part yield (i.e., $\alpha(t)S_0 + \beta(t)S_t$), for practitioners.$^a$

- The stock price follows geometric Brownian motion between adjacent ex-dividend dates.

- But they result in binomial trees that grow exponentially (recall p. 304).

- The binomial-trinomial tree can avoid this problem in most cases.

Options on a Stock That Pays Known Dividends (continued)

- Suppose that the known dividend is $D$ dollars and the ex-dividend date is at time $t$.
- So there are $m \triangleq t/\Delta t$ periods between time 0 and the ex-dividend date.\(^a\)
- To avoid negative stock prices, we need to make sure the lowest stock price at time $t$ is at least $D$, i.e.,
  $$S e^{- (t/\Delta t) \sigma \sqrt{\Delta t}} \geq D.$$
  - Or,
  $$\Delta t \geq \left[ \frac{t \sigma}{\ln(S/D)} \right]^2.$$

\(^a\)Or simply assume $m$ is an integer input and $\Delta t \triangleq t/m$. 
Options on a Stock That Pays Known Dividends (continued)

- Build a CRR tree from time 0 to time $t$ as before.

- Subtract $D$ from all the stock prices on the tree at time $t$ to represent the price drop on the ex-dividend date.

- Assume the top node’s price equals $S'$.
  - As usual, its two successor nodes will have prices $S'u$ and $S'u^{-1}$.

- The remaining nodes’ successor nodes will have prices $S'u^{-3}, S'u^{-5}, S'u^{-7}, \ldots$.

same as the CRR tree.
Options on a Stock That Pays Known Dividends (continued)

- For each node at time $t$ below the top node, we build the trinomial connection.

- Note that the binomial-trinomial structure remains valid in the special case when $\Delta t' = \Delta t$ on p. 703.
  - And even with the displacements $\pm 2\sigma \sqrt{\Delta t}$ (as on p. 731).
Options on a Stock That Pays Known Dividends (concluded)

- Hence the construction can be completed.

- From time $t + \Delta t$ onward, the standard binomial tree will be used until the maturity date or the next ex-dividend date when the procedure can be repeated.

- The resulting tree is called the stair tree.\(^a\)

\(^a\)Dai (B82506025, R86526008, D8852600) & Lyuu (2004); Dai (B82506025, R86526008, D8852600) (2009).
Other Applications of Binomial-Trinomial Trees

• Pricing guaranteed minimum withdrawal benefits.\(^a\)

• Option pricing with stochastic volatilities.\(^b\)

• Efficient Parisian option pricing.\(^c\)

• Option pricing with time-varying volatilities and time-varying barriers.\(^d\)

• Defaultable bond pricing.\(^e\)

\(^a\)H. Wu (R96723058) (2009).
\(^b\)C. Huang (R97922073) (2010).
\(^c\)Y. Huang (R97922081) (2010).
\(^d\)C. Chou (R97944012) (2010); C. Chen (R98922127) (2011).
General Properties of Trees$^a$

- Consider the Ito process,
  
  $$dX = a(X, t) \, dt + \sigma \, dW,$$

  where $a(X, t) = O(1)$ and $\sigma$ is a constant.

- The mean and volatility of the next move’s size are $O(\Delta t)$ and $O(\sqrt{\Delta t})$, respectively.

- Note that $\sqrt{\Delta t} \gg \Delta t$.

- The tree spacing must be in the order of $\sigma\sqrt{\Delta t}$ if the variance is to be matched.$^b$

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$^a$Chiu (R98723059) (2012); C. H. Wu (R99922149) (2012).

Merton’s Jump-Diffusion Model

- Empirically, stock returns tend to have fat tails, inconsistent with the Black-Scholes model’s assumptions.
- Stochastic volatility and jump processes have been proposed to address this problem.
- Merton’s (1976) jump-diffusion model is our focus.
Merton’s Jump-Diffusion Model (continued)

- This model superimposes a jump component on a diffusion component.
- The diffusion component is the familiar geometric Brownian motion.
- The jump component is composed of lognormal jumps driven by a Poisson process.
  - It models the sudden changes in the stock price because of the arrival of important new information.
Merton’s Jump-Diffusion Model (continued)

- Let $S_t$ be the stock price at time $t$.
- The risk-neutral jump-diffusion process for the stock price follows

$$\frac{dS_t}{S_t} = (r - \lambda \bar{k}) \, dt + \sigma \, dW_t + k \, dq_t. \quad (96)$$

- Above, $\sigma$ denotes the volatility of the diffusion component.
Merton’s Jump-Diffusion Model (continued)

- The jump event is governed by a compound Poisson process $q_t$ with intensity $\lambda$, where $k$ denotes the magnitude of the random jump.
  - The distribution of $k$ obeys
    \[ \ln(1 + k) \sim N(\gamma, \delta^2) \]
    with mean $\bar{k} \equiv E(k) = e^{\gamma + \delta^2/2} - 1$.

- The model with $\lambda = 0$ reduces to the Black-Scholes model.
Merton’s Jump-Diffusion Model (continued)

- The solution to Eq. (96) on p. 738 is

\[ S_t = S_0 e^{(r - \lambda \bar{k} - \sigma^2/2)t + \sigma W_t} U(n(t)), \tag{97} \]

where

\[ U(n(t)) = \prod_{i=0}^{n(t)} (1 + k_i). \]

- \( k_i \) is the magnitude of the \( i \)th jump with \( \ln(1 + k_i) \sim N(\gamma, \delta^2) \).
- \( k_0 = 0 \).
- \( n(t) \) is a Poisson process with intensity \( \lambda \).
Merton's Jump-Diffusion Model (concluded)

- Recall that $n(t)$ denotes the number of jumps that occur up to time $t$.
- As $k > -1$, stock prices will stay positive.
- The geometric Brownian motion, the lognormal jumps, and the Poisson process are assumed to be independent.
Tree for Merton’s Jump-Diffusion Model\textsuperscript{a}

- Define the $S$-logarithmic return of the stock price $S'$ as
  \[ \ln(S'/S). \]

- Define the logarithmic distance between stock prices $S'$ and $S$ as
  \[ | \ln(S') - \ln(S) | = | \ln(S'/S) |. \]

\textsuperscript{a}Dai (B82506025, R86526008, D8852600), C. Wang (F95922018), Lyuu, & Y. Liu (2010).
Tree for Merton’s Jump-Diffusion Model (continued)

- Take the logarithm of Eq. (97) on p. 740:

\[ M_t \triangleq \ln \left( \frac{S_t}{S_0} \right) = X_t + Y_t, \quad (98) \]

where

\[ X_t \triangleq \left( r - \lambda \bar{k} - \frac{\sigma^2}{2} \right) t + \sigma W_t, \quad (99) \]

\[ Y_t \triangleq \sum_{i=0}^{n(t)} \ln (1 + k_i). \quad (100) \]

- It decomposes the \( S_0 \)-logarithmic return of \( S_t \) into the diffusion component \( X_t \) and the jump component \( Y_t \).
Tree for Merton’s Jump-Diffusion Model (continued)

- Motivated by decomposition (98) on p. 743, the tree construction divides each period into a diffusion phase followed by a jump phase.

- In the diffusion phase, $X_t$ is approximated by the BOPM.

- So $X_t$ makes an up move to $X_t + \sigma \sqrt{\Delta t}$ with probability $p_u$ or a down move to $X_t - \sigma \sqrt{\Delta t}$ with probability $p_d$. 
Tree for Merton’s Jump-Diffusion Model (continued)

- According to BOPM,

\[ p_u = \frac{e^{\mu \Delta t} - d}{u - d}, \]
\[ p_d = 1 - p_u, \]

except that \( \mu = r - \lambda \bar{k} \) here.

- The diffusion component gives rise to diffusion nodes.

- They are spaced at \( 2\sigma \sqrt{\Delta t} \) apart such as the white nodes A, B, C, D, E, F, and G on p. 746.
White nodes are *diffusion nodes*. Gray nodes are *jump nodes*. In the diffusion phase, the solid black lines denote the binomial structure of BOPM; the dashed lines denote the trinomial structure. Here $m$ is set to one for simplicity. Only the double-circled nodes will remain after the construction. Note that $a$ and $b$ are diffusion nodes because no jump occurs in the jump phase.
Tree for Merton’s Jump-Diffusion Model (concluded)

- In the jump phase, $Y_{t+\Delta t}$ is approximated by moves from each diffusion node to $2m$ jump nodes that match the first $2m$ moments of the lognormal jump.

- The $m$ jump nodes above the diffusion node are spaced at $h \overset{\Delta}{=} \sqrt{\gamma^2 + \delta^2}$ apart.

- The same holds for the $m$ jump nodes below the diffusion node.

- The gray nodes at time $\ell\Delta t$ on p. 746 are jump nodes.

- The size of the tree is $O(n^{2.5})$. 
Multivariate Contingent Claims

- They depend on two or more underlying assets.
- The basket call on $m$ assets has the terminal payoff
  \[
  \max \left( \sum_{i=1}^{m} \alpha_i S_i(\tau) - X, 0 \right),
  \]
  where $\alpha_i$ is the percentage of asset $i$.
- Basket options are essentially options on a portfolio of stocks; they are index options.
- Option on the best of two risky assets and cash has a terminal payoff of $\max(S_1(\tau), S_2(\tau), X)$. 
### Multivariate Contingent Claims (concluded)\(^a\)

<table>
<thead>
<tr>
<th>Name</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exchange option</td>
<td>( \max(S_1(\tau) - S_2(\tau), 0) )</td>
</tr>
<tr>
<td>Better-off option</td>
<td>( \max(S_1(\tau), \ldots, S_k(\tau), 0) )</td>
</tr>
<tr>
<td>Worst-off option</td>
<td>( \min(S_1(\tau), \ldots, S_k(\tau), 0) )</td>
</tr>
<tr>
<td>Binary maximum option</td>
<td>( I{ \max(S_1(\tau), \ldots, S_k(\tau)) &gt; X } )</td>
</tr>
<tr>
<td>Maximum option</td>
<td>( \max(\max(S_1(\tau), \ldots, S_k(\tau)) - X, 0) )</td>
</tr>
<tr>
<td>Minimum option</td>
<td>( \max(\min(S_1(\tau), \ldots, S_k(\tau)) - X, 0) )</td>
</tr>
<tr>
<td>Spread option</td>
<td>( \max(S_1(\tau) - S_2(\tau) - X, 0) )</td>
</tr>
<tr>
<td>Basket average option</td>
<td>( \max((S_1(\tau) + \cdots + S_k(\tau))/k - X, 0) )</td>
</tr>
<tr>
<td>Multi-strike option</td>
<td>( \max(S_1(\tau) - X_1, \ldots, S_k(\tau) - X_k, 0) )</td>
</tr>
<tr>
<td>Pyramid rainbow option</td>
<td>( \max(</td>
</tr>
<tr>
<td>Madonna option</td>
<td>( \max(\sqrt{(S_1(\tau) - X_1)^2 + \cdots + (S_k(\tau) - X_k)^2} - X, 0) )</td>
</tr>
</tbody>
</table>

\(^a\)Lyuu & Teng (R91723054) (2011).
Correlated Trinomial Model\textsuperscript{a}

- Two risky assets $S_1$ and $S_2$ follow

$$\frac{dS_i}{S_i} = r \, dt + \sigma_i \, dW_i$$

in a risk-neutral economy, $i = 1, 2$.

- Let

$$M_i \triangleq e^{r \Delta t},$$

$$V_i \triangleq M_i^2 (e^{\sigma_i^2 \Delta t} - 1).$$

- $S_i M_i$ is the mean of $S_i$ at time $\Delta t$.
- $S_i^2 V_i$ the variance of $S_i$ at time $\Delta t$.

\textsuperscript{a}Boyle, Evnine, & Gibbs (1989).
Correlated Trinomial Model (continued)

- The value of $S_1 S_2$ at time $\Delta t$ has a joint lognormal distribution with mean $S_1 S_2 M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$, where $\rho$ is the correlation between $dW_1$ and $dW_2$.

- Next match the 1st and 2nd moments of the approximating discrete distribution to those of the continuous counterpart.

- At time $\Delta t$ from now, there are 5 distinct outcomes.
Correlated Trinomial Model (continued)

- The five-point probability distribution of the asset prices is

<table>
<thead>
<tr>
<th>Probability</th>
<th>Asset 1</th>
<th>Asset 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 )</td>
<td>( S_1 u_1 )</td>
<td>( S_2 u_2 )</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>( S_1 u_1 )</td>
<td>( S_2 d_2 )</td>
</tr>
<tr>
<td>( p_3 )</td>
<td>( S_1 d_1 )</td>
<td>( S_2 d_2 )</td>
</tr>
<tr>
<td>( p_4 )</td>
<td>( S_1 d_1 )</td>
<td>( S_2 u_2 )</td>
</tr>
<tr>
<td>( p_5 )</td>
<td>( S_1 )</td>
<td>( S_2 )</td>
</tr>
</tbody>
</table>

- As usual, impose \( u_i d_i = 1 \).
Correlated Trinomial Model (continued)

- The probabilities must sum to one, and the means must be matched:

\[
1 = p_1 + p_2 + p_3 + p_4 + p_5, \\
S_1 M_1 = (p_1 + p_2) S_1 u_1 + p_5 S_1 + (p_3 + p_4) S_1 d_1, \\
S_2 M_2 = (p_1 + p_4) S_2 u_2 + p_5 S_2 + (p_2 + p_3) S_2 d_2.
\]
Correlated Trinomial Model (concluded)

• Let $R \triangleq M_1 M_2 e^{\rho \sigma_1 \sigma_2 \Delta t}$.

• Match the variances and covariance:

\[
S_1^2 V_1 = (p_1 + p_2)((S_1 u_1)^2 - (S_1 M_1)^2) + p_5(S_1^2 - (S_1 M_1)^2) \\
\quad + (p_3 + p_4)((S_1 d_1)^2 - (S_1 M_1)^2),
\]

\[
S_2^2 V_2 = (p_1 + p_4)((S_2 u_2)^2 - (S_2 M_2)^2) + p_5(S_2^2 - (S_2 M_2)^2) \\
\quad + (p_2 + p_3)((S_2 d_2)^2 - (S_2 M_2)^2),
\]

\[
S_1 S_2 R = (p_1 u_1 u_2 + p_2 u_1 d_2 + p_3 d_1 d_2 + p_4 d_1 u_2 + p_5) S_1 S_2.
\]

• The solutions appear on p. 246 of the textbook.
Correlated Trinomial Model Simplified\(^a\)

- Let \(\mu'_i \equiv r - \sigma^2_i / 2\) and \(u_i \equiv e^{\lambda \sigma_i \sqrt{\Delta t}}\) for \(i = 1, 2\).
- The following simpler scheme is good enough:

\[
\begin{align*}
p_1 &= \frac{1}{4} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu'_1}{\sigma_1} - \frac{\mu'_2}{\sigma_2} \right) + \frac{\rho}{\lambda^2} \right], \\
p_2 &= \frac{1}{4} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu'_1}{\sigma_1} + \frac{\mu'_2}{\sigma_2} \right) + \frac{\rho}{\lambda^2} \right], \\
p_3 &= \frac{1}{4} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu'_1}{\sigma_1} - \frac{\mu'_2}{\sigma_2} \right) - \frac{\rho}{\lambda^2} \right], \\
p_4 &= \frac{1}{4} \left[ \frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda} \left( \frac{\mu'_1}{\sigma_1} + \frac{\mu'_2}{\sigma_2} \right) - \frac{\rho}{\lambda^2} \right], \\
p_5 &= 1 - \frac{1}{\lambda^2}. \end{align*}
\]

\(^a\)Madan, Milne, & Shefrin (1989).
Correlated Trinomial Model Simplified (continued)

- All of the probabilities lie between 0 and 1 if and only if
  \[-1 + \lambda \sqrt{\Delta t} \left| \frac{\mu'_1}{\sigma_1} + \frac{\mu'_2}{\sigma_2} \right| \leq \rho \leq 1 - \lambda \sqrt{\Delta t} \left| \frac{\mu'_1}{\sigma_1} - \frac{\mu'_2}{\sigma_2} \right|\]

  (101)

  \[1 \leq \lambda\]

  (102)

- We call a multivariate tree (correlation-) optimal if it guarantees valid probabilities as long as
  \[-1 + O(\sqrt{\Delta t}) < \rho < 1 - O(\sqrt{\Delta t}),\]

  such as the above one.\(^a\)

\(^a\)W. Kao (R98922093) (2011); W. Kao (R98922093), Lyuu, & Wen (D94922003) (2014).
Correlated Trinomial Model Simplified (concluded)

• But this model cannot price 2-asset 2-barrier options accurately.\(^a\)

• Few multivariate trees are both optimal and able to handle multiple barriers.\(^b\)

• An alternative is to use orthogonalization.\(^c\)

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\(^a\)See Y. Chang (B89704039, R93922034), Hsu (R7526001, D89922012), & Lyuu (2006); W. Kao (R98922093), Lyuu, & Wen (D94922003) (2014) for solutions.

\(^b\)See W. Kao (R98922093), Lyuu, & Wen (D94922003) (2014) for one.

\(^c\)Hull & White (1990); Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2012).
Extrapolation

- It is a method to speed up numerical convergence.
- Say $f(n)$ converges to an unknown limit $f$ at rate of $1/n$:

$$f(n) = f + \frac{c}{n} + o\left(\frac{1}{n}\right). \quad (103)$$

- Assume $c$ is an unknown constant independent of $n$.
  - Convergence is basically monotonic and smooth.
Extrapolation (concluded)

- From two approximations \( f(n_1) \) and \( f(n_2) \) and ignoring the smaller terms,

\[
\begin{align*}
  f(n_1) &= f + \frac{c}{n_1}, \\
  f(n_2) &= f + \frac{c}{n_2}.
\end{align*}
\]

- A better approximation to the desired \( f \) is

\[
f = \frac{n_1 f(n_1) - n_2 f(n_2)}{n_1 - n_2}.
\]

(104)

- This estimate should converge faster than \( 1/n \).\(^a\)

- The Richardson extrapolation uses \( n_2 = 2n_1 \).

\(^a\)It is identical to the forward rate formula (21) on p. 140!
Improving BOPM with Extrapolation

- Consider standard European options.
- Denote the option value under BOPM using $n$ time periods by $f(n)$.
- It is known that BOPM convergences at the rate of $1/n$, consistent with Eq. (103) on p. 758.
- But the plots on p. 288 (redrawn on next page) demonstrate that convergence to the true option value oscillates with $n$.
- Extrapolation is inapplicable at this stage.
Improving BOPM with Extrapolation (concluded)

- Take the at-the-money option in the left plot on p. 761.
- The sequence with odd $n$ turns out to be monotonic and smooth (see the left plot on p. 763).\(^a\)
- Apply extrapolation (104) on p. 759 with $n_2 = n_1 + 2$, where $n_1$ is odd.
- Result is shown in the right plot on p. 763.
- The convergence rate is amazing.
- See Exercise 9.3.8 of the text (p. 111) for ideas in the general case.

\(^a\)This can be proved (L. Chang & Palmer, 2007).
Numerical Methods
All science is dominated by the idea of approximation.
— Bertrand Russell
Finite-Difference Methods

• Place a grid of points on the space over which the desired function takes value.

• Then approximate the function value at each of these points (p. 767).

• Solve the equation numerically by introducing difference equations in place of derivatives.
Example: Poisson’s Equation

- It is $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = -\rho(x, y)$, which describes the electrostatic field.

- Replace second derivatives with finite differences through central difference.

- Introduce evenly spaced grid points with distance of $\Delta x$ along the $x$ axis and $\Delta y$ along the $y$ axis.

- The finite difference form is

$$
-\rho(x_i, y_j) = \frac{\theta(x_{i+1}, y_j) - 2\theta(x_i, y_j) + \theta(x_{i-1}, y_j)}{(\Delta x)^2} + \frac{\theta(x_i, y_{j+1}) - 2\theta(x_i, y_j) + \theta(x_i, y_{j-1})}{(\Delta y)^2}.
$$
Example: Poisson’s Equation (concluded)

• In the above, \( \Delta x \equiv x_i - x_{i-1} \) and \( \Delta y \equiv y_j - y_{j-1} \) for \( i, j = 1, 2, \ldots \).

• When the grid points are evenly spaced in both axes so that \( \Delta x = \Delta y = h \), the difference equation becomes

\[
-h^2 \rho(x_i, y_j) = \theta(x_{i+1}, y_j) + \theta(x_{i-1}, y_j) \\
+ \theta(x_i, y_{j+1}) + \theta(x_i, y_{j-1}) - 4\theta(x_i, y_j).
\]

• Given boundary values, we can solve for the \( x_i \)'s and the \( y_j \)'s within the square \( [\pm L, \pm L] \).

• From now on, \( \theta_{i,j} \) will denote the finite-difference approximation to the exact \( \theta(x_i, y_j) \).
Explicit Methods

• Consider the diffusion equation
  \[ D\left( \frac{\partial^2 \theta}{\partial x^2} \right) - \left( \frac{\partial \theta}{\partial t} \right) = 0, \quad D > 0. \]

• Use evenly spaced grid points \((x_i, t_j)\) with distances \(\Delta x\) and \(\Delta t\), where \(\Delta x \triangleq x_{i+1} - x_i\) and \(\Delta t \triangleq t_{j+1} - t_j\).

• Employ central difference for the second derivative and forward difference for the time derivative to obtain
  \[
  \left. \frac{\partial \theta(x, t)}{\partial t} \right|_{t=t_j} = \frac{\theta(x, t_{j+1}) - \theta(x, t_j)}{\Delta t} + \cdots, \quad (105)
  \]
  \[
  \left. \frac{\partial^2 \theta(x, t)}{\partial x^2} \right|_{x=x_i} = \frac{\theta(x_{i+1}, t) - 2\theta(x_i, t) + \theta(x_{i-1}, t)}{(\Delta x)^2} + \cdots \quad (106)
  \]
Explicit Methods (continued)

- Next, assemble Eqs. (105) and (106) into a single equation at \((x_i, t_j)\).
- But we need to decide how to evaluate \(x\) in the first equation and \(t\) in the second.
- Since central difference around \(x_i\) is used in Eq. (106), we might as well use \(x_i\) for \(x\) in Eq. (105).
- Two choices are possible for \(t\) in Eq. (106).
- The first choice uses \(t = t_j\) to yield the following finite-difference equation,

\[
\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2}.
\]

(107)
Explicit Methods (continued)

- The stencil of grid points involves four values, \( \theta_{i,j+1} \), \( \theta_{i,j} \), \( \theta_{i+1,j} \), and \( \theta_{i-1,j} \).

- Rearrange Eq. (107) on p. 771 as

\[
\theta_{i,j+1} = \frac{D \Delta t}{(\Delta x)^2} \theta_{i+1,j} + \left(1 - \frac{2D \Delta t}{(\Delta x)^2}\right) \theta_{i,j} + \frac{D \Delta t}{(\Delta x)^2} \theta_{i-1,j}.
\]

- We can calculate \( \theta_{i,j+1} \) from \( \theta_{i,j} \), \( \theta_{i+1,j} \), \( \theta_{i-1,j} \), at the previous time \( t_j \) (see exhibit (a) on next page).
Stencils

(a)

(b)
Explicit Methods (concluded)

• Starting from the initial conditions at \( t_0 \), that is, 
  \[ \theta_{i,0} = \theta(x_i, t_0), \quad i = 1, 2, \ldots, \]
  we calculate

  \[ \theta_{i,1}, \quad i = 1, 2, \ldots. \]

• And then

  \[ \theta_{i,2}, \quad i = 1, 2, \ldots. \]

• And so on.
Stability

• The explicit method is numerically unstable unless

\[ \Delta t \leq (\Delta x)^2/(2D). \]

− A numerical method is unstable if the solution is highly sensitive to changes in initial conditions.

• The stability condition may lead to high running times and memory requirements.

• For instance, halving \( \Delta x \) would imply quadrupling \( (\Delta t)^{-1} \), resulting in a running time 8 times as much.
Explicit Method and Trinomial Tree

• Recall that
\[ \theta_{i,j+1} = \frac{D\Delta t}{(\Delta x)^2} \theta_{i+1,j} + \left(1 - \frac{2D\Delta t}{(\Delta x)^2}\right) \theta_{i,j} + \frac{D\Delta t}{(\Delta x)^2} \theta_{i-1,j}. \]

• When the stability condition is satisfied, the three coefficients for \( \theta_{i+1,j}, \theta_{i,j}, \) and \( \theta_{i-1,j} \) all lie between zero and one and sum to one.

• They can be interpreted as probabilities.

• So the finite-difference equation becomes identical to backward induction on trinomial trees!
Explicit Method and Trinomial Tree (concluded)

• The freedom in choosing $\Delta x$ corresponds to similar freedom in the construction of trinomial trees.

• The explicit finite-difference equation is also identical to backward induction on a binomial tree.$^{a}$
  – Let the binomial tree take 2 steps each of length $\Delta t/2$.
  – It is now a trinomial tree.

$^{a}$Hilliard (2014).
Implicit Methods

• Suppose we use $t = t_{j+1}$ in Eq. (106) on p. 770 instead.

• The finite-difference equation becomes

\[
\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = D \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2}.
\]  

(108)

• The stencil involves $\theta_{i,j}$, $\theta_{i,j+1}$, $\theta_{i+1,j+1}$, and $\theta_{i-1,j+1}$.

• This method is implicit:
  – The value of any one of the three quantities at $t_{j+1}$ cannot be calculated unless the other two are known.
  – See exhibit (b) on p. 773.
Implicit Methods (continued)

- Equation (108) can be rearranged as

\[
\theta_{i-1,j+1} - (2 + \gamma) \theta_{i,j+1} + \theta_{i+1,j+1} = -\gamma \theta_{i,j},
\]

where \( \gamma \equiv (\Delta x)^2 / (D \Delta t) \).

- This equation is unconditionally stable.

- Suppose the boundary conditions are given at \( x = x_0 \) and \( x = x_{N+1} \).

- After \( \theta_{i,j} \) has been calculated for \( i = 1, 2, \ldots, N \), the values of \( \theta_{i,j+1} \) at time \( t_{j+1} \) can be computed as the solution to the following tridiagonal linear system,
Implicit Methods (continued)

\[
\begin{bmatrix}
  a & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
 1 & a & 1 & 0 & \cdots & \cdots & 0 \\
 0 & 1 & a & 1 & 0 & \cdots & 0 \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\begin{bmatrix}
  \theta_{1,j+1} \\
  \theta_{2,j+1} \\
  \theta_{3,j+1} \\
  \vdots \\
  \vdots \\
  \vdots \\
  \theta_{N,j+1} \\
\end{bmatrix}
= 
\begin{bmatrix}
  -\gamma \theta_{1,j} - \theta_{0,j+1} \\
  -\gamma \theta_{2,j} \\
  -\gamma \theta_{3,j} \\
  \vdots \\
  \vdots \\
  \vdots \\
  -\gamma \theta_{N-1,j} \\
  -\gamma \theta_{N,j} - \theta_{N+1,j+1} \\
\end{bmatrix},
\]

where \( a \triangleq -2 - \gamma \).
Implicit Methods (concluded)

- Tridiagonal systems can be solved in $O(N)$ time and $O(N)$ space.
  - Never invert a matrix to solve a tridiagonal system.
- The matrix above is nonsingular when $\gamma \geq 0$.
  - A square matrix is nonsingular if its inverse exists.
Crank-Nicolson Method

• Take the average of explicit method (107) on p. 771 and implicit method (108) on p. 778:

\[
\frac{\theta_{i,j+1} - \theta_{i,j}}{\Delta t} = \frac{1}{2} \left( D \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{(\Delta x)^2} + D \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{(\Delta x)^2} \right).
\]

• After rearrangement,

\[
\gamma \theta_{i,j+1} - \frac{\theta_{i+1,j+1} - 2\theta_{i,j+1} + \theta_{i-1,j+1}}{2} = \gamma \theta_{i,j} + \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{2}.
\]

• This is an unconditionally stable implicit method with excellent rates of convergence.
Numerically Solving the Black-Scholes PDE (82) on p. 629

- See text.
- Brennan and Schwartz (1978) analyze the stability of the implicit method.
Monte Carlo Simulation\textsuperscript{a}

- Monte Carlo simulation is a sampling scheme.
- In many important applications within finance and without, Monte Carlo is one of the few feasible tools.
- When the time evolution of a stochastic process is not easy to describe analytically, Monte Carlo may very well be the only strategy that succeeds consistently.

\textsuperscript{a}A top 10 algorithm (Dongarra & Sullivan, 2000).
The Big Idea

• Assume $X_1, X_2, \ldots, X_n$ have a joint distribution.

• $\theta \triangleq E[g(X_1, X_2, \ldots, X_n)]$ for some function $g$ is desired.

• We generate

$$
\left( x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)} \right), \quad 1 \leq i \leq N
$$

independently with the same joint distribution as $(X_1, X_2, \ldots, X_n)$.

• Set

$$
Y_i \triangleq g \left( x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)} \right).
$$
The Big Idea (concluded)

- $Y_1, Y_2, \ldots, Y_N$ are independent and identically distributed random variables.

- Each $Y_i$ has the same distribution as $Y \overset{\Delta}{=} g(X_1, X_2, \ldots, X_n)$.

- Since the average of these $N$ random variables, $\overline{Y}$, satisfies $E[\overline{Y}] = \theta$, it can be used to estimate $\theta$.

- The strong law of large numbers says that this procedure converges almost surely.

- The number of replications (or independent trials), $N$, is called the sample size.
Accuracy

- The Monte Carlo estimate and true value may differ owing to two reasons:
  1. Sampling variation.
  2. The discreteness of the sample paths.\(^a\)
- The first can be controlled by the number of replications.
- The second can be controlled by the number of observations along the sample path.

\(^a\)This may not be an issue if the financial derivative only requires discrete sampling along the time dimension, such as the discrete barrier option.
Accuracy and Number of Replications

- The statistical error of the sample mean $\overline{Y}$ of the random variable $Y$ grows as $1/\sqrt{N}$.
  - Because $\text{Var}[\overline{Y}] = \text{Var}[Y]/N$.

- In fact, this convergence rate is asymptotically optimal.\(^a\)

- So the variance of the estimator $\overline{Y}$ can be reduced by a factor of $1/N$ by doing $N$ times as much work.

- This is amazing because the same order of convergence holds independently of the dimension $n$.

\(^a\)The Berry-Esseen theorem.
Accuracy and Number of Replications (concluded)

- In contrast, classic numerical integration schemes have an error bound of $O(N^{-c/n})$ for some constant $c > 0$.
  - $n$ is the dimension.

- The required number of evaluations thus grows exponentially in $n$ to achieve a given level of accuracy.
  - The curse of dimensionality.

- The Monte Carlo method is more efficient than alternative procedures for multivariate derivatives when $n$ is large.
Monte Carlo Option Pricing

• For the pricing of European options on a dividend-paying stock, we may proceed as follows.

• Assume

\[
\frac{dS}{S} = \mu \, dt + \sigma \, dW.
\]

• Stock prices \( S_1, S_2, S_3, \ldots \) at times \( \Delta t, 2\Delta t, 3\Delta t, \ldots \) can be generated via

\[
S_{i+1} = S_i e^{(\mu - \sigma^2/2) \Delta t + \sigma \sqrt{\Delta t} \xi}, \quad \xi \sim N(0, 1).
\]

(109)
Monte Carlo Option Pricing (continued)

- If we discretize $dS/S = \mu \, dt + \sigma \, dW$ directly, we will obtain

$$S_{i+1} = S_i + S_i \mu \Delta t + S_i \sigma \sqrt{\Delta t} \, \xi.$$

- But this is locally normally distributed, not lognormally, hence biased.\(^a\)

- In practice, this is not expected to be a major problem as long as $\Delta t$ is sufficiently small.

\(^a\)Contributed by Mr. Tai, Hui-Chin (R97723028) on April 22, 2009.
Monte Carlo Option Pricing (continued)

- Non-dividend-paying stock prices in a risk-neutral economy can be generated by setting $\mu = r$ and $\Delta t = T$.

1: $C := 0$; \{Accumulated terminal option value.\}
2: for $i = 1, 2, 3, \ldots , N$ do
3: \quad $P := S \times e^{(r-\sigma^2/2)T+\sigma \sqrt{T}\xi}$, $\xi \sim N(0, 1)$;
4: \quad $C := C + \max(P - X, 0)$;
5: end for
6: return $Ce^{-rT}/N$;
Monte Carlo Option Pricing (concluded)

- Pricing Asian options is also easy.

1. $C := 0$;
2. for $i = 1, 2, 3, \ldots, N$ do
3. \hspace{1em} $P := S$; $M := S$;
4. \hspace{1em} for $j = 1, 2, 3, \ldots, n$ do
5. \hspace{2em} $P := P \times e^{(r-\sigma^2/2)(T/n)+\sigma\sqrt{T/n} \xi}$;
6. \hspace{2em} $M := M + P$;
7. \hspace{1em} end for
8. \hspace{1em} $C := C + \max(M/(n+1) - X, 0)$;
9. end for
10. return $Ce^{-rT}/N$;
How about American Options?

- Standard Monte Carlo simulation is inappropriate for American options because of early exercise (why?).
  - Given a sample path $S_0, S_1, \ldots, S_n$, how to decide which $S_i$ is an early-exercise point?
  - What is the option price at each $S_i$ if the option is not exercised?

- It is difficult to determine the early-exercise point based on one single path.

- But Monte Carlo simulation can be modified to price American options with small biases (pp. 847ff).\(^a\)

\(^a\)Longstaff & Schwartz (2001).