Futures Price under the BOPM

- Futures prices form a martingale under the risk-neutral probability.
  - The expected futures price in the next period is\(^a\)
    \[
    p_F F u + (1 - p_F) F d = F \left( \frac{1 - d}{u - d} u + \frac{u - 1}{u - d} d \right) = F.
    \]
- Can be generalized to
  \[
  F_i = E_i^\pi[F_k], \quad i \leq k,
  \]
  where \(F_i\) is the futures price at time \(i\).
- This equation holds under stochastic interest rates, too.\(^b\)

\(^a\)Recall p. 476.
\(^b\)See Exercise 13.2.11 of the textbook.
Martingale Pricing and Numeraire

- The martingale pricing formula (67) on p. 518 uses the money market account as numeraire.
  - It expresses the price of any asset relative to the money market account.

- The money market account is not the only choice for numeraire.

- Suppose asset $S$’s value is positive at all times.

---

*a* John Law (1671–1729), “Money to be qualified for exchanging goods and for payments need not be certain in its value.”

*b* Leon Walras (1834–1910).
Martingale Pricing and Numeraire (concluded)

- Choose $S$ as numeraire.

- Martingale pricing says there exists a risk-neutral probability $\pi$ under which the relative price of any asset $C$ is a martingale:

$$
\frac{C(i)}{S(i)} = E^\pi_i \left[ \frac{C(k)}{S(k)} \right], \quad i \leq k.
$$

- $S(j)$ denotes the price of $S$ at time $j$.

- So the discount process remains a martingale.$^a$

---

$^a$This result is related to Girsanov’s theorem (1960).
Example

- Take the binomial model with two assets.
- In a period, asset one’s price can go from $S$ to $S_1$ or $S_2$.
- In a period, asset two’s price can go from $P$ to $P_1$ or $P_2$.
- Both assets must move up or down at the same time.
- Assume

\[
\frac{S_1}{P_1} < \frac{S}{P} < \frac{S_2}{P_2}
\]  

(68)

to rule out arbitrage opportunities.
Example (continued)

• For any derivative security, let \( C_1 \) be its price at time one if asset one’s price moves to \( S_1 \).

• Let \( C_2 \) be its price at time one if asset one’s price moves to \( S_2 \).

• Replicate the derivative by solving

\[
\begin{align*}
\alpha S_1 + \beta P_1 &= C_1, \\
\alpha S_2 + \beta P_2 &= C_2,
\end{align*}
\]

using \( \alpha \) units of asset one and \( \beta \) units of asset two.
Example (continued)

• By Eqs. (68) on p. 524, $\alpha$ and $\beta$ have unique solutions.

• In fact,

$$
\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2} \quad \text{and} \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}.
$$

• The derivative costs

$$
C = \alpha S + \beta P
= \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2.
$$
Example (continued)

• It is easy to verify that

$$\frac{C}{P} = p \frac{C_1}{P_1} + (1 - p) \frac{C_2}{P_2}.$$ 

  – Above,

$$p \triangleq \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}.$$ 

  – By Eqs. (68) on p. 524, $0 < p < 1$.

• $C$’s price using asset two as numeraire (i.e., $C/P$) is a martingale under the risk-neutral probability $p$.

• The expected returns of the two assets are \textit{irrelevant}. 
Example (concluded)

• In the BOPM, $S$ is the stock and $P$ is the bond.

• Furthermore, $p$ assumes the bond is the numeraire.

• In the binomial option pricing formula (p. 255), the $S \sum b(j; n, pu/R)$ term uses the stock as the numeraire.
  
  – It results in a different probability measure $pu/R$.

• In the limit, $SN(x)$ for the call and $SN(-x)$ for the put in the Black-Scholes formula (p. 285) use the stock as the numeraire.a

---

aSee Exercise 13.2.12 of the textbook.
Brownian Motion$^a$

- Brownian motion is a stochastic process \( \{ X(t), t \geq 0 \} \) with the following properties.

1. \( X(0) = 0 \), unless stated otherwise.
2. for any \( 0 \leq t_0 < t_1 < \cdots < t_n \), the random variables \( X(t_k) - X(t_{k-1}) \) for \( 1 \leq k \leq n \) are independent.$^b$
3. for \( 0 \leq s < t \), \( X(t) - X(s) \) is normally distributed with mean \( \mu(t - s) \) and variance \( \sigma^2(t - s) \), where \( \mu \) and \( \sigma \neq 0 \) are real numbers.

$^a$Robert Brown (1773–1858).

$^b$So \( X(t) - X(s) \) is independent of \( X(r) \) for \( r \leq s < t \).
Brownian Motion (concluded)

- The existence and uniqueness of such a process is guaranteed by Wiener’s theorem.\(^a\)

- This process will be called a \((\mu, \sigma)\) Brownian motion with drift \(\mu\) and variance \(\sigma^2\).

- Although Brownian motion is a continuous function of \(t\) with probability one, it is almost nowhere differentiable.

- The \((0, 1)\) Brownian motion is called the Wiener process.

- If condition 3 is replaced by “\(X(t) - X(s)\) depends only on \(t - s\),” we have the more general Levy process.\(^b\)

---

\(^a\)Norbert Wiener (1894–1964). He received his Ph.D. from Harvard in 1912.

\(^b\)Paul Levy (1886–1971).
Example

• If \( \{ X(t), t \geq 0 \} \) is the Wiener process, then

\[
X(t) - X(s) \sim N(0, t - s).
\]

• A \((\mu, \sigma)\) Brownian motion \( Y = \{ Y(t), t \geq 0 \} \) can be expressed in terms of the Wiener process:

\[
Y(t) = \mu t + \sigma X(t).
\]  \hfill (69)

• Note that

\[
Y(t + s) - Y(t) \sim N(\mu s, \sigma^2 s).
\]
Brownian Motion as Limit of Random Walk

Claim 1 A $(\mu, \sigma)$ Brownian motion is the limiting case of random walk.

- A particle moves $\Delta x$ to the right with probability $p$ after $\Delta t$ time.
- It moves $\Delta x$ to the left with probability $1 - p$.
- Define

  $$X_i \triangleq \begin{cases} 
+1 & \text{if the } i\text{th move is to the right,} \\
-1 & \text{if the } i\text{th move is to the left.}
\end{cases}$$

  $X_i$ are independent with

  $$\text{Prob}[X_i = 1] = p = 1 - \text{Prob}[X_i = -1].$$
Brownian Motion as Limit of Random Walk (continued)

• Assume \( n \triangleq t/\Delta t \) is an integer.

• Its position at time \( t \) is

\[
Y(t) \triangleq \Delta x (X_1 + X_2 + \cdots + X_n).
\]

• Recall

\[
E[X_i] = 2p - 1,
\]

\[
\text{Var}[X_i] = 1 - (2p - 1)^2.
\]
Brownian Motion as Limit of Random Walk (continued)

- Therefore,

\[ E[Y(t)] = n(\Delta x)(2p - 1), \]
\[ \text{Var}[Y(t)] = n(\Delta x)^2 \left[ 1 - (2p - 1)^2 \right]. \]

- With \( \Delta x \overset{\Delta}{=} \sigma \sqrt{\Delta t} \) and \( p \overset{\Delta}{=} \left[ 1 + (\mu/\sigma)\sqrt{\Delta t} \right]/2, \)

\[ E[Y(t)] = n\sigma \sqrt{\Delta t} (\mu/\sigma)\sqrt{\Delta t} = \mu t, \]
\[ \text{Var}[Y(t)] = n\sigma^2 \Delta t \left[ 1 - (\mu/\sigma)^2 \Delta t \right] \to \sigma^2 t, \]

as \( \Delta t \to 0. \)

\(^a\)Identical to Eq. (38) on p. 278!
Brownian Motion as Limit of Random Walk (concluded)

• Thus, \( \{ Y(t), t \geq 0 \} \) converges to a \((\mu, \sigma)\) Brownian motion by the central limit theorem.

• Brownian motion with zero drift is the limiting case of symmetric random walk by choosing \( \mu = 0 \).

• Similarity to the BOPM: The \( p \) is identical to the probability in Eq. (38) on p. 278 and \( \Delta x = \ln u \).

• Note that

\[
\text{Var}[Y(t + \Delta t) - Y(t)] = \text{Var}[\Delta x X_{n+1}] = (\Delta x)^2 \times \text{Var}[X_{n+1}] \rightarrow \sigma^2 \Delta t.
\]
Geometric Brownian Motion

• Let \( X \overset{\Delta}{=} \{ X(t), t \geq 0 \} \) be a Brownian motion process.

• The process

\[
\{ Y(t) \overset{\Delta}{=} e^{X(t)}, t \geq 0 \},
\]

is called geometric Brownian motion.

• Suppose further that \( X \) is a \((\mu, \sigma)\) Brownian motion.

• \( X(t) \sim N(\mu t, \sigma^2 t) \) with moment generating function

\[
E \left[ e^{sX(t)} \right] = E \left[ Y(t)^s \right] = e^{\mu ts + (\sigma^2 ts^2 / 2)}
\]

from Eq. (25) on p 158.
Geometric Brownian Motion (concluded)

• In particular,

\[ E[Y(t)] = e^{\mu t + (\sigma^2 t/2)}, \]
\[ \text{Var}[Y(t)] = E\left[Y(t)^2\right] - E[Y(t)]^2 = e^{2\mu t + \sigma^2 t} \left(e^{\sigma^2 t} - 1\right). \]
A Case for Long-Term Investment

• Suppose the stock follows the geometric Brownian motion

\[ S(t) = S(0) e^{N(\mu t, \sigma^2 t)} = S(0) e^{tN(\mu, \sigma^2 / t)}, \quad t \geq 0, \]

where \( \mu > 0 \).

• The annual rate of return has a normal distribution:

\[ N \left( \mu, \frac{\sigma^2}{t} \right). \]

• The larger the \( t \), the likelier the return is positive.

• The smaller the \( t \), the likelier the return is negative.

Continuous-Time Financial Mathematics
A proof is that which convinces a reasonable man; a rigorous proof is that which convinces an unreasonable man.
— Mark Kac (1914–1984)

The pursuit of mathematics is a divine madness of the human spirit.
— Alfred North Whitehead (1861–1947), *Science and the Modern World*
Stochastic Integrals

- Use $W \triangleq \{ W(t), t \geq 0 \}$ to denote the Wiener process.

- The goal is to develop integrals of $X$ from a class of stochastic processes,

$$I_t(X) \triangleq \int_0^t X \, dW, \quad t \geq 0.$$ 

- $I_t(X)$ is a random variable called the stochastic integral of $X$ with respect to $W$.

- The stochastic process $\{I_t(X), t \geq 0\}$ will be denoted by $\int X \, dW$.

\[\text{footnote}{Kiyoshi Ito (1915–2008).}\]
Stochastic Integrals (concluded)

- Typical requirements for $X$ in financial applications are:
  - $\text{Prob}[\int_0^t X^2(s) \, ds < \infty] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E[X^2(s)] \, ds < \infty$.
  - The information set at time $t$ includes the history of $X$ and $W$ up to that point in time.
  - But it contains nothing about the evolution of $X$ or $W$ after $t$ (nonanticipating, so to speak).
  - The future cannot influence the present.
Ito Integral

• A theory of stochastic integration.

• As with calculus, it starts with step functions.

• A stochastic process \( \{ X(t) \} \) is simple if there exist

\[
0 = t_0 < t_1 < \cdots
\]

such that

\[
X(t) = X(t_{k-1}) \quad \text{for } t \in [t_{k-1}, t_k), \; k = 1, 2, \ldots
\]

for any realization (see figure on next page).
Ito Integral (continued)

- The Ito integral of a simple process is defined as

\[ I_t(X) \triangleq \sum_{k=0}^{n-1} X(t_k)[W(t_{k+1}) - W(t_k)], \]

(70)

where \( t_n = t \).

- The integrand \( X \) is evaluated at \( t_k \), not \( t_{k+1} \).

- Define the Ito integral of more general processes as a limiting random variable of the Ito integral of simple stochastic processes.
Ito Integral (continued)

• Let \( X = \{ X(t), t \geq 0 \} \) be a general stochastic process.

• Then there exists a random variable \( I_t(X) \), unique almost certainly, such that \( I_t(X_n) \) converges in probability to \( I_t(X) \) for each sequence of simple stochastic processes \( X_1, X_2, \ldots \) such that \( X_n \) converges in probability to \( X \).

• If \( X \) is continuous with probability one, then \( I_t(X_n) \) converges in probability to \( I_t(X) \) as

\[
\delta_n \triangleq \max_{1 \leq k \leq n} (t_k - t_{k-1})
\]

goes to zero.
Ito Integral (concluded)

- It is a fundamental fact that $\int X \, dW$ is continuous almost surely.

- The following theorem says the Ito integral is a martingale.$^a$

**Theorem 19** *The Ito integral $\int X \, dW$ is a martingale.*

- A corollary is the mean value formula

$$E \left[ \int_a^b X \, dW \right] = 0.$$  

$^a$See Exercise 14.1.1 for simple stochastic processes.
Discrete Approximation

• Recall Eq. (70) on p. 546.

• The following simple stochastic process \{ \hat{X}(t) \} can be used in place of \( X \) to approximate \( \int_0^t X \, dW \),

\[
\hat{X}(s) \triangleq X(t_{k-1}) \text{ for } s \in [t_{k-1}, t_k), \quad k = 1, 2, \ldots, n.
\]

• Note the nonanticipating feature of \( \hat{X} \).
  – The information up to time \( s \),

\[
\{ \hat{X}(t), W(t), 0 \leq t \leq s \},
\]

cannot determine the future evolution of \( X \) or \( W \).
Discrete Approximation (concluded)

- Suppose we defined the stochastic integral as

\[ \sum_{k=0}^{n-1} X(t_{k+1})[W(t_{k+1}) - W(t_k)]. \]

- Then we would be using the following different simple stochastic process in the approximation,

\[ \hat{Y}(s) \triangleq X(t_k) \text{ for } s \in [t_{k-1}, t_k), k = 1, 2, \ldots, n. \]

- This clearly anticipates the future evolution of \( X \).\(^a\)

\(^a\)See Exercise 14.1.2 of the textbook for an example where it matters.
Ito Process

- The stochastic process $X = \{X_t, t \geq 0\}$ that solves

$$X_t = X_0 + \int_0^t a(X_s, s) \, ds + \int_0^t b(X_s, s) \, dW_s, \quad t \geq 0$$

is called an Ito process.

- $X_0$ is a scalar starting point.

- $\{a(X_t, t) : t \geq 0\}$ and $\{b(X_t, t) : t \geq 0\}$ are stochastic processes satisfying certain regularity conditions.

- $a(X_t, t)$: the drift.

- $b(X_t, t)$: the diffusion.
Ito Process (continued)

• A shorthand\textsuperscript{a} is the following stochastic differential equation for the Ito differential $dX_t$,

$$dX_t = a(X_t, t) \, dt + b(X_t, t) \, dW_t. \quad (71)$$

– Or simply

$$dX_t = a_t \, dt + b_t \, dW_t.$$  

– This is Brownian motion with an \textit{instantaneous} drift $a_t$ and an \textit{instantaneous} variance $b_t^2$.

• $X$ is a martingale if $a_t = 0$ (Theorem 19 on p. 548).

\textsuperscript{a}Paul Langevin (1872–1946) in 1904.
Ito Process (concluded)

- \( dW \) is normally distributed with mean zero and variance \( dt \).

- An equivalent form of Eq. (71) is

\[
dX_t = a_t \, dt + b_t \sqrt{dt} \, \xi, \tag{72}
\]

where \( \xi \sim N(0, 1) \).
Euler Approximation

- Define \( t_n \overset{\Delta}{=} n\Delta t \).
- The following approximation follows from Eq. (72),

\[
\hat{X}(t_{n+1}) = \hat{X}(t_n) + a(\hat{X}(t_n), t_n) \Delta t + b(\hat{X}(t_n), t_n) \Delta W(t_n).
\]  

(73)

- It is called the Euler or Euler-Maruyama method.
- Recall that \( \Delta W(t_n) \) should be interpreted as

\[
W(t_{n+1}) - W(t_n),
\]

not \( W(t_n) - W(t_{n-1})! \)!
Euler Approximation (concluded)

• With the Euler method, one can obtain a sample path

\[ \hat{X}(t_1), \hat{X}(t_2), \hat{X}(t_3), \ldots \]

from a sample path

\[ W(t_0), W(t_1), W(t_2), \ldots . \]

• Under mild conditions, \( \hat{X}(t_n) \) converges to \( X(t_n) \).
More Discrete Approximations

- Under fairly loose regularity conditions, Eq. (73) on p. 555 can be replaced by

\[
\hat{X}(t_{n+1}) \\
= \hat{X}(t_n) + a(\hat{X}(t_n), t_n) \Delta t + b(\hat{X}(t_n), t_n) \sqrt{\Delta t} Y(t_n).
\]

- \(Y(t_0), Y(t_1), \ldots\) are independent and identically distributed with zero mean and unit variance.
More Discrete Approximations (concluded)

- An even simpler discrete approximation scheme:

\[
\hat{X}(t_{n+1}) = \hat{X}(t_n) + a(\hat{X}(t_n), t_n) \Delta t + b(\hat{X}(t_n), t_n) \sqrt{\Delta t} \xi.
\]

- \( \text{Prob}[\xi = 1] = \text{Prob}[\xi = -1] = 1/2. \)
- Note that \( E[\xi] = 0 \) and \( \text{Var}[\xi] = 1. \)

- This is a binomial model.

- As \( \Delta t \) goes to zero, \( \hat{X} \) converges to \( X. \)

\textsuperscript{a}He (1990).
Trading and the Ito Integral

- Consider an Ito process
  \[ dS_t = \mu_t \, dt + \sigma_t \, dW_t. \]
  - \( S_t \) is the vector of security prices at time \( t \).
- Let \( \phi_t \) be a trading strategy denoting the quantity of each type of security held at time \( t \).
  - Hence the stochastic process \( \phi_t S_t \) is the value of the portfolio \( \phi_t \) at time \( t \).
- \( \phi_t \, dS_t \triangleq \phi_t (\mu_t \, dt + \sigma_t \, dW_t) \) represents the change in the value from security price changes occurring at time \( t \).
Trading and the Ito Integral (concluded)

- The equivalent Ito integral,

\[ G_T(\phi) \triangleq \int_0^T \phi_t \, dS_t = \int_0^T \phi_t \mu_t \, dt + \int_0^T \phi_t \sigma_t \, dW_t, \]

measures the gains realized by the trading strategy over the period \([0, T]\).
Ito’s Lemma

A smooth function of an Ito process is itself an Ito process.

**Theorem 20** Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $dX = a_t \, dt + b_t \, dW$. Then $f(X)$ is the Ito process,

$$
\begin{align*}
  f(X_t) &= f(X_0) + \int_0^t f'(X_s) \, a_s \, ds + \int_0^t f'(X_s) \, b_s \, dW \\
  &\quad + \frac{1}{2} \int_0^t f''(X_s) \, b_s^2 \, ds
\end{align*}
$$

for $t \geq 0$.

---

*Ito (1944).*
Ito’s Lemma (continued)

• In differential form, Ito’s lemma becomes

\[
df(X) = f'(X) \, adt + f'(X) \, b \, dW + \frac{1}{2} f''(X) \, b^2 \, dt. \tag{74}
\]

• Compared with calculus, the interesting part is the third term on the right-hand side.

• A convenient formulation of Ito’s lemma is

\[
df(X) = f'(X) \, dX + \frac{1}{2} f''(X) (dX)^2.
\]
Ito’s Lemma (continued)

• We are supposed to multiply out

\[(dX)^2 = (a \, dt + b \, dW)^2\] symbolically according to

<table>
<thead>
<tr>
<th>×</th>
<th>dW</th>
<th>dt</th>
</tr>
</thead>
<tbody>
<tr>
<td>dW</td>
<td>dt</td>
<td>0</td>
</tr>
<tr>
<td>dt</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- The \((dW)^2 = dt\) entry is justified by a known result.

• Hence \((dX)^2 = (a \, dt + b \, dW)^2 = b^2 \, dt\).

• This form is easy to remember because of its similarity to the Taylor expansion.
Ito’s Lemma (continued)

**Theorem 21 (Higher-Dimensional Ito’s Lemma)** Let $W_1, W_2, \ldots, W_n$ be independent Wiener processes and $X \triangleq (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is twice continuously differentiable and $X_i$ is an Ito process with $dX_i = a_i \, dt + \sum_{j=1}^n b_{ij} \, dW_j$. Then $df(X)$ is an Ito process with the differential,

$$df(X) = \sum_{i=1}^m f_i(X) \, dX_i + \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m f_{ik}(X) \, dX_i \, dX_k,$$

where $f_i \triangleq \partial f / \partial X_i$ and $f_{ik} \triangleq \partial^2 f / \partial X_i \partial X_k$. 

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Ito’s Lemma (continued)

- The multiplication table for Theorem 21 is

<table>
<thead>
<tr>
<th>×</th>
<th>$dW_i$</th>
<th>$dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$dW_k$</td>
<td>$\delta_{ik} , dt$</td>
<td>0</td>
</tr>
<tr>
<td>$dt$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

in which

$$\delta_{ik} = \begin{cases} 
1, & \text{if } i = k, \\
0, & \text{otherwise}.
\end{cases}$$
Ito’s Lemma (continued)

• In applying the higher-dimensional Ito’s lemma, usually one of the variables, say $X_1$, is time $t$ and $dX_1 = dt$.

• In this case, $b_{1j} = 0$ for all $j$ and $a_1 = 1$.

• As an example, let

$$dX_t = a_t \, dt + b_t \, dW_t.$$ 

• Consider the process $f(X_t, t)$. 
Ito’s Lemma (continued)

- Then

\[
\begin{align*}
    df &= \frac{\partial f}{\partial X_t} dX_t + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (dX_t)^2 \\
    &= \frac{\partial f}{\partial X_t} (a_t \, dt + b_t \, dW_t) + \frac{\partial f}{\partial t} dt \\
    &\quad + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (a_t \, dt + b_t \, dW_t)^2 \\
    &= \left( \frac{\partial f}{\partial X_t} a_t + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} b_t^2 \right) dt \\
    &\quad + \frac{\partial f}{\partial X_t} b_t \, dW_t.
\end{align*}
\]  

(75)
Ito’s Lemma (continued)

Theorem 22 (Alternative Ito’s Lemma) Let $W_1, W_2, \ldots, W_m$ be Wiener processes and $X \triangleq (X_1, X_2, \ldots, X_m)$ be a vector process. Suppose $f : R^m \rightarrow R$ is twice continuously differentiable and $X_i$ is an Ito process with $dX_i = a_i \, dt + b_i \, dW_i$. Then $df(X)$ is the following Ito process,

$$
    df(X) = \sum_{i=1}^{m} f_i(X) \, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X) \, dX_i \, dX_k.
$$
Ito’s Lemma (concluded)

• The multiplication table for Theorem 22 is

<table>
<thead>
<tr>
<th>×</th>
<th>(dW_i)</th>
<th>(dt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(dW_k)</td>
<td>(\rho_{ik} \ dt)</td>
<td>0</td>
</tr>
<tr>
<td>(dt)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

• Above, \(\rho_{ik}\) denotes the correlation between \(dW_i\) and \(dW_k\).
Geometric Brownian Motion

• Consider geometric Brownian motion

\[ Y(t) \overset{\Delta}{=} e^{X(t)}. \]

- \( X(t) \) is a \((\mu, \sigma)\) Brownian motion.
- By Eq. (69) on p. 531,

\[ dX = \mu \, dt + \sigma \, dW. \]

• Note that

\[
\frac{\partial Y}{\partial X} = Y, \\
\frac{\partial^2 Y}{\partial X^2} = Y.
\]
Geometric Brownian Motion (continued)

- Ito’s formula (74) on p. 562 implies

\[
\begin{align*}
    dY &= Y \, dX + \left( \frac{1}{2} \right) Y \, (dX)^2 \\
    &= Y \left( \mu \, dt + \sigma \, dW \right) + \left( \frac{1}{2} \right) Y \left( \mu \, dt + \sigma \, dW \right)^2 \\
    &= Y \left( \mu \, dt + \sigma \, dW \right) + \left( \frac{1}{2} \right) Y \sigma^2 \, dt.
\end{align*}
\]

- Hence

\[
\frac{dY}{Y} = \left( \mu + \frac{\sigma^2}{2} \right) \, dt + \sigma \, dW. \tag{76}
\]

- The annualized instantaneous rate of return is \( \mu + \sigma^2 / 2 \) (not \( \mu \)).

\[\text{a} \text{Consistent with Lemma 11 (p. 283).}\]
Geometric Brownian Motion (concluded)

• Similarly, suppose

\[ \frac{dY}{Y} = \mu \, dt + \sigma \, dW. \]

• Then \( X(t) \triangleq \ln Y(t) \) follows

\[ dX = (\mu - \sigma^2/2) \, dt + \sigma \, dW. \]
Product of Geometric Brownian Motion Processes

• Let

\[
\frac{dY}{Y} = a \, dt + b \, dW_Y, \\
\frac{dZ}{Z} = f \, dt + g \, dW_Z.
\]

• Assume \( dW_Y \) and \( dW_Z \) have correlation \( \rho \).

• Consider the Ito process

\[ U \overset{\Delta}{=} YZ. \]
Product of Geometric Brownian Motion Processes (continued)

- Apply Ito’s lemma (Theorem 22 on p. 568):

\[ dU = Z dY + Y dZ + dY dZ \]
\[ = ZY (a \, dt + b \, dW_Y) + YZ (f \, dt + g \, dW_Z) + YZ (a \, dt + b \, dW_Y) (f \, dt + g \, dW_Z) \]
\[ = U (a + f + bg\rho) \, dt + Ub \, dW_Y + Ug \, dW_Z. \]

- The product of correlated geometric Brownian motion processes thus remains geometric Brownian motion.
Product of Geometric Brownian Motion Processes (continued)

• Note that

\[ Y = \exp \left[ (a - b^2/2) \, dt + b \, dW_Y \right], \]
\[ Z = \exp \left[ (f - g^2/2) \, dt + g \, dW_Z \right], \]
\[ U = \exp \left[ (a + f - (b^2 + g^2)/2) \, dt + b \, dW_Y + g \, dW_Z \right]. \]

– There is no \( bg \rho \) term in \( U \)!
Product of Geometric Brownian Motion Processes (concluded)

- $\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$.
- This holds even if $Y$ and $Z$ are correlated.
- Finally, $\ln Y$ and $\ln Z$ have correlation $\rho$. 
Quotients of Geometric Brownian Motion Processes

• Suppose $Y$ and $Z$ are drawn from p. 573.

• Let

$$U \triangleq Y/Z.$$  

• We now show that

$$\frac{dU}{U} = (a - f + g^2 - bg\rho)\,dt + b\,dW_Y - g\,dW_Z.$$  \hspace{1cm} (77)

• Keep in mind that $dW_Y$ and $dW_Z$ have correlation $\rho$.

---

$^a$Exercise 14.3.6 of the textbook is erroneous.
Quotients of Geometric Brownian Motion Processes (concluded)

• The multidimensional Ito’s lemma (Theorem 22 on p. 568) can be employed to show that

\[ dU = \left( \frac{1}{Z} \right) dY - \left( \frac{Y}{Z^2} \right) dZ - \left( \frac{1}{Z^2} \right) dY dZ + \left( \frac{Y}{Z^3} \right) (dZ)^2 \]

\[ = \left( \frac{1}{Z} \right) (aY dt + bY dW_Y) - \left( \frac{Y}{Z^2} \right) (fZ dt + gZ dW_Z) \]

\[ - \left( \frac{1}{Z^2} \right) (bgY Z \rho dt) + \left( \frac{Y}{Z^3} \right) (g^2 Z^2 dt) \]

\[ = U(a dt + b dW_Y) - U(f dt + g dW_Z) \]

\[ - U(bg \rho dt) + U(g^2 dt) \]

\[ = U(a - f + g^2 - bg \rho) dt + Ub dW_Y - Ug dW_Z. \]
Forward Price

• Suppose $S$ follows

$$\frac{dS}{S} = \mu \, dt + \sigma \, dW.$$  

• Consider $F(S, t) \triangleq Se^{y(T-t)}$ for some constants $y$ and $T$.

• As $F$ is a function of two variables, we need the various partial derivatives of $F(S, t)$ with respect to $S$ and $t$.

• Note that in partial differentiation with respect to one variable, other variables are held constant.\(^{a}\)

\(^{a}\)Contributed by Mr. Sun, Ao (R05922147) on April 26, 2017.
Forward Prices (continued)

• Now,

\[
\frac{\partial F}{\partial S} = e^{y(T-t)},
\]

\[
\frac{\partial^2 F}{\partial S^2} = 0,
\]

\[
\frac{\partial F}{\partial t} = -ySe^{y(T-t)}.
\]

• Then

\[
dF = e^{y(T-t)} dS - ySe^{y(T-t)} dt
\]

\[
= Se^{y(T-t)} (\mu dt + \sigma dW) - ySe^{y(T-t)} dt
\]

\[
= F(\mu - y) dt + F\sigma dW.
\]
Forward Prices (concluded)

• One can also prove it by Eq. (75) on p. 567.

• Thus $F$ follows

\[
\frac{dF}{F} = (\mu - y) \, dt + \sigma \, dW.
\]

• This result has applications in forward and futures contracts.

• In Eq. (52) on p. 446, $\mu = r = y$.

• So

\[
\frac{dF}{F} = \sigma \, dW,
\]

a martingale.\textsuperscript{a}

\textsuperscript{a}It is also consistent with p. 521.
Ornstein-Uhlenbeck (OU) Process

• The OU process:

\[ dX = -\kappa X \, dt + \sigma \, dW, \]

where \( \kappa, \sigma \geq 0 \).

• For \( t_0 \leq s \leq t \) and \( X(t_0) = x_0 \), it is known that

\[
E[X(t)] = e^{-\kappa(t-t_0)} E[x_0],
\]

\[
\text{Var}[X(t)] = \frac{\sigma^2}{2\kappa} \left(1 - e^{-2\kappa(t-t_0)}\right) + e^{-2\kappa(t-t_0)} \text{Var}[x_0],
\]

\[
\text{Cov}[X(s), X(t)] = \frac{\sigma^2}{2\kappa} e^{-\kappa(t-s)} \left[1 - e^{-2\kappa(s-t_0)}\right] + e^{-\kappa(t+s-2t_0)} \text{Var}[x_0].
\]
Ornstein-Uhlenbeck Process (continued)

• $X(t)$ is normally distributed if $x_0$ is a constant or normally distributed.

• $X$ is said to be a normal process.

• $E[x_0] = x_0$ and $\text{Var}[x_0] = 0$ if $x_0$ is a constant.

• The OU process has the following mean reversion property.
  – When $X > 0$, $X$ is pulled toward zero.
  – When $X < 0$, it is pulled toward zero again.
Ornstein-Uhlenbeck Process (continued)

- A generalized version:

\[ dX = \kappa (\mu - X) \, dt + \sigma \, dW, \]

where \( \kappa, \sigma \geq 0 \).

- Given \( X(t_0) = x_0 \), a constant, it is known that

\[
E[X(t)] = \mu + (x_0 - \mu) e^{-\kappa(t-t_0)}, \quad (78)
\]

\[
\text{Var}[X(t)] = \frac{\sigma^2}{2\kappa} \left[ 1 - e^{-2\kappa(t-t_0)} \right],
\]

for \( t_0 \leq t \).
Ornstein-Uhlenbeck Process (concluded)

- The mean and standard deviation are roughly $\mu$ and $\sigma/\sqrt{2\kappa}$, respectively.

- For large $t$, the probability of $X < 0$ is extremely unlikely in any finite time interval when $\mu > 0$ is large relative to $\sigma/\sqrt{2\kappa}$.

- The process is mean-reverting.
  - $X$ tends to move toward $\mu$.
  - Useful for modeling term structure, stock price volatility, and stock price return.
Square-Root Process

• Suppose $X$ is an OU process.

• Consider

  \[ V \triangleq X^2. \]

• Ito’s lemma says $V$ has the differential,

  \[
  dV = 2X \, dX + (dX)^2 \\
  = 2\sqrt{V} \left( -\kappa \sqrt{V} \, dt + \sigma \, dW \right) + \sigma^2 \, dt \\
  = \left( -2\kappa V + \sigma^2 \right) \, dt + 2\sigma \sqrt{V} \, dW,
  \]

  a square-root process.
Square-Root Process (continued)

- In general, the square-root process has the stochastic differential equation,

\[ dX = \kappa (\mu - X) \, dt + \sigma \sqrt{X} \, dW, \]

where \( \kappa, \sigma \geq 0 \) and \( X(0) \) is a nonnegative constant.

- Like the OU process, it possesses mean reversion: \( X \) tends to move toward \( \mu \), but the volatility is proportional to \( \sqrt{X} \) instead of a constant.
Square-Root Process (continued)

• When $X$ hits zero and $\mu \geq 0$, the probability is one that it will not move below zero.
  – Zero is a reflecting boundary.

• Hence, the square-root process is a good candidate for modeling interest rates.$^a$

• The OU process, in contrast, allows negative interest rates.$^b$

• The two processes are related (see p. 586).

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$^a$Cox, Ingersoll, & Ross (1985).
$^b$But some rates have gone negative in Europe in 2015!
Square-Root Process (concluded)

- The random variable $2cX(t)$ follows the noncentral chi-square distribution,\(^a\)

$$\chi \left( \frac{4\kappa \mu}{\sigma^2}, 2cX(0) e^{-\kappa t} \right),$$

where $c \triangleq (2\kappa/\sigma^2)(1 - e^{-\kappa t})^{-1}$.

- Given $X(0) = x_0$, a constant,

$$E[X(t)] = x_0 e^{-\kappa t} + \mu \left( 1 - e^{-\kappa t} \right),$$

$$\text{Var}[X(t)] = x_0 \frac{\sigma^2}{\kappa} (e^{-\kappa t} - e^{-2\kappa t}) + \mu \frac{\sigma^2}{2\kappa} \left( 1 - e^{-\kappa t} \right)^2,$$

for $t \geq 0$.