Known Dividends

• Constant dividends introduce complications.

• Use $D$ to denote the amount of the dividend.

• Suppose an ex-dividend date falls in the first period.

• At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.

• Follow the stock price one more period.

• The number of possible stock prices is not three but four: $(Su - D)u$, $(Su - D)d$, $(Sd - D)u$, $(Sd - D)d$.
  – The binomial tree no longer combines.
An Ad-Hoc Approximation

• Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.\(^a\)

• This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.

• The riskless component at any time is the PV of future dividends during the life of the option.
  – Then, \(\sigma\) is the volatility of the process followed by the risky component.

• The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

\(^a\)Roll (1977).
An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all future dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.
The Ad-Hoc Approximation vs. P. 304 (Step 1)

\[
(S - D/R)u^2
\]

\[
(S - D/R)u
\]

\[
S - D/R
\]

\[
(S - D/R)ud
\]

\[
(S - D/R)d
\]

\[
(S - D/R)d^2
\]

\[
(S - D/R)u^2
\]
The Ad-Hoc Approximation vs. P. 304 (Step 2)

\[(S - D/R)u^2\]

\[(S - D/R)u\]

\[(S - D/R) + D/R = S\]

\[(S - D/R)ud\]

\[(S - D/R)d\]

\[(S - D/R)d^2\]
The Ad-Hoc Approximation vs. P. 304\textsuperscript{a}

- The trees are different.
- The stock prices at maturity are also different.
  \[ (Su - D)u, (Su - D)d, (Sd - D)u, (Sd - D)d \]
  (p. 304).
  \[ (S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2 \] (ad hoc).
- Note that, as \( d < R < u \),
  \[ (Su - D)u > (S - D/R)u^2, \]
  \[ (Sd - D)d < (S - D/R)d^2, \]

\textsuperscript{a}Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.
The Ad-Hoc Approximation vs. P. 304 (concluded)

- So the ad hoc approximation has a smaller dynamic range.

- This explains why in practice the volatility is usually increased when using the ad hoc approximation.
A General Approach

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 722ff).
- Other approaches include adjusting $\sigma$ and approximating the known dividend with a dividend yield.\(^b\)


\(^{b}\)Geske & Shastri (1985). It works well for American options but not European options (Dai, 2009).
Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.

- The payment of a continuous dividend yield at rate $q$ reduces the growth rate of the stock price by $q$.
  - A stock that grows from $S$ to $S_\tau$ with a continuous dividend yield of $q$ would grow from $S$ to $S_\tau e^{q\tau}$ without the dividends.

- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.\(^a\)

\(^a\)In pricing European options, only the distribution of $S_\tau$ matters.
Continuous Dividend Yields (continued)

- So the Black-Scholes formulas hold with $S$ replaced by $Se^{-q\tau}$:

$$
C = Se^{-q\tau} N(x) - X e^{-r\tau} N(x - \sigma \sqrt{\tau}), \quad (39)
$$

$$
P = X e^{-r\tau} N(-x + \sigma \sqrt{\tau}) - Se^{-q\tau} N(-x), \quad (39')
$$

where

$$
x \equiv \ln(S/X) + (r - q + \sigma^2/2) \tau / \sigma \sqrt{\tau}.
$$

- Formulas (39) and (39') remain valid as long as the dividend yield is predictable.

---

\(^a\)Merton (1973).
Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace $u$ with $ue^{-q\Delta t}$ and $d$ with $de^{-q\Delta t}$, where $\Delta t \equiv \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- Other than the changes, binomial tree algorithms stay the same.
  - In particular, $p$ should use the original $u$ and $d$!\(^a\)

\(^a\)Contributed by Ms. Wang, Chuan-Ju (F95922018) on May 2, 2007.
Continuous Dividend Yields (concluded)

- Alternatively, pick the risk-neutral probability as

\[ \frac{e^{(r-q)\Delta t} - d}{u - d}, \tag{40} \]

where \( \Delta t \equiv \tau/n \).

- The reason: The stock price grows at an expected rate of \( r - q \) in a risk-neutral economy.

- The \( u \) and \( d \) remain unchanged.

- Other than the change in Eq. (40), binomial tree algorithms stay the same as if there were no dividends.
Exercise Boundaries of American Options\textsuperscript{a}

- The exercise boundary is a nondecreasing function of $t$ for American puts (see the plot next page).
- The exercise boundary is a nonincreasing function of $t$ for American calls.

\textsuperscript{a}See Exercise 15.2.7 of the textbook.
Risk Reversals\textsuperscript{a}

- From formulas (39) and (39′) on p. 313, one can verify that $C = P$ when

$$X = Se^{(r-q)\tau}.$$ 

- A risk reversal consists of a short out-of-the-money put and a long out-of-the-money call with the same maturity.\textsuperscript{b}

- Furthermore, the portfolio has zero value.

- A short risk reversal position is also called a collar.\textsuperscript{c}

\begin{itemize}
  \item \textsuperscript{a}Neftci (2008).
  \item \textsuperscript{b}Thus their strike prices must be distinct.
  \item \textsuperscript{c}Bennett (2014).
\end{itemize}
Sensitivity Analysis of Options
Cleopatra’s nose, had it been shorter, 
the whole face of the world 
would have been changed. 
— Blaise Pascal (1623–1662)
Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.

- Let \( x \equiv \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}} \) (recall p. 285).

- Recall that
  \[
  N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0, \\
  \]
  the density function of standard normal distribution.
Delta

• Defined as
  \[ \Delta \equiv \frac{\partial f}{\partial S}. \]
  
  – \( f \) is the price of the derivative.
  – \( S \) is the price of the underlying asset.

• The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.\(^a\)

• The delta used in the BOPM (p. 230) is the discrete analog.

• The delta of a long stock is apparently 1.

\(^a\)Elementary calculus.
Delta (continued)

- The delta of a European call on a non-dividend-paying stock equals
  \[ \frac{\partial C}{\partial S} = N(x) > 0. \]

- The delta of a European put equals
  \[ \frac{\partial P}{\partial S} = N(x) - 1 = -N(-x) < 0. \]

- So the deltas of a call and an otherwise identical put cancel each other when
  \( N(x) = 1/2 \), i.e., when\(^a\)
  \[ X = Se^{(r+\sigma^2/2)\tau}. \]  \( (41) \)

\(^a\)The straddle (p. 195) \( C + P \) then has zero delta!
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money options.
Dashed curves: out-of-the-money calls or in-the-money puts.
Delta (continued)

- Suppose the stock pays a continuous dividend yield of \( q \).

- Let

\[
x \equiv \frac{\ln(S/X) + (r - q + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}
\]  

(42)

(recall p. 313).

- Then

\[
\frac{\partial C}{\partial S} = e^{-q\tau} N(x) > 0,
\]

\[
\frac{\partial P}{\partial S} = -e^{-q\tau} N(-x) < 0.
\]
Delta (continued)

• Consider an $X_1$-strike call and an $X_2$-strike put, $X_1 \geq X_2$.
• They are otherwise identical.
• Let

$$x_i \equiv \ln(S/X_i) + \left( r - q + \sigma^2/2 \right) \tau / \sigma \sqrt{\tau}.$$  \hspace{1cm} (43)

• Then their deltas sum to zero when $x_1 = -x_2$.
• That implies

$$\frac{S}{X_1} = \frac{X_2}{S} e^{-\left(2r - 2q + \sigma^2\right) \tau}.$$  \hspace{1cm} (44)

\[a\text{The strangle (p. 197) } C + P \text{ then has zero delta!}\]
Delta (continued)

- Suppose we demand \( X_1 = X_2 = X \) and have a straddle.
- Then

\[
X = S e^{(r-q+\sigma^2/2)\tau}
\]

leads to a straddle with zero delta.

- This generalizes Eq. (41) on p. 323.

- When \( C(X_1) \)'s delta and \( P(X_2) \)'s delta sum to zero, does the portfolio \( C(X_1) - P(X_2) \) have zero value?
Delta (concluded)

• This portfolio $C(X_1) - P(X_2)$ has value

\[
Se^{-q\tau} N(x_1) - X_1 e^{-r\tau} N(x_1 - \sigma \sqrt{\tau}) \\
- X_2 e^{-r\tau} N(-x_2 + \sigma \sqrt{\tau}) + Se^{-q\tau} N(-x_2) \\
= 2Se^{-q\tau} N(x_1) - X_1 e^{-r\tau} N(x_1 - \sigma \sqrt{\tau}) - X_2 e^{-r\tau} N(x_1 + \sigma \sqrt{\tau}) \\
= 2Se^{-q\tau} N(x_1) - X_1 e^{-r\tau} N(x_1 - \sigma \sqrt{\tau}) \\
- \frac{S^2}{X_1} e^{(r-2q+\sigma^2)\tau} N(x_1 + \sigma \sqrt{\tau}).
\]

• This is not identically zero so not a risk reversal (p. 318).

• E.g., with $r = q = 0$ and $\tau$ large, it is about

\[
2S - (S^2 / X_1) e^{\sigma^2 \tau} = 2S - X_2.
\]
Delta Neutrality

• A position with a total delta equal to 0 is delta-neutral.
  – A delta-neutral portfolio is immune to small price changes in the underlying asset.

• Creating one serves for hedging purposes.
  – A portfolio consisting of a call and \(-\Delta\) shares of stock is delta-neutral.
  – Short \(\Delta\) shares of stock to hedge a long call.
  – Long \(\Delta\) shares of stock to hedge a short call.

• In general, hedge a position in a security with delta \(\Delta_1\) by shorting \(\Delta_1/\Delta_2\) units of a security with delta \(\Delta_2\).
Theta (Time Decay)

- Defined as the rate of change of a security’s value with respect to time, or \( \Theta \equiv -\partial f / \partial \tau = \partial f / \partial t. \)

- For a European call on a non-dividend-paying stock,
  \[
  \Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} - rX e^{-r\tau} N(x - \sigma \sqrt{\tau}) < 0.
  \]
  - The call loses value with the passage of time.

- For a European put,
  \[
  \Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} + rX e^{-r\tau} N(-x + \sigma \sqrt{\tau}).
  \]
  - Can be negative or positive.
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money options.
Dashed curve: out-of-the-money call or in-the-money put.
Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or \( \Gamma \equiv \partial^2 \Pi / \partial S^2 \).
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs to be rebalanced more often to maintain delta neutrality.
- Roughly, delta \( \sim \) duration, and gamma \( \sim \) convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

\[
N'(x)/(S\sigma\sqrt{\tau}) > 0.
\]
Dotted lines: in-the-money call or out-of-the-money put. 
Solid lines: at-the-money option. 
Dashed lines: out-of-the-money call or in-the-money put.
Vega\textsuperscript{a} (Lambda, Kappa, Sigma)

- Defined as the rate of change of a security's value with respect to the volatility of the underlying asset

\[ \Lambda \equiv \frac{\partial f}{\partial \sigma}. \]

- Volatility often changes over time.

- A security with a high vega is very sensitive to small changes or estimation error in volatility.

- The vega of a European call or put on a non-dividend-paying stock is \( S \sqrt{\tau} N'(x) > 0. \)
  - So higher volatility always increases the option value.

\textsuperscript{a}Vega is not Greek.
Vega (continued)

• If the stock pays a continuous dividend yield of $q$, then
  \[
  \Lambda = S e^{-q\tau} \sqrt{\tau} N'(x),
  \]
  where $x$ is defined in Eq. (42) on p. 325.

• Vega is maximized when $x = 0$, i.e., when
  \[
  S = X e^{-(r-q+\sigma^2/2)\tau}.
  \]

• Vega declines very fast as $S$ moves away from that peak.
Vega (continued)

• Now consider a portfolio consisting of an $X_1$-strike call $C$ and a short $X_2$-strike put $P$, $X_1 \geq X_2$.

• The options’ vegas cancel out when

\[ x_1 = -x_2, \]

where $x_i$ are defined in Eq. (43) on p. 326.

• This leads to Eq. (44) on p. 326.
  – The same condition leads to zero delta for the strangle $C + P$ (p. 326).
Vega (concluded)

• Note that if \( S \neq X, \tau \to 0 \) implies
  \[ \Lambda \to 0 \]
  (which answers the question on p. 290 for the Black-Scholes model).

• The Black-Scholes formula (p. 285) implies
  \[
  C \to S, \\
  P \to Xe^{-r\tau},
  \]
  as \( \sigma \to \infty \).

• These boundary conditions may be handy for certain numerical methods.
Variance Vega\textsuperscript{a}

- Defined as the rate of change of a security's value with respect to the variance (square of volatility) of the underlying asset

\[ V \equiv \frac{\partial f}{\partial \sigma^2}. \]

- It is easy to verify that

\[ V = \frac{\Lambda}{2\sigma}. \]

\textsuperscript{a}Demeterfi, Derman, Kamal, & Zou (1999).
Rho

• Defined as the rate of change in its value with respect to interest rates
  \[ \rho \equiv \frac{\partial f}{\partial r}. \]

• The rho of a European call on a non-dividend-paying stock is
  \[ X \tau e^{-r \tau} N(x - \sigma \sqrt{\tau}) > 0. \]

• The rho of a European put on a non-dividend-paying stock is
  \[ -X \tau e^{-r \tau} N(-x + \sigma \sqrt{\tau}) < 0. \]
Dotted curves: in-the-money call or out-of-the-money put.
Solid curves: at-the-money option.
Dashed curves: out-of-the-money call or in-the-money put.
Numerical Greeks

• Needed when closed-form formulas do not exist.

• Take delta as an example.

• A standard method computes the finite difference,

\[ \frac{f(S + \Delta S) - f(S - \Delta S)}{2\Delta S} \]

• The computation time roughly doubles that for evaluating the derivative security itself.
An Alternative Numerical Delta\textsuperscript{a}

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, $f_u$ and $f_d$ are computed.
- These values correspond to derivative values at stock prices $S_u$ and $S_d$, respectively.
- Delta is approximated by
  \[ \frac{f_u - f_d}{S_u - S_d}. \]
- Almost zero extra computational effort.

\textsuperscript{a}Pelsser & Vorst (1994).
Numerical Gamma

- At the stock price \((Suu + Sud)/2\), delta is approximately \((f_{uu} - f_{ud})/(Suu - Sud)\).

- At the stock price \((Sud + Sdd)/2\), delta is approximately \((f_{ud} - f_{dd})/(Sud - Sdd)\).

- Gamma is the rate of change in deltas between \((Suu + Sud)/2\) and \((Sud + Sdd)/2\), that is,

\[
\frac{\frac{f_{uu} - f_{ud}}{Suu - Sud} - \frac{f_{ud} - f_{dd}}{Sud - Sdd}}{(Suu - Sdd)/2}.
\]

- Alternative formulas exist (p. 628).
Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

\[ \frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}. \]

- It does not work (see text for the reason).

- In general, calculating gamma is a hard problem numerically.

- But why did the binomial tree version work?
Other Numerical Greeks

• The theta can be computed as

\[ \frac{f_{ud} - f}{2(\tau/n)}. \]

– In fact, the theta of a European option can be derived from delta and gamma (p. 627).

• For vega and rho, there seems no alternative but to run the binomial tree algorithm twice.\(^a\)

\(^a\)But see pp. 974ff.
Extensions of Options Theory
As I never learnt mathematics, so I have had to think.
— Joan Robinson (1903–1983)
Pricing Corporate Securities\textsuperscript{a}

- Interpret the underlying asset as the total value of the firm.

- The option pricing methodology can be applied to pricing corporate securities.
  - The result is called the structural model.

- Assumptions:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

\textsuperscript{a}Black & Scholes (1973); Merton (1974).
Risky Zero-Coupon Bonds and Stock

• Consider XYZ.com.

• Capital structure:
  – $n$ shares of its own common stock, $S$.
  – Zero-coupon bonds with an aggregate par value of $X$.

• What is the value of the bonds, $B$?

• What is the value of the XYZ.com stock?
Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds’ maturity date, suppose the total value of the firm $V^*$ is less than the bondholders’ claim $X$.
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain $X$ and the stockholders $V^* - X$.

<table>
<thead>
<tr>
<th></th>
<th>$V^*$</th>
<th>$X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stock</td>
<td>0</td>
<td>$V^* - X$</td>
</tr>
</tbody>
</table>
Risky Zero-Coupon Bonds and Stock (continued)

- The stock has the same payoff as a call!
- It is a call on the total value of the firm with a strike price of $X$ and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call\(^a\) on the total value of the firm.
- Let $V$ stand for the total value of the firm.
- Let $C$ stand for a call on $V$.

\(^a\)See p. 186.
Risky Zero-Coupon Bonds and Stock (continued)

• Thus

\[ nS = C, \]
\[ B = V - C. \]

• Knowing \( C \) amounts to knowing how the value of the firm is divided between stockholders and bondholders.

• Whatever the value of \( C \), the total value of the stock and bonds at maturity remains \( V^* \).

• The relative size of debt and equity is irrelevant to the firm’s current value \( V \).
Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 12 (p. 285) and the put-call parity,

\[
\begin{align*}
    nS &= VN(x) - X e^{-r\tau} N(x - \sigma\sqrt{\tau}), \\
    B &= VN(-x) + X e^{-r\tau} N(x - \sigma\sqrt{\tau}).
\end{align*}
\]

- Above,

\[
x \equiv \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
\]

- The continuously compounded yield to maturity of the firm’s bond is

\[
\frac{\ln(X/B)}{\tau}.
\]

\(^a\)Merton (1974).
Risky Zero-Coupon Bonds and Stock (continued)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

\[
\frac{\ln(X/B)}{\tau} - r
\]

\[
= -\frac{1}{\tau} \ln \left( N(-z) + \frac{1}{\omega} N(z - \sigma \sqrt{\tau}) \right).
\]

- \( \omega \equiv X e^{-r \tau} / V. \)

- \( z \equiv (\ln \omega) / (\sigma \sqrt{\tau}) + (1/2) \sigma \sqrt{\tau} = -x + \sigma \sqrt{\tau}. \)

- Note that \( \omega \) is the debt-to-total-value ratio.
Risky Zero-Coupon Bonds and Stock (concluded)

- In general, suppose the firm has a dividend yield at rate $q$ and the bankruptcy costs are a constant proportion $\alpha$ of the remaining firm value.

- Then Eqs. (46)–(47) on p. 355 become, respectively,

\[
\begin{align*}
nS &= Ve^{-q\tau}N(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \\
B &= (1 - \alpha)Ve^{-q\tau}N(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}).
\end{align*}
\]

- Above,

\[
x \equiv \frac{\ln(V/X) + (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
\]
A Numerical Example

- XYZ.com’s assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck’s market value per share is $44.5.
- XYZ.com’s securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay $1,000 at maturity.
- \( n = 1,000, \ V = 44.5 \times n = 44,500 \), and \( X = 30 \times 1,000 = 30,000 \).
<table>
<thead>
<tr>
<th>Option</th>
<th>Strike</th>
<th>Exp.</th>
<th>Vol.</th>
<th>Last</th>
<th>Vol.</th>
<th>Last</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merck</td>
<td>30</td>
<td>Jul</td>
<td>328</td>
<td>15 1/4</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>441/2</td>
<td>35</td>
<td>Jul</td>
<td>150</td>
<td>91/2</td>
<td>10</td>
<td>1/16</td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Apr</td>
<td>887</td>
<td>43/4</td>
<td>136</td>
<td>1/16</td>
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<tr>
<td>441/2</td>
<td>40</td>
<td>Jul</td>
<td>220</td>
<td>51/2</td>
<td>297</td>
<td>1/4</td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Oct</td>
<td>58</td>
<td>6</td>
<td>10</td>
<td>1/2</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Apr</td>
<td>3050</td>
<td>7/8</td>
<td>100</td>
<td>11/8</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>May</td>
<td>462</td>
<td>13/8</td>
<td>50</td>
<td>13/8</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Jul</td>
<td>883</td>
<td>1 15/16</td>
<td>147</td>
<td>13/4</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Oct</td>
<td>367</td>
<td>23/4</td>
<td>188</td>
<td>21/16</td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for $15.25$.
- So XYZ.com’s stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

\[ B = 44,500 - 15,250 = 29,250 \]

dollars.

- Or $975$ per bond.
A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $X$ par value plus $n$ written European puts on Merck at a strike price of $30$.
  - By the put-call parity.$^a$

- The difference between $B$ and the price of the default-free bond is the value of these puts.

- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts $X$.

$^a$See p. 209.
<table>
<thead>
<tr>
<th>Promised payment to bondholders</th>
<th>Current market value of bonds</th>
<th>Current market value of stock</th>
<th>Current total value of firm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$B$</td>
<td>$nS$</td>
<td>$V$</td>
</tr>
<tr>
<td>30,000</td>
<td>29,250.0</td>
<td>15,250.0</td>
<td>44,500</td>
</tr>
<tr>
<td>35,000</td>
<td>35,000.0</td>
<td>9,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>40,000</td>
<td>39,000.0</td>
<td>5,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>45,000</td>
<td>42,562.5</td>
<td>1,937.5</td>
<td>44,500</td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

• Suppose the promised payment to bondholders is $45,000.

• Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.

• Since that option is selling for $115/16$, the market value of the XYZ.com stock is 
  $(1 + 15/16) \times n = 1,937.5$ dollars.

• The market value of the stock decreases as the debt-equity ratio increases.
A Numerical Example (continued)

• There are conflicts between stockholders and bondholders.

• An option’s terms cannot be changed after issuance.

• But a firm can change its capital structure.

• There lies one key difference between options and corporate securities.
  – Parameters such volatility, dividend, and strike price are under partial control of the stockholders.
A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.

- The total debt is now $X = 45,000$ dollars.

- The table on p. 362 says the total market value of the bonds should be $42,562.5$.

- The new bondholders pay

  \[ 42,562.5 \times \frac{15}{45} = 14,187.5 \] dollars.

- The remaining stock is worth $1,937.5$. 
A Numerical Example (continued)

- The stockholders therefore gain

\[ 14,187.5 + 1,937.5 - 15,250 = 875 \]

dollars.

- The *original* bondholders lose an equal amount,

\[ 29,250 - \frac{30}{45} \times 42,562.5 = 875. \]

  - This is called claim dilution.\(^a\)

\(^a\text{Fama & Miller (1972).}\)
A Numerical Example (continued)

- Suppose the stockholders sell \((1/3) \times n\) Merck shares to fund a $14,833.3 cash dividend.

- They now have $14,833.3 in cash plus a call on \((2/3) \times n\) Merck shares.

- The strike price remains \(X = 30,000\).

- This is equivalent to owning \(2/3\) of a call on \(n\) Merck shares with a strike price of $45,000.

- \(n\) such calls are worth $1,937.5 (p. 362).

- So the total market value of the XYZ.com stock is \((2/3) \times 1,937.5 = 1,291.67\) dollars.
A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence

\[(2/3) \times n \times 44.5 - 1,291.67 = 28,375\]

dollars.

- Hence the stockholders gain

\[14,833.3 + 1,291.67 - 15,250 \approx 875\]

dollars.

- The bondholders watch their value drop from $29,250 to $28,375, a loss of $875.
Further Topics

• Other Examples:
  – Subordinated debts as bull call spreads.
  – Warrants as calls.
  – Callable bonds as American calls with 2 strike prices.
  – Convertible bonds.

• Securities with a complex liability structure must be solved by trees.\(^a\)

\(^a\)Dai (B82506025, R86526008, D8852600), Lyuu, & C. Wang (F95922018) (2010).
Barrier Options

- Their payoff depends on whether the underlying asset’s price reaches a certain price level $H$ throughout its life.

- A knock-out option is an ordinary European option which ceases to exist if the barrier $H$ is reached by the price of its underlying asset.

- A call knock-out option is sometimes called a down-and-out option if $H < S$.

- A put knock-out option is sometimes called an up-and-out option when $H > S$.

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A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She was working for Lehman Brothers in Hong Kong as of April, 2006.
Barrier Options (concluded)

- A knock-in option comes into existence if a certain barrier is reached.

- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.

- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.

- Formulas exist for all the possible barrier options mentioned above.\(^a\)

\(^a\)Haug (2006).
A Formula for Down-and-In Calls$^a$

• Assume $X \geq H$.

• The value of a European down-and-in call on a stock paying a dividend yield of $q$ is

$$Se^{-q\tau} \left( \frac{H}{S} \right)^{2\lambda} N(x) - Xe^{-r\tau} \left( \frac{H}{S} \right)^{2\lambda-2} N(x - \sigma \sqrt{\tau}),$$

(48)

- $x \equiv \frac{\ln(H^2/(SX))+(r-q+\sigma^2/2)\tau}{\sigma \sqrt{\tau}}$.

- $\lambda \equiv (r-q+\sigma^2/2)/\sigma^2$.

• A European down-and-out call can be priced via the in-out parity (see text).

$^a$Merton (1973). See Exercise 17.1.6 of the textbook for a proof.
A Formula for Up-and-In Puts\textsuperscript{a}

- Assume $X \leq H$.
- The value of a European up-and-in put is
  \[
  X e^{-r\tau} \left( \frac{H}{S} \right)^{2\lambda-2} N(-x + \sigma \sqrt{\tau}) - S e^{-q\tau} \left( \frac{H}{S} \right)^{2\lambda} N(-x).
  \]
- Again, a European up-and-out put can be priced via the in-out parity.

\textsuperscript{a}Merton (1973).
Are American Options Barrier Options?\textsuperscript{a}

- American options are barrier options with the exercise boundary as the barrier and the payoff as the rebate?
- One salient difference is that the exercise boundary must be derived during backward induction.
- But the barrier in a barrier option is given a priori.

\textsuperscript{a}Contributed by Mr. Yang, Jui-Chung (D97723002) on March 25, 2009.
Interesting Observations

- Assume $H < X$.
- Replace $S$ in the pricing formula Eq. (39) on p. 313 for the call with $H^2/S$.
- Equation (48) on p. 373 for the down-and-in call becomes Eq. (39) when $r - q = \sigma^2/2$.
- Equation (48) becomes $S/H$ times Eq. (39) when $r - q = 0$. 
Interesting Observations (concluded)

- Replace $S$ in the pricing formula for the down-and-in call, Eq. (48), with $H^2/S$.

- Equation (48) becomes Eq. (39) when $r - q = \sigma^2/2$.

- Equation (48) becomes $H/S$ times Eq. (39) when $r - q = 0$.\(^a\)

- Why?\(^b\)

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\(^a\) Contributed by Mr. Chou, Ming-Hsin (R02723073) on April 24, 2014.

\(^b\) Apply the reflection principle (p. 656), Eq. (38) on p. 278, and Lemma 11 (p. 283).
Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.

- Pricing down-and-in options is subtler.
\[ S = 8, \quad X = 6, \quad H = 4, \quad R = 1.25, \quad u = 2, \text{ and } d = 0.5. \]

Backward-induction: \[ C = \frac{0.5 \times C_u + 0.5 \times C_d}{1.25}. \]
Binomial Tree Algorithms (continued)

• But convergence is erratic because $H$ is not at a price level on the tree (see plot on next page).\(^a\)
  – The barrier $H$ is moved lower\(^b\) to a node price.
  – This “effective barrier” changes as $n$ increases.

• In fact, the binomial tree is $O(1/\sqrt{n})$ convergent.\(^c\)

• Solutions will be presented later.

\(^a\)Boyle & Lau (1994).
\(^b\)Higher is an alternative
\(^c\)J. Lin (R95221010) (2008).
Binomial Tree Algorithms (concluded)\textsuperscript{a}

Down-and-in call value

\textsuperscript{a}Lyuu (1998).