Option on a Non-Dividend-Paying Stock: Multi-Period

• Consider a call with two periods remaining before expiration.

• Under the binomial model, the stock can take on three possible prices at time two: $S_{uu}$, $S_{ud}$, and $S_{dd}$.
  – There are 4 paths.
  – But the tree combines or recombines.

• At any node, the next two stock prices only depend on the current price, not the prices of earlier times.\textsuperscript{a}

\textsuperscript{a}It is Markovian.
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let $C_{uu}$ be the call’s value at time two if the stock price is $S_{uu}$.
- Thus,
  
  $$C_{uu} = \max(0, S_{uu} - X).$$

- $C_{ud}$ and $C_{dd}$ can be calculated analogously,

  $$C_{ud} = \max(0, S_{ud} - X),$$

  $$C_{dd} = \max(0, S_{dd} - X).$$
\[ C_{uu} = \max(0, S_{uu} - X) \]

\[ C_{ud} = \max(0, S_{ud} - X) \]

\[ C_{dd} = \max(0, S_{dd} - X) \]
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

\[
C_u = \frac{pC_{uu} + (1 - p)C_{ud}}{R},
\]
\[
C_d = \frac{pC_{ud} + (1 - p)C_{dd}}{R}.
\]

- Deltas can be derived from Eq. (28) on p. 232.
- For example, the delta at \( C_u \) is

\[
\frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}}.
\]

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Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute

\[ \frac{pC_u + (1 - p)C_d}{R} \]

as the option price.
- The values of delta \( h \) and \( B \) can be derived from Eqs. (28)–(29) on p. 232.
Early Exercise

- Since the call will not be exercised at time 1 even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.

- Therefore,
  \[
  hS + B = \frac{pC_u + (1-p)C_d}{R} \geq \frac{[pu + (1-p)d] S - X}{R} \\
  = S - \frac{X}{R} > S - X.
  \]
  - The call again will not be exercised at present.a

- So
  \[
  C = hS + B = \frac{pC_u + (1-p)C_d}{R}.
  \]

---

aConsistent with Theorem 5 (p. 215).
Backward Induction\(^a\)

- The above expression calculates \( C \) from the two successor nodes \( C_u \) and \( C_d \) and none beyond.
- The same computation happened at \( C_u \) and \( C_d \), too, as demonstrated in Eq. (32) on p. 242.
- This recursive procedure is called backward induction.
- \( C \) equals

\[
\begin{align*}
    &\left[p^2 C_{uu} + 2p(1 - p) C_{ud} + (1 - p)^2 C_{dd}\right] (1/R^2) \\
    &= \left[p^2 \max(0, Su^2 - X) + 2p(1 - p) \max(0, Sud - X) \right. \\
    &\left. + (1 - p)^2 \max(0, Sd^2 - X)\right] / R^2.
\end{align*}
\]

\(^a\)Ernst Zermelo (1871–1953).
Backward Induction (continued)

• In the \( n \)-period case,

\[
C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max \left( 0, Su^j d^{n-j} - X \right)}{R^n}.
\]

- The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

• Similarly,

\[
P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \times \max \left( 0, X - Su^j d^{n-j} \right)}{R^n}.
\]
Backward Induction (concluded)

• Note that

\[ p_j = \binom{n}{j} p^j (1 - p)^{n-j} \]

\[ \frac{R^n}{R^n} \]

is the state price\(^a\) for the state \( Su^j d^{n-j} \), \( j = 0, 1, \ldots, n \).

• In general,

\[ \text{option price} = \sum_j p_j \times \text{payoff at state } j. \]

\(^a\)Recall p. 194. One can obtain the undiscounted state price \( \binom{n}{j} p^j (1 - p)^{n-j} \)—the risk-neutral probability—for the state \( Su^j d^{n-j} \) with \( (X_M - X_L)^{-1} \) units of the butterfly spread where \( X_L = Su^{j-1} d^{n-j+1} \), \( X_M = Su^j d^{n-j} \), and \( X_H = Su^{j-1+1} d^{n-j-1} \). See Bahra (1997).
Risk-Neutral Pricing Methodology

• Every derivative can be priced as if the economy were risk-neutral.

• For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-rn} E^\pi [ \mathcal{D} ].$$ (33)

  – $E^\pi$ means the expectation is taken under the risk-neutral probability.

• The “equivalence” between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.
Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- But it does not depend on predicting future stock prices.
- The portfolio’s value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.\(^a\)
  - Changes in value are due entirely to capital gains.

\(^a\)Except at the beginning, of course, when you have to put up the option value \(C\) or \(P\) before the replication starts.
Hakansson’s Paradox\(^a\)

- If options can be replicated, why are they needed at all?

\(^a\)Hakansson (1979).
Can You Figure Out $u, d$ without Knowing $q$?\(^a\)

- Yes, you can, under BOPM.
- Let us observe the time series of past stock prices, e.g.,
  
  $u$ is available
  \[
  \underbrace{S, Su, Su^2, Su^3, Su^3d, \ldots} \quad d \text{ is available}
  \]

- So with sufficiently long history, you will figure out $u$ and $d$ without knowing $q$.

\(^a\)Contributed by Mr. Hsu, Jia-Shuo (D97945003) on March 11, 2009.
Binomial Distribution

• Denote the binomial distribution with parameters $n$ and $p$ by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1 - p)^{n-j} = \frac{n!}{j! (n - j)!} p^j (1 - p)^{n-j}.$$

  - $n! = 1 \times 2 \times \cdots \times n$.
  - Convention: $0! = 1$.

• Suppose you flip a coin $n$ times with $p$ being the probability of getting heads.

• Then $b(j; n, p)$ is the probability of getting $j$ heads.
The Binomial Option Pricing Formula

• The stock prices at time $n$ are 
  $$Su^n, Su^{n-1}d, \ldots, Sd^n.$$ 

• Let $a$ be the minimum number of upward price moves for the call to finish in the money.

• So $a$ is the smallest nonnegative integer $j$ such that 
  $$Su^j d^{n-j} \geq X,$$
  or, equivalently,
  $$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$
The Binomial Option Pricing Formula (concluded)

• Hence,

\[
C = \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \left( S u^j d^{n-j} - X \right) \frac{R^n}{R^n}
\]

\[
= S \sum_{j=a}^{n} \binom{n}{j} (pu)^j \left[ (1 - p) d \right]^{n-j} \frac{R^n}{R^n}
\]

\[= \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j} \]

\[= S \sum_{j=a}^{n} b(j; n, pu/R) - X e^{-rn} \sum_{j=a}^{n} b(j; n, p). \tag{35}\]
Numerical Examples

- A non-dividend-paying stock is selling for $160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is $85.069$ by backward induction.
- Or, the PV of the expected payoff at expiration:
  \[
  \frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.
  \]
Binomial process for the stock price
(probabilities in parentheses)

Binomial process for the call price
(hedge ratios in parentheses)
Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for $90 instead.
- Sell the call for $90 and invest $85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,
  
  \[ 90 - 85.069 = 4.931 \text{ dollars}, \]

  is the arbitrage profit as we will see.
Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to $240.
- The new delta is 0.90625.
- Buy
  
  \[ 0.90625 - 0.82031 = 0.08594 \]
  
  more shares at the cost of \( 0.08594 \times 240 = 20.6256 \) dollars financed by borrowing.

- Debt now totals \( 20.6256 + 46.1806 \times 1.2 = 76.04232 \) dollars.
Numerical Examples (continued)

• The trading strategy is self-financing because the portfolio has a value of

\[ 0.90625 \times 240 - 76.04232 = 141.45768. \]

• It matches the corresponding call value!
Numerical Examples (continued)

Time 2:

• Suppose the stock price plunges to $120.
• The new delta is 0.25.
• Sell $0.90625 - 0.25 = 0.65625$ shares.
• This generates an income of $0.65625 \times 120 = 78.75$ dollars.
• Use this income to reduce the debt to

\[
76.04232 \times 1.2 - 78.75 = 12.5
\]

dollars.
Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to $180.
- The call we wrote finishes in the money.
- For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.
Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to $60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of 
  \[0.25 \times 60 = 15\]  
dollars.
- Use it to repay the debt of \(12.5 \times 1.2 = 15\) dollars.
Applications besides Exploiting Arbitrage Opportunities\textsuperscript{a}

- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with $85.069.

- Hedge the options we issued.
  - Use $85.069 to set up a portfolio to replicate the call to counterbalance its values exactly.\textsuperscript{b}

- …

- Without hedge, one may end up forking out $390 in the worst case!\textsuperscript{c}

\textsuperscript{a}Thanks to a lively class discussion on March 16, 2011.

\textsuperscript{b}Hedging and replication are mirror images.

\textsuperscript{c}Thanks to a lively class discussion on March 16, 2016.
Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.

- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.

- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.\(^a\)

- To price European puts, simply replace the payoff.

\(^a\)But watch out for the proper updating of array entries.
\[
\begin{align*}
C[0][0] & \quad 1 - p
\end{align*}
\]

\[
\begin{align*}
C[1][0] & \quad p \quad 1 - p
\end{align*}
\]

\[
\begin{align*}
C[2][0] & \quad p \quad 1 - p
\end{align*}
\]

\[
\begin{align*}
\max(0, Su^3 - X)
\end{align*}
\]

\[
\begin{align*}
C[1][1] & \quad 1 - p
\end{align*}
\]

\[
\begin{align*}
C[2][1] & \quad p \quad 1 - p
\end{align*}
\]

\[
\begin{align*}
\max(0, Su^2 d - X)
\end{align*}
\]

\[
\begin{align*}
C[2][2] & \quad 1 - p
\end{align*}
\]

\[
\begin{align*}
\max(0, Sud^2 - X)
\end{align*}
\]

\[
\begin{align*}
\max(0, Sd^3 - X)
\end{align*}
\]
Further Time Improvement for Calls
Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.

- Note that

$$b(j; n, p) = \frac{p(n - j + 1)}{(1 - p) j} b(j - 1; n, p).$$
Optimal Algorithm (continued)

- The following program computes $b(j; n, p)$ in $b[j]$:
- It runs in $O(n)$ steps.

1: $b[a] := \binom{n}{a} p^a (1 - p)^{n-a}$;
2: for $j = a + 1, a + 2, \ldots , n$ do
3: $b[j] := b[j - 1] \times p \times (n - j + 1)/((1 - p) \times j)$;
4: end for
Optimal Algorithm (concluded)

- With the $b(j; n, p)$ available, the risk-neutral valuation formula (34) on p. 255 is trivial to compute.
- But we only need a single variable to store the $b(j; n, p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique cannot be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.
The Bushy Tree

\[ Su^3 \]
\[ Su^2 \]
\[ Su^2d \]
\[ Su \]
\[ Sud \]
\[ Sud^2 \]
\[ Sd \]
\[ Sdu \]
\[ Sud^2 \]
\[ Sd^2 \]
\[ Sud^2 \]
\[ Sd^3 \]

\[ 2^n \]

\[ n \]
Toward the Black-Scholes Formula

• The binomial model seems to suffer from two unrealistic assumptions.
  – The stock price takes on only two values in a period.
  – Trading occurs at discrete points in time.

• As \( n \) increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.\(^a\)

• Need to calibrate the BOPM’s parameters \( u, d, \) and \( R \) to make it converge to the continuous-time model.

• We now skim through the proof.

\(^a\)Continuous-time trading may create arbitrage opportunities in practice (Budish, Cramton, & Shim, 2015)!
Toward the Black-Scholes Formula (continued)

• Let $\tau$ denote the time to expiration of the option measured in years.

• Let $r$ be the continuously compounded annual rate.

• With $n$ periods during the option’s life, each period represents a time interval of $\tau/n$.

• Need to adjust the period-based $u$, $d$, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$. 
Toward the Black-Scholes Formula (continued)

• First, $\hat{r} = r\tau/n$.
  – Each period is $\tau/n$ years long.
  – The period gross return $R = e^{\hat{r}}$.

• Let
  \[
  \hat{\mu} \equiv \frac{1}{n} E \left[ \ln \frac{S_\tau}{S} \right]
  \]
  denote the expected value of the continuously compounded rate of return per period.

• Let
  \[
  \hat{\sigma}^2 \equiv \frac{1}{n} \text{Var} \left[ \ln \frac{S_\tau}{S} \right]
  \]
  denote the variance of that return.
Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that

\[
\hat{\mu} = q \ln(u/d) + \ln d,
\]
\[
\hat{\sigma}^2 = q(1 - q) \ln^2(u/d).
\]

- Assume the stock’s *true* continuously compounded rate of return over \( \tau \) years has mean \( \mu \tau \) and variance \( \sigma^2 \tau \).

- Call \( \sigma \) the stock’s (annualized) volatility.
Toward the Black-Scholes Formula (continued)

• The BOPM converges to the distribution only if

\[ n\hat{\mu} = n[q \ln(u/d) + \ln d] \to \mu \tau, \]
\[ n\hat{\sigma}^2 = nq(1 - q) \ln^2(u/d) \to \sigma^2 \tau. \]

• We need one more condition to have a solution for \( u, d, q \).
Toward the Black-Scholes Formula (continued)

• Impose

\[ ud = 1. \]

– It makes nodes at the same horizontal level of the tree have identical price (review p. 267).
– Other choices are possible (see text).

• Exact solutions for \( u, d, q \) are feasible if Eqs. (36)–(37) are replaced by equations: 3 equations for 3 variables.\(^a\)

\(^a\)Chance (2008).
Toward the Black-Scholes Formula (continued)

• The above requirements can be satisfied by

\[ u = e^{\sigma \sqrt{\tau/n}}, \quad d = e^{-\sigma \sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2} \frac{\mu}{\sigma} \sqrt{\frac{\tau}{n}}. \quad (38) \]

• With Eqs. (38), it can be checked that

\[ n\hat{\mu} = \mu \tau, \]
\[ n\hat{\sigma}^2 = \left[ 1 - \left( \frac{\mu}{\sigma} \right)^2 \frac{\tau}{n} \right] \sigma^2 \tau \to \sigma^2 \tau. \]
Toward the Black-Scholes Formula (continued)

- The choices (38) result in the CRR binomial model.\(^a\)

- With the above choice, even if \(u\) and \(d\) are not calibrated, the mean is still matched!\(^b\)

\(^a\)Cox, Ross, and Rubinstein (1979).
\(^b\)Recall Eq. (31) on p. 237. So \(u\) and \(d\) are related to volatility.
Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities \( d < R < u \) may not hold under Eqs. (38) on p. 278 or Eq. (30) on p. 236.
  - If this happens, the probabilities lie outside \([0, 1]\).\(^a\)

- The problem disappears when \( n \) satisfies
  \[
  e^{\sigma \sqrt{r/\tau}} > e^{r \tau/n},
  \]
  i.e., when \( n > r^2 \tau / \sigma^2 \) (check it).
  - So it goes away if \( n \) is large enough.
  - Other solutions will be presented later.

\(^a\)Many papers and programs forget to check this condition!
Toward the Black-Scholes Formula (continued)

• What is the limiting probabilistic distribution of the continuously compounded rate of return \( \ln(S_\tau/S) \)?

• The central limit theorem says \( \ln(S_\tau/S) \) converges to \( N(\mu \tau, \sigma^2 \tau) \).\(^a\)

• So \( \ln S_\tau \) approaches \( N(\mu \tau + \ln S, \sigma^2 \tau) \).

• Conclusion: \( S_\tau \) has a lognormal distribution in the limit.

\(^a\)The normal distribution with mean \( \mu \tau \) and variance \( \sigma^2 \tau \).
Lemma 11 \textit{The continuously compounded rate of return} 
\[ \ln\left(\frac{S_\tau}{S}\right) \] \textit{approaches the normal distribution with mean} 
\[ (r - \sigma^2/2)\tau \] \textit{and variance} \[ \sigma^2\tau \] \textit{in a risk-neutral economy.}

- Let \( q \) equal the risk-neutral probability
  \[ p \equiv \frac{(e^{r\tau/n} - d)}{(u - d)}. \]
- Let \( n \to \infty. \)

\footnote{See Lemma 9.3.3 of the textbook.}
Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is\(^a\)

\[ Se^{r\tau}. \]

- The stock’s expected annual rate of return\(^b\) is thus the riskless rate \( r \).

\(^a\)By Lemma 11 (p. 283) and Eq. (26) on p. 165.

\(^b\)In the sense of \((1/\tau) \ln E[S_\tau/S]\) (arithmetic average rate of return) not \((1/\tau) E[\ln(S_\tau/S)]\) (geometric average rate of return). In the latter case, it would be \( r - \sigma^2/2 \) by Lemma 11.
Theorem 12 (The Black-Scholes Formula)

\[
C = SN(x) - X e^{-r\tau} N(x - \sigma \sqrt{\tau}),
\]
\[
P = X e^{-r\tau} N(-x + \sigma \sqrt{\tau}) - SN(-x),
\]

where

\[
x \equiv \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}}.
\]

\(^{\text{a}}\text{On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!}\)
Toward the Black-Scholes Formula (concluded)

- See Eq. (35) on p. 255 for the meaning of $x$.
- See Exercise 13.2.12 of the textbook for an interpretation of the probability measure associated with $N(x)$ and $N(-x)$. 
BOPM and Black-Scholes Model

• The Black-Scholes formula needs 5 parameters: $S, X, \sigma, \tau,$ and $r$.

• Binomial tree algorithms take 6 inputs: $S, X, u, d, \hat{r},$ and $n$.

• The connections are

$$u = e^{\sigma\sqrt{\tau/n}},$$

$$d = e^{-\sigma\sqrt{\tau/n}},$$

$$\hat{r} = r\tau/n.$$
• $S = 100$, $X = 100$ (left), and $X = 95$ (right).
BOPM and Black-Scholes Model (concluded)

• The binomial tree algorithms converge reasonably fast.

• The error is $O(1/n)$.\textsuperscript{a}

• Oscillations are inherent, however.

• Oscillations can be dealt with by the judicious choices of $u$ and $d$.\textsuperscript{b}

\textsuperscript{a}Chang and Palmer (2007).
\textsuperscript{b}See Exercise 9.3.8 of the textbook.
Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market’s opinion of the volatility.\(^a\)
  - Solve for \(\sigma\) given the option price, \(S\), \(X\), \(\tau\), and \(r\) with numerical methods.
  - How about American options?

\(^a\)Implied volatility is hard to compute when \(\tau\) is small (why?).
Implied Volatility (concluded)

• Implied volatility is the wrong number to put in the wrong formula to get the right price of plain-vanilla options.\textsuperscript{a}

• Implied volatility is often preferred to historical volatility in practice.
  – Using the historical volatility is like driving a car with your eyes on the rearview mirror?

\textsuperscript{a}Rebonato (2004).
Problems; the Smile

• Options written on the same underlying asset usually do not produce the same implied volatility.

• A typical pattern is a “smile” in relation to the strike price.
  – The implied volatility is lowest for at-the-money options.
  – It becomes higher the further the option is in- or out-of-the-money.

• Other patterns have also been observed.
Problems; the Smile (concluded)

• To address this issue, volatilities are often combined to produce a composite implied volatility.

• This practice is not sound theoretically.

• The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

• So?
Binomial Tree Algorithms for American Puts

• Early exercise has to be considered.

• The binomial tree algorithm starts with the terminal payoffs

\[ \max(0, X - Su^j d^{m-j}) \]

and applies backward induction.

• At each intermediate node, it compares the payoff if exercised and the continuation value.

• It keeps the larger one.
Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.
Time-Dependent Instantaneous Volatility$^a$

- Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of $\sigma$.

- In the limit, the variance of $\ln(S_\tau/S)$ is
  \[ \int_0^\tau \sigma^2(t) \, dt \]
  rather than $\sigma^2 \tau$.

- The annualized volatility to be used in the Black-Scholes formula should now be
  \[ \sqrt{\frac{\int_0^\tau \sigma^2(t) \, dt}{\tau}}. \]

$^a$Merton (1973).
Time-Dependent Instantaneous Volatility (concluded)

- There is no guarantee that the implied volatility is constant.

- For the binomial model, $u$ and $d$ depend on time:

\[
  u = e^{\sigma(t) \sqrt{\tau/n}}, \quad d = e^{-\sigma(t) \sqrt{\tau/n}}.
\]

- How to make the binomial tree combine?\(^a\)

\(^a\)Amin (1991); Chen (R98922127) (2011).
Time-Dependent Short Rates

• Suppose the short rate (i.e., the one-period spot rate) changes over time but otherwise predictable.

• The riskless rate $r$ in the Black-Scholes formula should be the spot rate with a time to maturity equal to $\tau$.

• In other words,

$$ r = \sum_{i=0}^{n-1} \frac{r_i}{\tau}, $$

where $r_i$ is the continuously compounded short rate measured in periods for period $i$.\(^a\)

• Will the binomial tree fail to combine?

\(^a\)That is, one-period forward rate.
Trading Days and Calendar Days

• Interest accrues based on the calendar day.

• But $\sigma$ is usually calculated based on trading days only.
  
  – Stock price seems to have lower volatilities when the exchange is closed.\textsuperscript{a}

• How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?\textsuperscript{b}

\textsuperscript{a}Fama (1965); K. French (1980); K. French & Roll (1986).

\textsuperscript{b}Recall p. 150 about dating issues.
Trading Days and Calendar Days (continued)

- Think of $\sigma$ as measuring the *annualized* volatility of stock price *one year from now*.

- Suppose a year has $m$ (say 253) trading days.

- We can replace $\sigma$ in the Black-Scholes formula with

$$\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}.$$ 

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\(^{a}\)D. French (1984).
Trading Days and Calendar Days (concluded)

- This works only for European options.
- How about binomial tree algorithms?\(^a\)

\(^a\)Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.
Options on a Stock That Pays Dividends

• Early exercise must be considered.

• Proportional dividend payout model is tractable (see text).
  – The dividend amount is a constant proportion of the prevailing stock price.

• In general, the corporate dividend policy is a complex issue.