Day Count Conventions: Actual/Actual

• The first “actual” refers to the actual number of days in a month.

• The second refers to the actual number of days in a coupon period.

• The number of days between June 17, 1992, and October 1, 1992, is 106.
  – 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.
Day Count Conventions: 30/360

• Each month has 30 days and each year 360 days.

• The number of days between June 17, 1992, and October 1, 1992, is 104.
  – 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.

• In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_2, m_2, d_2)$ is

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1)$$

• But if $d_1$ or $d_2$ is 31, we need to change it to 30 before applying the above formula.
Day Count Conventions: 30/360 (concluded)

- An equivalent formula without any adjustment is (check it)

\[
360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1) \\
+ \max(30 - d_1, 0) + \min(d_2, 30).
\]

- Many variations regarding 31, Feb 28, and Feb 29.
Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.

- Let

\[
\omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}.
\]

(11)
Full Price (continued)

\[ C(1 - \omega) \]

coupon payment date \( (1 - \omega) \) \( \omega \) coupon payment date
Full Price (concluded)

- The price is now calculated by

\[
PV = \frac{C}{(1 + \frac{r}{m})^\omega} + \frac{C}{(1 + \frac{r}{m})^{\omega+1}} \cdots
\]

\[
= \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}.
\]  \hspace{1cm} (12)
Accrued Interest

- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.
- The buyer pays the invoice price: the quoted price plus the accrued interest (AI).
- The accrued interest equals

\[
C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega).
\]
Accrued Interest (concluded)

- The yield to maturity is the \( r \) satisfying Eq. (12) on p. 76 when PV is the invoice price:

\[
\text{clean price} + \text{AI} = \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}.
\]
Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
- The accrued interest is \((10/2) \times (1 - \frac{60}{180}) = 3.3333\) per $100 of par value.
Example (“30/360”) (concluded)

• The yield to maturity is 3%.

• This can be verified by Eq. (12) on p. 76 with
  \[ \omega = \frac{60}{180}, \]
  \[ n = 4, \]
  \[ m = 2, \]
  \[ F = 100, \]
  \[ C = 5, \]
  \[ PV = 111.2891 + 3.3333, \]
  \[ r = 0.03. \]
Price Behavior (2) Revisited

• Before: A bond selling at par if the yield to maturity equals the coupon rate.

• But it assumed that the settlement date is on a coupon payment date.

• Now suppose the settlement date for a bond selling at par (i.e., the quoted price is equal to the par value) falls between two coupon payment dates.

• Then its yield to maturity is less than the coupon rate.
  – The short reason: Exponential growth to $C$ is replaced by linear growth, hence “overpaying.”
Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy
Price Volatility

• Volatility measures how bond prices respond to interest rate changes.

• It is key to the risk management of interest rate-sensitive securities.
Price Volatility (concluded)

• What is the sensitivity of the percentage price change to changes in interest rates?

• Define price volatility by

\[- \frac{\partial P}{\partial y} \cdot \frac{P}{P} \cdot (13)\]
Price Volatility of Bonds

• The price volatility of a level-coupon bond is

\[
- \frac{(C/y) n - (C/y^2) \left( (1 + y)^{n+1} - (1 + y) \right) - nF}{(C/y) \left( (1 + y)^{n+1} - (1 + y) \right) + F(1 + y)}.
\]

  - $F$ is the par value.
  - $C$ is the coupon payment per period.
  - Formula can be simplified a bit with $C = Fc/m$.

• For bonds without embedded options,

\[
- \frac{\partial P}{\partial y} > 0.
\]

• What is the volatility of the bond in Eq. (12) on p. 76?
Macaulay Duration\textsuperscript{a}

- The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.

- The weights are the cash flows’ PVs divided by the asset’s price.

- Formally,

\[
\text{MD} \equiv \frac{1}{P} \sum_{i=1}^{n} i \frac{C_i}{(1+y)^i}.
\]

- The Macaulay duration, in periods, is equal to

\[
\text{MD} = -(1+y) \frac{\partial P}{\partial y} \frac{1}{P}.
\] \hfill (14)

\textsuperscript{a}Macaulay (1938).
MD of Bonds

• The MD of a level-coupon bond is

\[
MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1 + y)^i} + \frac{nF}{(1 + y)^n} \right]. \quad (15)
\]

• It can be simplified to

\[
MD = \frac{c(1 + y) \left[ (1 + y)^n - 1 \right] + ny(y - c)}{cy \left[ (1 + y)^n - 1 \right] + y^2},
\]

where \( c \) is the period coupon rate.

• The MD of a zero-coupon bond equals \( n \), its term to maturity.

• The MD of a level-coupon bond is less than \( n \).
Remarks

• Equations (14) on p. 87 and (15) on p. 88 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  – That is, if the cash flow is independent of yields.

• To see this point, suppose the market yield declines.

• The MD will be lengthened.

• But for securities whose maturity actually decreases as a result, the price volatility$^a$ may decrease.

$^a$As originally defined in Eq. (13) on p. 85.
How Not To Think about MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But it should be seen mainly as measuring *price volatility*.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.
Conversion

- For the MD to be year-based, modify Eq. (15) on p. 88 to
  \[
  \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} C \left(1 + \frac{y}{k}\right)^i + \frac{n}{k} \frac{F}{(1 + \frac{y}{k})^n} \right],
  \]
  where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

- Equation (14) on p. 87 also becomes
  \[
  MD = - \left(1 + \frac{y}{k}\right) \frac{\partial P}{\partial y} \frac{1}{P}.
  \]

- By definition, \( MD \) (in years) = \( \frac{MD \text{ (in periods)}}{k} \).  
Modified Duration

• Modified duration is defined as

\[\text{modified duration} \equiv -\frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}.\]  \hspace{1cm} (16)

• By the Taylor expansion,

percent price change \(\approx -\text{modified duration} \times \text{yield change}.\)
Example

• Consider a bond whose modified duration is 11.54 with a yield of 10%.

• If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

\[-11.54 \times 0.001 = -0.01154 = -1.154\%.
\]
Modified Duration of a Portfolio

• The modified duration of a portfolio equals

\[ \sum_{i} \omega_i D_i. \]

- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

- Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.
- We need a general numerical formula for volatility.
- The effective duration is defined as

\[
\frac{P_- - P_+}{P_0(y_+ - y_-)}.
\]

- \(P_-\) is the price if the yield is decreased by \(\Delta y\).
- \(P_+\) is the price if the yield is increased by \(\Delta y\).
- \(P_0\) is the initial price, \(y\) is the initial yield.
- \(\Delta y\) is small.
Effective Duration (concluded)

- One can compute the effective duration of just about any financial instrument.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use
  \[
  \frac{P_0 - P_+}{P_0 \Delta y}.
  \]
  - More economical but theoretically less accurate.
The Practices

• Duration is usually expressed in percentage terms — call it $D\%$ — for quick mental calculation.\(^{a}\)

• The percentage price change expressed in percentage terms is then approximated by

$$-D\% \times \Delta r$$

when the yield increases instantaneously by $\Delta r\%$.

– Price will drop by 20% if $D\% = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.

• $D\%$ in fact equals modified duration (prove it!).

\(^{a}\)Neftci (2008), “Market professionals do not like to use decimal points.”
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

$$\text{modified duration} \times \text{price} = -\frac{\partial P}{\partial y}.$$ 

- The approximate dollar price change is

$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$ 

- One can hedge a bond with a dollar duration $D$ by bonds with a dollar duration $-D$. 
Convexity

• Convexity is defined as

\[
\text{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.
\]

• The convexity of a level-coupon bond is positive (prove it!).

• For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

• So between two bonds with the same price and duration, the one with a higher convexity is more valuable.\(^a\)

\(^a\)Do you spot a problem here (Christensen & Sørensen, 1994)?
Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

\[
\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}
\]

when there are \( k \) periods per annum.
Use of Convexity

- The approximation $\frac{\Delta P}{P} \approx -\text{duration} \times \text{yield change}$ works for small yield changes.

- For larger yield changes, use

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2$$

$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$

- Recall the figure on p. 101.
The Practices

• Convexity is usually expressed in percentage terms — call it $C\%$ — for quick mental calculation.

• The percentage price change expressed in percentage terms is approximated by

$$-D\% \times \Delta r + C\% \times (\Delta r)^2 / 2$$

when the yield increases instantaneously by $\Delta r\%$.

- Price will drop by 17% if $D\% = 10$, $C\% = 1.5$, and $\Delta r = 2$ because

$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$

• $C\%$ equals convexity divided by 100 (prove it!).
Effective Convexity

• The effective convexity is defined as

\[
\frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2},
\]

- \( P_- \) is the price if the yield is decreased by \( \Delta y \).
- \( P_+ \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.

• Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

• Numerically, choosing the right \( \Delta y \) is a delicate matter.
Approximate \( d^2 f(x)^2 / dx^2 \) at \( x = 1 \), \( f(x) = x^2 \)

- The difference of \( ((1 + \Delta x)^2 + (1 - \Delta x)^2 - 2)/(\Delta x)^2 \) and 2:

- This numerical issue is common in financial engineering but does not admit general solutions yet (see pp. 792ff).
Interest Rates and Bond Prices: Which Determines Which?\textsuperscript{a}

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

\textsuperscript{a}Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest. Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.

- The set of yields to maturity for bonds form the term structure.
  - The bonds must be of equal quality.
  - They differ solely in their terms to maturity.

- The term structure is fundamental to the valuation of fixed-income securities.
Term Structure of Interest Rates (concluded)

• Term structure often refers exclusively to the yields of zero-coupon bonds.

• A yield curve plots the yields to maturity of coupon bonds against maturity.

• A par yield curve is constructed from bonds trading near par.
Yield Curve as of July 24, 2015

Yield (%) vs. Year

©2017 Prof. Yuh-Dauh Lyuu, National Taiwan University
Four Typical Shapes

• A normal yield curve is upward sloping.
• An inverted yield curve is downward sloping.
• A flat yield curve is flat.
• A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

- The $i$-period spot rate $S(i)$ is the yield to maturity of an $i$-period zero-coupon bond.
- The PV of one dollar $i$ periods from now is by definition 
  \[ [1 + S(i)]^{-i}. \]
  - It is the price of an $i$-period zero-coupon bond.\(^a\)
- The one-period spot rate is called the short rate.
- Spot rate curve: Plot of spot rates against maturity:
  \[ S(1), S(2), \ldots, S(n). \]

\(^a\)Recall Eq. (9) on p. 61.
Problems with the PV Formula

- In the bond price formula (3) on p. 35,

\[ \sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n}, \]

every cash flow is discounted at the same yield \( y \).

- Consider two riskless bonds with different yields to maturity because of their different cash flow streams:

\[ PV_1 = \sum_{i=1}^{n_1} \frac{C}{(1+y_1)^i} + \frac{F}{(1+y_1)^{n_1}}, \]
\[ PV_2 = \sum_{i=1}^{n_2} \frac{C}{(1+y_2)^i} + \frac{F}{(1+y_2)^{n_2}}. \]
Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their *contemporaneous* cash flows with *different* rates.
- But shouldn’t they be discounted at the *same* rate?
Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the $i$th bond paying $C_i$ dollars at time $i$. 

\[ \begin{align*} 
C_1 & \uparrow \\
1 & \downarrow \\
C_2 & \uparrow \\
2 & \downarrow \\
C_3 & \uparrow \\
3 & \downarrow \\
\vdots & \vdots \\
C_n & \uparrow \\
n & \downarrow 
\end{align*} \]
Spot Rate Discount Methodology (concluded)

• So a level-coupon bond has the price

\[ P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}. \]  \hspace{1cm} (17)

• This pricing method incorporates information from the term structure.

• It discounts each cash flow at the corresponding spot rate.
Discount Factors

• In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

– Above, $d(i) \equiv [1 + S(i)]^{-i}$, $i = 1, 2, \ldots, n$, are called the discount factors.

– $d(i)$ is the PV of one dollar $i$ periods from now.

– This formula—now just a definition—will be justified on p. 204.

• The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

• Start with the short rate $S(1)$.
  – Note that short-term Treasuries are zero-coupon bonds.

• Compute $S(2)$ from the two-period coupon bond price $P$ by solving

$$P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.$$
Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price $P$ of the $n$-period coupon bond and

  $$S(1), S(2), \ldots, S(n - 1).$$

- Then $S(n)$ can be computed from Eq. (17) on p. 118, repeated below,

  $$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$  

- The running time can be made to be $O(n)$ (see text).
- The procedure is called bootstrapping.
Some Problems

• Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).

• Some maturities might be missing from the data points (the incompleteness problem).

• Treasuries might not be of the same quality.

• Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.\(^a\)

\(^a\)Any economic justifications?
Which One?