Foundations of Term Structure Modeling
[Meriwether] scoring especially high marks in mathematics — an indispensable subject for a bond trader.

[The] fixed-income traders I knew seemed smarter than the equity trader […] there’s no competitive edge to being smart in the equities business[.]


Bond market terminology was designed less to convey meaning than to bewilder outsiders.

Terminology

• A period denotes a unit of elapsed time.
  – Viewed at time $t$, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case.

• Bonds will be assumed to have a par value of one — unless stated otherwise.

• The time unit for continuous-time models will usually be measured by the year.
Standard Notations

The following notation will be used throughout.

\( t \): a point in time.

\( r(t) \): the one-period riskless rate prevailing at time \( t \) for repayment one period later.\(^a\)

\( P(t, T) \): the present value at time \( t \) of one dollar at time \( T \).

\(^a\)Alternatively, the instantaneous spot rate, or short rate, at time \( t \).
Standard Notations (continued)

$r(t, T)$: the $(T - t)$-period interest rate prevailing at time $t$ stated on a per-period basis and compounded once per period.\(^a\)

$F(t, T, M)$: the forward price at time $t$ of a forward contract that delivers at time $T$ a zero-coupon bond maturing at time $M \geq T$.

\(^a\)In other words, the $(T - t)$-period spot rate at time $t$. 
Standard Notations (concluded)

\( f(t, T, L) \): the \( L \)-period forward rate at time \( T \) implied at time \( t \) stated on a per-period basis and compounded once per period.

\( f(t, T) \): the one-period or instantaneous forward rate at time \( T \) as seen at time \( t \) stated on a per period basis and compounded once per period.

- It is \( f(t, T, 1) \) in the discrete-time model and \( f(t, T, dt) \) in the continuous-time model.
- Note that \( f(t, t) \) equals the short rate \( r(t) \).
Fundamental Relations

- The price of a zero-coupon bond equals

\[ P(t, T) = \begin{cases} 
   (1 + r(t, T))^{-(T-t)}, & \text{in discrete time,} \\
   e^{-r(t,T)(T-t)}, & \text{in continuous time.}
\end{cases} \]

- \( r(t, T) \) as a function of \( T \) defines the spot rate curve at time \( t \).

- By definition,

\[ f(t,t) = \begin{cases} 
   r(t, t + 1), & \text{in discrete time,} \\
   r(t, t), & \text{in continuous time.}
\end{cases} \]
Fundamental Relations (continued)

• Forward prices and zero-coupon bond prices are related:

\[ F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M. \]  \hspace{1cm} (115)

- The forward price equals the future value at time \( T \) of the underlying asset.\(^a\)

• Equation (115) holds whether the model is discrete-time or continuous-time.

\(^a\)See Exercise 24.2.1 of the textbook for proof.
Fundamental Relations (continued)

- Forward rates and forward prices are related definitionally by

\[
f(t, T, L) = \left( \frac{1}{F(t, T, T+L)} \right)^{1/L} - 1 = \left( \frac{P(t, T)}{P(t, T+L)} \right)^{1/L} - 1
\]

(116)

in discrete time.

- The analog to Eq. (116) under simple compounding is

\[
f(t, T, L) = \frac{1}{L} \left( \frac{P(t, T)}{P(t, T+L)} - 1 \right).
\]
Fundamental Relations (continued)

- In continuous time,

\[ f(t, T, L) = - \frac{\ln F(t, T, T + L)}{L} = \frac{\ln(P(t, T)/P(t, T + L))}{L} \]

by Eq. (115) on p. 969.

- Furthermore,

\[
\begin{align*}
f(t, T, \Delta t) & = \frac{\ln(P(t, T)/P(t, T + \Delta t))}{\Delta t} \to - \frac{\partial \ln P(t, T)}{\partial T} \\
& = - \frac{\partial P(t, T)/\partial T}{P(t, T)}.
\end{align*}
\]
Fundamental Relations (continued)

• So

\[ f(t, T) \equiv \lim_{\Delta t \to 0} f(t, T, \Delta t) = -\frac{\partial P(t, T) / \partial T}{P(t, T)}, \quad t \leq T. \]

(118)

• Because Eq. (118) is equivalent to

\[ P(t, T) = e^{-\int_t^T f(t, s) \, ds}, \]

(119)

the spot rate curve is

\[ r(t, T) = \frac{\int_t^T f(t, s) \, ds}{T - t}. \]
Fundamental Relations (concluded)

- The discrete analog to Eq. (119) is

\[ P(t, T) = \frac{1}{(1 + r(t))(1 + f(t, t + 1)) \cdots (1 + f(t, T - 1))}. \]

- The short rate and the market discount function are related by

\[ r(t) = - \frac{\partial P(t, T)}{\partial T} \bigg|_{T=t}. \]
Risk-Neutral Pricing

• Assume the local expectations theory.

• The expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate.
  – For all $t + 1 < T$,
    \[
    \frac{E_t[ P(t + 1, T) ]}{P(t, T)} = 1 + r(t). \tag{120}
    \]
    
    – Relation (120) in fact follows from the risk-neutral valuation principle.\(^a\)

\(^a\)Theorem 17 on p. 503.
Risk-Neutral Pricing (continued)

- The local expectations theory is thus a consequence of the existence of a risk-neutral probability \( \pi \).

- Rewrite Eq. (120) as

\[
E_t^\pi \left[ \frac{P(t+1, T)}{1 + r(t)} \right] = P(t, T).
\]

  - It says the current market discount function equals the expected market discount function one period from now discounted by the short rate.
Risk-Neutral Pricing (continued)

• Apply the above equality iteratively to obtain

\[
P(t, T) = E_t^\pi \left[ P(t + 1, T) \right] \frac{1}{1 + r(t)}
\]

\[
= E_t^\pi \left[ E_{t+1}^\pi \left[ P(t + 2, T) \right] \right] \frac{1}{1 + r(t)(1 + r(t + 1))} = \cdots
\]

\[
= E_t^\pi \left[ \frac{1}{(1 + r(t))(1 + r(t + 1)) \cdots (1 + r(T - 1))} \right]. \quad (121)
\]
Risk-Neutral Pricing (concluded)

• Equation (120) on p. 974 can also be expressed as

\[ E_t[P(t+1, T)] = F(t, t+1, T). \]

– Verify that with, e.g., Eq. (115) on p. 969.

• Hence the forward price for the next period is an unbiased estimator of the expected bond price.\(^a\)

\(^a\)But the forward rate is not an unbiased estimator of the expected future short rate (p. 925).
Continuous-Time Risk-Neutral Pricing

• In continuous time, the local expectations theory implies

\[ P(t, T) = E_t \left[ e^{\int_t^T r(s) \, ds} \right], \quad t < T. \quad (122) \]

• Note that \( e^{\int_t^T r(s) \, ds} \) is the bank account process, which denotes the rolled-over money market account.
Interest Rate Swaps

• Consider an interest rate swap made at time $t$ (now) with payments to be exchanged at times $t_1, t_2, \ldots, t_n$.

• The fixed rate is $c$ per annum.

• The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$.

• For simplicity, assume $t_{i+1} - t_i$ is a fixed constant $\Delta t$ for all $i$, and the notional principal is one dollar.

• If $t < t_0$, we have a forward interest rate swap.

• The ordinary swap corresponds to $t = t_0$. 
Interest Rate Swaps (continued)

- The amount to be paid out at time $t_{i+1}$ is $(f_i - c) \Delta t$ for the floating-rate payer.
- Simple rates are adopted here.
- Hence $f_i$ satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$
Interest Rate Swaps (continued)

- The value of the swap at time \( t \) is thus

\[
\sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t_i}^{t} r(s) \, ds} \left( f_{i-1} - c \right) \Delta t \right]
\]

\[
= \sum_{i=1}^{n} E_t^\pi \left[ e^{-\int_{t_i}^{t} r(s) \, ds} \left( \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right) \right]
\]

\[
= \sum_{i=1}^{n} \left[ P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i) \right]
\]

\[
= P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\]
Interest Rate Swaps (concluded)

- So a swap can be replicated as a portfolio of bonds.
- In fact, it can be priced by simple present value calculations.
Swap Rate

• The swap rate, which gives the swap zero value, equals

\[ S_n(t) \equiv \frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i) \Delta t}. \]  

(123)

• The swap rate is the fixed rate that equates the present values of the fixed payments and the floating payments.

• For an ordinary swap, \( P(t, t_0) = 1 \).
The Term Structure Equation

• Let us start with the zero-coupon bonds and the money market account.

• Let the zero-coupon bond price $P(r, t, T)$ follow

$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$ 

• At time $t$, short one unit of a bond maturing at time $s_1$ and buy $\alpha$ units of a bond maturing at time $s_2$. 
The Term Structure Equation (continued)

- The net wealth change follows

\[-dP(r, t, s_1) + \alpha dP(r, t, s_2)\]

\[= (-P(r, t, s_1) \mu_p(r, t, s_1) + \alpha P(r, t, s_2) \mu_p(r, t, s_2)) \, dt\]

\[+ (-P(r, t, s_1) \sigma_p(r, t, s_1) + \alpha P(r, t, s_2) \sigma_p(r, t, s_2)) \, dW.\]

- Pick

\[\alpha \equiv \frac{P(r, t, s_1) \sigma_p(r, t, s_1)}{P(r, t, s_2) \sigma_p(r, t, s_2)}.\]
The Term Structure Equation (continued)

- Then the net wealth has no volatility and must earn the riskless return:

$$
\frac{-P(r,t,s_1) \mu_p(r,t,s_1) + \alpha P(r,t,s_2) \mu_p(r,t,s_2)}{-P(r,t,s_1) + \alpha P(r,t,s_2)} = r.
$$

- Simplify the above to obtain

$$
\frac{\sigma_p(r,t,s_1) \mu_p(r,t,s_2) - \sigma_p(r,t,s_2) \mu_p(r,t,s_1)}{\sigma_p(r,t,s_1) - \sigma_p(r,t,s_2)} = r.
$$

- This becomes

$$
\frac{\mu_p(r,t,s_2) - r}{\sigma_p(r,t,s_2)} = \frac{\mu_p(r,t,s_1) - r}{\sigma_p(r,t,s_1)}
$$

after rearrangement.
The Term Structure Equation (continued)

Since the above equality holds for any $s_1$ and $s_2$,

$$\frac{\mu_p(r, t, s) - r}{\sigma_p(r, t, s)} \equiv \lambda(r, t)$$

for some $\lambda$ independent of the bond maturity $s$.

As $\mu_p = r + \lambda \sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset’s volatility.

The term $\lambda(r, t)$ is called the market price of risk.

The market price of risk must be the same for all bonds to preclude arbitrage opportunities.
The Term Structure Equation (continued)

- Assume a Markovian short rate model,

\[ dr = \mu(r, t) \, dt + \sigma(r, t) \, dW. \]

- Then the bond price process is also Markovian.

- By Eq. (14.15) on p. 202 of the textbook,

\[
\mu_p = \left( -\frac{\partial P}{\partial T} + \mu(r, t) \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right)/P,
\]

\[ (125) \]

\[
\sigma_p = \left( \sigma(r, t) \frac{\partial P}{\partial r} \right)/P,
\]

\[ (125') \]

subject to \( P(\cdot, T, T) = 1. \)
The Term Structure Equation (concluded)

• Substitute $\mu_p$ and $\sigma_p$ into Eq. (124) on p. 987 to obtain

$$\frac{\partial P}{\partial T} + \left[ \mu(r, t) - \lambda(r, t) \sigma(r, t) \right] \frac{\partial P}{\partial r} + \frac{1}{2} \sigma(r, t)^2 \frac{\partial^2 P}{\partial r^2} = rP.$$  \hspace{1cm} (126)

• This is called the term structure equation.

• Once $P$ is available, the spot rate curve emerges via

$$r(t, T) = -\frac{\ln P(t, T)}{T - t}.$$  

• Equation (126) applies to all interest rate derivatives, the difference being the terminal and the boundary conditions.
The Binomial Model

• The analytical framework can be nicely illustrated with the binomial model.

• Suppose the bond price $P$ can move with probability $q$ to $Pu$ and probability $1 - q$ to $Pd$, where $u > d$:
The Binomial Model (continued)

- Over the period, the bond’s expected rate of return is

\[
\hat{\mu} \equiv \frac{q P u + (1 - q) P d}{P} - 1 = q u + (1 - q) d - 1.
\]  

(127)

- The variance of that return rate is

\[
\hat{\sigma}^2 \equiv q (1 - q) (u - d)^2.
\]

(128)
The Binomial Model (continued)

- In particular, the bond whose maturity is one period away will move from a price of $1/(1+r)$ to its par value $1$.

- This is the money market account modeled by the short rate $r$.

- The market price of risk is defined as $\lambda \equiv (\hat{\mu} - r)/\hat{\sigma}$.

- As in the continuous-time case, it can be shown that $\lambda$ is independent of the maturity of the bond (see text).
The Binomial Model (concluded)

• Now change the probability from $q$ to

$$p \equiv q - \lambda \sqrt{q(1-q)} = \frac{(1+r) - d}{u - d}, \quad (129)$$

which is independent of bond maturity and $q$.

– Recall the BOPM.

• The bond’s expected rate of return becomes

$$\frac{pP_r + (1-p)P_d}{P} - 1 = pu + (1-p)d - 1 = r.$$

• The local expectations theory hence holds under the new probability measure $p$. 
Numerical Examples

- Assume this spot rate curve:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot rate</td>
<td>4%</td>
<td>5%</td>
</tr>
</tbody>
</table>

- Assume the one-year rate (short rate) can move up to 8% or down to 2% after a year:

4% \leftrightarrow 8% \\
4% \leftrightarrow 2%
Numerical Examples (continued)

• *No* real-world probabilities are specified.

• The prices of one- and two-year zero-coupon bonds are, respectively,

\[
\frac{100}{1.04} = 96.154, \\
\frac{100}{(1.05)^2} = 90.703.
\]

• They follow the binomial processes on p. 996.
Numerical Examples (continued)

\[
\begin{align*}
90.703 & \quad 92.593 \ (= 100/1.08) & \quad 96.154 & \quad 100 \\
& \quad 98.039 \ (= 100/1.02) & \quad & \quad 100
\end{align*}
\]

The price process of the two-year zero-coupon bond is on the left; that of the one-year zero-coupon bond is on the right.
Numerical Examples (continued)

- The pricing of derivatives can be simplified by assuming investors are risk-neutral.
- Suppose all securities have the same expected one-period rate of return, the riskless rate.
- Then
  \[ (1 - p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\% , \]
  where \( p \) denotes the risk-neutral probability of a down move in rates.
Numerical Examples (concluded)

- Solving the equation leads to $p = 0.319$.
- Interest rate contingent claims can be priced under this probability.
Numerical Examples: Fixed-Income Options

- A one-year European call on the two-year zero with a $95 strike price has the payoffs,

\[ C \begin{cases} 0.000 \\ 3.039 \end{cases} \]

- To solve for the option value \( C \), we replicate the call by a portfolio of \( x \) one-year and \( y \) two-year zeros.
Numerical Examples: Fixed-Income Options (continued)

• This leads to the simultaneous equations,

\[ x \times 100 + y \times 92.593 = 0.000, \]
\[ x \times 100 + y \times 98.039 = 3.039. \]

• They give \( x = -0.5167 \) and \( y = 0.5580 \).

• Consequently,

\[ C = x \times 96.154 + y \times 90.703 \approx 0.93 \]

to prevent arbitrage.
Numerical Examples: Fixed-Income Options (continued)

- This price is derived without assuming any version of an expectations theory.
- Instead, the arbitrage-free price is derived by replication.
- The price of an interest rate contingent claim does not depend directly on the real-world probabilities.
- The dependence holds only indirectly via the current bond prices.
Numerical Examples: Fixed-Income Options (concluded)

- An equivalent method is to utilize risk-neutral pricing.
- The above call option is worth
  \[ C = \frac{(1 - p) \times 0 + p \times 3.039}{1.04} \approx 0.93, \]
  the same as before.
- This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.
Numerical Examples: Futures and Forward Prices

• A one-year futures contract on the one-year rate has a payoff of $100 - r$, where $r$ is the one-year rate at maturity:

\[ F = 92 \ (= 100 - 8) \]
\[ F = 98 \ (= 100 - 2) \]

• As the futures price $F$ is the expected future payoff,\(^a\)

\[ F = (1 - p) \times 92 + p \times 98 = 93.914. \]

\(^a\)See Exercise 13.2.11 of the textbook or p. 504.
Numerical Examples: Futures and Forward Prices (concluded)

- The forward price for a one-year forward contract on a one-year zero-coupon bond is\(^a\)
  \[
  \frac{90.703}{96.154} = 94.331\%.
  \]

- The forward price exceeds the futures price.\(^b\)

\(^{a}\)By Eq. (115) on p. 969.
\(^{b}\)Recall p. 448.
Equilibrium Term Structure Models
8. What’s your problem? Any moron can understand bond pricing models.

— Top Ten Lies Finance Professors Tell Their Students
Introduction

• This chapter surveys equilibrium models.

• Since the spot rates satisfy

\[ r(t, T) = -\frac{\ln P(t, T)}{T - t}, \]

the discount function \( P(t, T) \) suffices to establish the spot rate curve.

• All models to follow are short rate models.

• Unless stated otherwise, the processes are risk-neutral.
The Vasicek Model\textsuperscript{a}

- The short rate follows

\[ dr = \beta(\mu - r) \, dt + \sigma \, dW. \]

- The short rate is pulled to the long-term mean level \( \mu \) at rate \( \beta \).

- Superimposed on this “pull” is a normally distributed stochastic term \( \sigma \, dW \).

- Since the process is an Ornstein-Uhlenbeck process,

\[ E[r(T) \mid r(t) = r] = \mu + (r - \mu) e^{-\beta(T-t)} \]

from Eq. (64) on p. 562.

\textsuperscript{a}Vasicek (1977).
The Vasicek Model (continued)

- The price of a zero-coupon bond paying one dollar at maturity can be shown to be

\[ P(t, T) = A(t, T) e^{-B(t,T) r(t)}, \]  

(130)

where

\[ A(t, T) = \begin{cases} 
\exp \left[ \frac{(B(t,T) - T + t)(\beta^2 \mu - \sigma^2/2)}{\beta^2} - \frac{\sigma^2 B(t,T)^2}{4\beta} \right] & \text{if } \beta \neq 0, \\
\exp \left[ \frac{\sigma^2 (T-t)^3}{6} \right] & \text{if } \beta = 0.
\end{cases} \]

and

\[ B(t, T) = \begin{cases} 
\frac{1 - e^{-\beta(T-t)}}{\beta} & \text{if } \beta \neq 0, \\
T - t & \text{if } \beta = 0.
\end{cases} \]
The Vasicek Model (concluded)

- If $\beta = 0$, then $P$ goes to infinity as $T \to \infty$.
- Sensibly, $P$ goes to zero as $T \to \infty$ if $\beta \neq 0$.
- Even if $\beta \neq 0$, $P$ may exceed one for a finite $T$.
- The spot rate volatility structure is the curve

$$
(\partial r(t, T)/\partial r) \sigma = \sigma B(t, T)/(T - t).
$$

- When $\beta > 0$, the curve tends to decline with maturity.
- The speed of mean reversion, $\beta$, controls the shape of the curve.
- Indeed, higher $\beta$ leads to greater attenuation of volatility with maturity.
The graph shows the yield as a function of term length. The yield is plotted on the vertical axis, ranging from 0.05 to 0.2, while the term length is on the horizontal axis, ranging from 2 to 10.

Three types of yield patterns are depicted:

- **Normal**: A linear increase followed by a plateau.
- **Humped**: A peak at a certain term length, followed by a decrease.
- **Inverted**: A decrease followed by an increase.

The yield increases with term length until it reaches a peak, after which it decreases or remains constant, depending on the pattern.
The Vasicek Model: Options on Zeros\textsuperscript{a}

- Consider a European call with strike price $X$ expiring at time $T$ on a zero-coupon bond with par value $1$ and maturing at time $s > T$.

- Its price is given by

$$P(t, s) N(x) - XP(t, T) N(x - \sigma_v).$$

\textsuperscript{a}Jamshidian (1989).
The Vasicek Model: Options on Zeros (concluded)

- Above

\[ x \equiv \frac{1}{\sigma_v} \ln \left( \frac{P(t, s)}{P(t, T) X} \right) + \frac{\sigma_v}{2}, \]

\[ \sigma_v \equiv v(t, T) B(T, s), \]

\[ v(t, T)^2 \equiv \begin{cases} 
\frac{\sigma^2 [1 - e^{-2\beta(T-t)}]}{2\beta}, & \text{if } \beta \neq 0 \\
\sigma^2(T-t), & \text{if } \beta = 0
\end{cases}. \]

- By the put-call parity, the price of a European put is

\[ X P(t, T) N(-x + \sigma_v) - P(t, s) N(-x). \]
Binomial Vasicek

• Consider a binomial model for the short rate in the time interval $[0, T]$ divided into $n$ identical pieces.

• Let $\Delta t \equiv T/n$ and

$$p(r) \equiv \frac{1}{2} + \frac{\beta(\mu - r) \sqrt{\Delta t}}{2\sigma}.$$ 

• The following binomial model converges to the Vasicek model,\(^a\)

$$r(k + 1) = r(k) + \sigma \sqrt{\Delta t} \xi(k), \quad 0 \leq k < n.$$ 

\(^a\)Nelson and Ramaswamy (1990).
Binomial Vasicek (continued)

- Above, $\xi(k) = \pm 1$ with

$$\text{Prob}[\xi(k) = 1] = \begin{cases} p(r(k)) & \text{if } 0 \leq p(r(k)) \leq 1 \\ 0 & \text{if } p(r(k)) < 0 \\ 1 & \text{if } 1 < p(r(k)) \end{cases}.$$ 

- Observe that the probability of an up move, $p$, is a decreasing function of the interest rate $r$.

- This is consistent with mean reversion.
Binomial Vasicek (concluded)

- The rate is the same whether it is the result of an up move followed by a down move or a down move followed by an up move.
- The binomial tree combines.
- The key feature of the model that makes it happen is its constant volatility, $\sigma$. 
The Cox-Ingersoll-Ross Model\textsuperscript{a}

- It is the following square-root short rate model:

\[ dr = \beta (\mu - r) \, dt + \sigma \sqrt{r} \, dW. \] (131)

- The diffusion differs from the Vasicek model by a multiplicative factor $\sqrt{r}$.

- The parameter $\beta$ determines the speed of adjustment.

- The short rate can reach zero only if $2\beta \mu < \sigma^2$.

- See text for the bond pricing formula.

\textsuperscript{a}Cox, Ingersoll, and Ross (1985).
Binomial CIR

- We want to approximate the short rate process in the time interval $[0, T]$.
- Divide it into $n$ periods of duration $\Delta t \equiv T/n$.
- Assume $\mu, \beta \geq 0$.
- A direct discretization of the process is problematic because the resulting binomial tree will not combine.
Binomial CIR (continued)

• Instead, consider the transformed process

\[ x(r) \equiv 2\sqrt{r}/\sigma. \]

• It follows

\[ dx = m(x) \, dt + dW, \]

where

\[ m(x) \equiv 2\beta\mu/(\sigma^2 x) - (\beta x/2) - 1/(2x). \]

• This new process has a constant volatility, and its associated binomial tree combines.
Binomial CIR (continued)

- Construct the combining tree for $r$ as follows.
- First, construct a tree for $x$.
- Then transform each node of the tree into one for $r$ via the inverse transformation

$$r = f(x) \equiv \frac{x^2 \sigma^2}{4}$$

(see p. 1021).
- When $x \approx 0$ (so $r \approx 0$), the moments may not be matched well.\textsuperscript{a}

\textsuperscript{a}Nawalkha and Beliaeva (2007).
\[ x + 2\sqrt{\Delta t} \quad f(x + 2\sqrt{\Delta t}) \\
 x + \sqrt{\Delta t} \quad f(x + \sqrt{\Delta t}) \\
 x \quad f(x) \\
 x - \sqrt{\Delta t} \quad f(x - \sqrt{\Delta t}) \\
 x - 2\sqrt{\Delta t} \quad f(x - 2\sqrt{\Delta t}) \]
Binomial CIR (concluded)

• The probability of an up move at each node $r$ is

$$p(r) \equiv \frac{\beta(\mu - r) \Delta t + r - r^-}{r^+ - r^-}. \quad (132)$$

- $r^+ \equiv f(x + \sqrt{\Delta t})$ denotes the result of an up move from $r$.
- $r^- \equiv f(x - \sqrt{\Delta t})$ the result of a down move.

• Finally, set the probability $p(r)$ to one as $r$ goes to zero to make the probability stay between zero and one.
Numerical Examples

• Consider the process,

\[ 0.2 (0.04 - r) \, dt + 0.1 \sqrt{r} \, dW, \]

for the time interval \([0, 1]\) given the initial rate \(r(0) = 0.04\).

• We shall use \(\Delta t = 0.2\ \text{year}\) for the binomial approximation.

• See p. 1024(a) for the resulting binomial short rate tree with the up-move probabilities in parentheses.
Numerical Examples (continued)

• Consider the node which is the result of an up move from the root.

• Since the root has \( x = 2\sqrt{r(0)/\sigma} = 4 \), this particular node’s \( x \) value equals \( 4 + \sqrt{\Delta t} = 4.4472135955 \).

• Use the inverse transformation to obtain the short rate

\[
\frac{x^2 \times (0.1)^2}{4} \approx 0.0494442719102.
\]
Numerical Examples (concluded)

• Once the short rates are in place, computing the probabilities is easy.

• Note that the up-move probability decreases as interest rates increase and decreases as interest rates decline.
  – I suspect that

\[ p(r) = A \sqrt{\frac{\Delta t}{r}} + B - C \sqrt{r \Delta t} \]

for some \( A, B, C > 0 \).\(^a\)

• This phenomenon agrees with mean reversion.

• Convergence is quite good (see text).

\(^a\)Thanks to a lively class discussion on May 28, 2014.
A General Method for Constructing Binomial Models

- We are given a continuous-time process,

\[ dy = \alpha(y, t) \, dt + \sigma(y, t) \, dW. \]

- Need to make sure the binomial model’s drift and diffusion converge to the above process.

- Set the probability of an up move to

\[ \frac{\alpha(y, t) \, \Delta t + y - y_d}{y_u - y_d}. \]

- Here \( y_u \equiv y + \sigma(y, t) \sqrt{\Delta t} \) and \( y_d \equiv y - \sigma(y, t) \sqrt{\Delta t} \) represent the two rates that follow the current rate \( y \).

---

\[ ^{a} \text{Nelson and Ramaswamy (1990).} \]
A General Method (continued)

- The displacements are identical, at \( \sigma(y, t) \sqrt{\Delta t} \).
- But the binomial tree may not combine as

\[
\sigma(y, t) \sqrt{\Delta t} - \sigma(y_u, t + \Delta t) \sqrt{\Delta t} \\
\neq -\sigma(y, t) \sqrt{\Delta t} + \sigma(y_d, t + \Delta t) \sqrt{\Delta t}
\]

in general.

- When \( \sigma(y, t) \) is a constant independent of \( y \), equality holds and the tree combines.
A General Method (continued)

- To achieve this, define the transformation
  \[ x(y, t) \equiv \int_y^y \sigma(z, t)^{-1} \, dz. \]

- Then \( x \) follows
  \[ dx = m(y, t) \, dt + dW \]
  for some \( m(y, t) \).\(^a\)

- The diffusion term is now a constant, and the binomial tree for \( x \) combines.

\(^a\)See Exercise 25.2.13 of the textbook.
A General Method (concluded)

- The transformation is unique.a

- The probability of an up move remains

\[
\frac{\alpha(y(x, t), t) \Delta t + y(x, t) - y_d(x, t)}{y_u(x, t) - y_d(x, t)},
\]

where \( y(x, t) \) is the inverse transformation of \( x(y, t) \) from \( x \) back to \( y \).

- Note that

\[
y_u(x, t) \equiv y(x + \sqrt{\Delta t}, t + \Delta t),
\]
\[
y_d(x, t) \equiv y(x - \sqrt{\Delta t}, t + \Delta t).
\]

---

aChiu (R98723059) (2012).
Examples

• The transformation is

\[
\int^r (\sigma \sqrt{z})^{-1} \, dz = \frac{2\sqrt{r}}{\sigma}
\]

for the CIR model.

• The transformation is

\[
\int^S (\sigma z)^{-1} \, dz = \frac{\ln S}{\sigma}
\]

for the Black-Scholes model.

• The familiar binomial option pricing model in fact discretizes \( \ln S \) not \( S \).
On One-Factor Short Rate Models

• By using only the short rate, they ignore other rates on the yield curve.

• Such models also restrict the volatility to be a function of interest rate *levels* only.

• The prices of all bonds move in the same direction at the same time (their magnitudes may differ).

• The returns on all bonds thus become highly correlated.

• In reality, there seems to be a certain amount of independence between short- and long-term rates.
On One-Factor Short Rate Models (continued)

- One-factor models therefore cannot accommodate nondegenerate correlation structures across maturities.
- Derivatives whose values depend on the correlation structure will be mispriced.
- The calibrated models may not generate term structures as concave as the data suggest.
- The term structure empirically changes in slope and curvature as well as makes parallel moves.
- This is inconsistent with the restriction that all segments of the term structure be perfectly correlated.
On One-Factor Short Rate Models (concluded)

- Multi-factor models lead to families of yield curves that can take a greater variety of shapes and can better represent reality.
- But they are much harder to think about and work with.
- They also take much more computer time—the curse of dimensionality.
- These practical concerns limit the use of multifactor models to two- or three-factor ones.
Options on Coupon Bonds\textsuperscript{a}

- Assume a one-factor short rate model.
- The price of a European option on a coupon bond can be calculated from those on zero-coupon bonds.
- Consider a European call expiring at time $T$ on a bond with par value $1$.
- Let $X$ denote the strike price.
- The bond has cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$, where $t_i > T$ for all $i$.

\textsuperscript{a}Jamshidian (1989).
Options on Coupon Bonds (continued)

• The payoff for the option is

\[
\max \left\{ \left[ \sum_{i=1}^{n} c_i P(r(T), T, t_i) \right] - X, 0 \right\}.
\]

• At time \( T \), there is a unique value \( r^* \) for \( r(T) \) that renders the coupon bond’s price equal the strike price \( X \).

• This \( r^* \) can be obtained by solving

\[
X = \sum_{i=1}^{n} c_i P(r, T, t_i)
\]

numerically for \( r \).
The solution is unique for one-factor models whose bond price is a monotonically decreasing function of $r$.

Let

$$X_i \equiv P(r^*, T, t_i),$$

the value at time $T$ of a zero-coupon bond with par value $1$ and maturing at time $t_i$ if $r(T) = r^*$.

Note that $P(r, T, t_i) \geq X_i$ if and only if $\leq r^*$. 
Options on Coupon Bonds (concluded)

- As $X = \sum_i c_i X_i$, the option’s payoff equals
  
  $\max \left\{ \left[ \sum_{i=1}^n c_i P(r(T), T, t_i) \right] - \left[ \sum_i c_i X_i \right], 0 \right\}$

  $= \sum_{i=1}^n c_i \times \max(P(r(T), T, t_i) - X_i, 0)$.

- Thus the call is a package of $n$ options on the underlying zero-coupon bond.

- Why can’t we do the same thing for Asian options?\(^a\)

\(^a\)Contributed by Mr. Yang, Jui-Chung (D97723002) on May 20, 2009.
No-Arbitrage Term Structure Models
How much of the structure of our theories really tells us about things in nature, and how much do we contribute ourselves?
— Arthur Eddington (1882–1944)
Motivations

- Recall the difficulties facing equilibrium models mentioned earlier.
  - They usually require the estimation of the market price of risk.
  - They cannot fit the market term structure.
  - But consistency with the market is often mandatory in practice.
No-Arbitrage Models

- No-arbitrage models utilize the full information of the term structure.
- They accept the observed term structure as consistent with an unobserved and unspecified equilibrium.
- From there, arbitrage-free movements of interest rates or bond prices over time are modeled.
- By definition, the market price of risk must be reflected in the current term structure; hence the resulting interest rate process is risk-neutral.

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Ho and Lee (1986). Thomas Lee is a “billionaire founder” of Thomas H. Lee Partners LP, according to Bloomberg on May 26, 2012.
No-Arbitrage Models (concluded)

- No-arbitrage models can specify the dynamics of zero-coupon bond prices, forward rates, or the short rate.

- Bond price and forward rate models are usually non-Markovian (path dependent).

- In contrast, short rate models are generally constructed to be explicitly Markovian (path independent).

- Markovian models are easier to handle computationally.
The Ho-Lee Model\textsuperscript{a}

- The short rates at any given time are evenly spaced.
- Let $p$ denote the risk-neutral probability that the short rate makes an up move.
- We shall adopt continuous compounding.

\textsuperscript{a}Ho and Lee (1986).
\[ r_3 \]

\[ r_2 \]

\[ r_1 \]

\[ r_2 + v_2 \]

\[ r_3 + v_3 \]

\[ r_3 + 2v_3 \]
The Ho-Lee Model (continued)

- The Ho-Lee model starts with zero-coupon bond prices $P(t, t + 1), P(t, t + 2), \ldots$ at time $t$ identified with the root of the tree.
- Let the discount factors in the next period be
  
  $P_d(t + 1, t + 2), P_d(t + 1, t + 3), \ldots$ if short rate moves down
  $P_u(t + 1, t + 2), P_u(t + 1, t + 3), \ldots$ if short rate moves up

- By backward induction, it is not hard to see that for $n \geq 2$,

  $$P_u(t + 1, t + n) = P_d(t + 1, t + n) e^{-(v_2 + \ldots + v_n)}$$

  (133)

  (see p. 376 of the textbook).
The Ho-Lee Model (continued)

- It is also not hard to check that the $n$-period zero-coupon bond has yields

$$y_d(n) \equiv -\frac{\ln P_d(t + 1, t + n)}{n - 1}$$

$$y_u(n) \equiv -\frac{\ln P_u(t + 1, t + n)}{n - 1} = y_d(n) + \frac{v_2 + \cdots + v_n}{n - 1}$$

- The volatility of the yield to maturity for this bond is therefore

$$\kappa_n \equiv \sqrt{py_u(n)^2 + (1 - p) y_d(n)^2 - \left[ py_u(n) + (1 - p) y_d(n) \right]^2}$$

$$= \sqrt{p(1 - p) \left( y_u(n) - y_d(n) \right)}$$

$$= \sqrt{p(1 - p)} \frac{v_2 + \cdots + v_n}{n - 1}.$$
The Ho-Lee Model (concluded)

• In particular, the short rate volatility is determined by taking $n = 2$:

$$\sigma = \sqrt{p(1-p)} \nu_2. \quad (134)$$

• The variance of the short rate therefore equals

$$p(1 - p)(r_u - r_d)^2,$$

where $r_u$ and $r_d$ are the two successor rates.\(^a\)

\(^a\)Contrast this with the lognormal model (108) on p. 909.
The Ho-Lee Model: Volatility Term Structure

- The volatility term structure is composed of

  \[ \kappa_2, \kappa_3, \ldots \]

  - It is independent of

  \[ r_2, r_3, \ldots \]

- It is easy to compute the \( v_i \)s from the volatility structure, and vice versa (review p. 1047).

- The \( r_i \)s can be computed by forward induction.

- The volatility structure is supplied by the market.
The Ho-Lee Model: Bond Price Process

• In a risk-neutral economy, the initial discount factors satisfy

\[ P(t, t+n) = (p P_u(t+1, t+n) + (1-p) P_d(t+1, t+n)) P(t, t+1). \]

• Combine the above with Eq. (133) on p. 1046 and assume \( p = 1/2 \) to obtain\(^a\)

\[
\begin{align*}
P_d(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2 \times \exp[v_2 + \cdots + v_n]}{1 + \exp[v_2 + \cdots + v_n]}, \\
P_u(t+1, t+n) &= \frac{P(t, t+n)}{P(t, t+1)} \frac{2}{1 + \exp[v_2 + \cdots + v_n]}.
\end{align*}
\]

\(^a\)In the limit, only the volatility matters.
The Ho-Lee Model: Bond Price Process (concluded)

- The bond price tree combines.
- Suppose all \( v_i \) equal some constant \( v \) and \( \delta \equiv e^v > 0 \).
- Then

\[
P_d(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2\delta^{n-1}}{1 + \delta^{n-1}},
\]

\[
P_u(t + 1, t + n) = \frac{P(t, t + n)}{P(t, t + 1)} \frac{2}{1 + \delta^{n-1}}.
\]

- Short rate volatility \( \sigma = v/2 \) by Eq. (134) on p. 1048.
- Price derivatives by taking expectations under the risk-neutral probability.
The Ho-Lee Model: Yields and Their Covariances

- The one-period rate of return of an \( n \)-period zero-coupon bond is

\[
r(t, t + n) \equiv \ln \left( \frac{P(t + 1, t + n)}{P(t, t + n)} \right).
\]

- Its value is either \( \ln \frac{P_a(t+1,t+n)}{P(t,t+n)} \) or \( \ln \frac{P_u(t+1,t+n)}{P(t,t+n)} \).

- Thus the variance of return is

\[
\text{Var}[r(t, t + n)] = p(1 - p)((n - 1) \nu)^2 = (n - 1)^2 \sigma^2.
\]
The Ho-Lee Model: Yields and Their Covariances (concluded)

- The covariance between $r(t, t + n)$ and $r(t, t + m)$ is\(^a\)
  $$(n - 1)(m - 1) \sigma^2.$$ 
- As a result, the correlation between any two one-period rates of return is unity.
- Strong correlation between rates is inherent in all one-factor Markovian models.

\(^a\)See Exercise 26.2.7 of the textbook.
The Ho-Lee Model: Short Rate Process

• The continuous-time limit of the Ho-Lee model is

\[ dr = \theta(t) \, dt + \sigma \, dW. \]

• This is Vasicek’s model with the mean-reverting drift replaced by a deterministic, time-dependent drift.

• A nonflat term structure of volatilities can be achieved if the short rate volatility is also made time varying,

\[ dr = \theta(t) \, dt + \sigma(t) \, dW. \]

• This corresponds to the discrete-time model in which \( v_i \) are not all identical.
The Ho-Lee Model: Some Problems

- Future (nominal) interest rates may be negative.
- The short rate volatility is independent of the rate level.
Problems with No-Arbitrage Models in General

• Interest rate movements should reflect shifts in the model’s state variables (factors) not its parameters.

• Model parameters, such as the drift $\theta(t)$ in the continuous-time Ho-Lee model, should be stable over time.

• But in practice, no-arbitrage models capture yield curve shifts through the recalibration of parameters.
  - A new model is thus born everyday.
Problems with No-Arbitrage Models in General (concluded)

- This in effect says the model estimated at some time does not describe the term structure of interest rates and their volatilities at other times.

- Consequently, a model’s intertemporal behavior is suspect, and using it for hedging and risk management may be unreliable.