Suppose the (instantaneous) volatility can change over time but otherwise predictable: $\sigma(t)$ instead of $\sigma$.

In the limit, the variance of $\ln(S_\tau/S)$ is

$$\int_0^\tau \sigma^2(t) \, dt$$

rather than $\sigma^2 \tau$.

The annualized volatility to be used in the Black-Scholes formula should now be

$$\sqrt{\frac{\int_0^\tau \sigma^2(t) \, dt}{\tau}}.$$

---

\(^{a}\)Merton (1973).
Time-Dependent Instantaneous Volatility (concluded)

- There is no guarantee that the implied volatility is constant.

- For the binomial model, $u$ and $d$ depend on time:
  \[
  u = e^{\sigma(t)\sqrt{\tau/n}}, \\
  d = e^{-\sigma(t)\sqrt{\tau/n}}.
  \]
Time-Dependent Short Rates

- Suppose the short rate (i.e., the one-period spot rate) changes over time but otherwise predictable.
- The riskless rate $r$ in the Black-Scholes formula should be the spot rate with a time to maturity equal to $\tau$.
- In other words,

$$ r = \frac{\sum_{i=0}^{n-1} r_i}{\tau}, $$

where $r_i$ is the continuously compounded, short rate measured in periods for period $i$. 
Trading Days and Calendar Days

• Interest accrues based on the calendar day.

• But $\sigma$ is usually calculated based on trading days only.
  – Stock price seems to have lower volatilities when the exchange is closed.\textsuperscript{a}

• How to harmonize these two different times into the Black-Scholes formula and binomial tree algorithms?\textsuperscript{b}

\textsuperscript{a}Fama (1965); French (1980); French and Roll (1986).
\textsuperscript{b}Recall p. 145 about dating issues.
Trading Days and Calendar Days (continued)

- Think of $\sigma$ as measuring the *annualized* volatility of stock price *one year from now*.

- Suppose a year has $m$ (say 253) trading days.

- We can replace $\sigma$ in the Black-Scholes formula with\textsuperscript{a}

$$\sigma \sqrt{\frac{365}{m} \times \frac{\text{number of trading days to expiration}}{\text{number of calendar days to expiration}}}.$$\textsuperscript{a}

\textsuperscript{a}French (1984).
Trading Days and Calendar Days (concluded)

• This works only for European options.

• How about binomial tree algorithms?

\[\text{Contributed by Mr. Lu, Zheng-Liang (D00922011) in 2015.}\]
Options on a Stock That Pays Dividends

• Early exercise must be considered.

• Proportional dividend payout model is tractable (see text).
  – The dividend amount is a constant proportion of the prevailing stock price.

• In general, the corporate dividend policy is a complex issue.
Known Dividends

- Constant dividends introduce complications.
- Use $D$ to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)u$, $(Su - D)d$, $(Sd - D)u$, $(Sd - D)d$.
  - The binomial tree no longer combines.
\[(Su - D)u\]  
\[(Su - D)d\]  
\[Su - D\]  
\[S\]  
\[(Sd - D)u\]  
\[(Sd - D)d\]  
\[Sd - D\]
An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends.\(^a\)

- This essentially decomposes the stock price into a riskless one paying known dividends and a risky one.

- The riskless component at any time is the PV of future dividends during the life of the option.
  - Then, \(\sigma\) is the volatility of the process followed by the \textit{risky} component.

- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

\(^a\)Roll (1977).
An Ad-Hoc Approximation (concluded)

• Start with the current stock price minus the PV of future dividends before expiration.

• Develop the binomial tree for the new stock price as if there were no dividends.

• Then add to each stock price on the tree the PV of all future dividends before expiration.

• American option prices can be computed as before on this tree of stock prices.
The Ad-Hoc Approximation vs. P. 298 (Step 1)

\[ S - \frac{D}{R} \]

\[ (S - \frac{D}{R})u^2 \]

\[ (S - \frac{D}{R})u \]

\[ (S - \frac{D}{R})ud \]

\[ (S - \frac{D}{R})d \]

\[ (S - \frac{D}{R})d^2 \]
The Ad-Hoc Approximation vs. P. 298 (Step 2)

\[
\begin{align*}
(S - D/R)u^2 \\
(S - D/R)u \\
(S - D/R) + D/R &= S \\
(S - D/R)ud \\
(S - D/R)d \\
(S - D/R)d^2
\end{align*}
\]
The Ad-Hoc Approximation vs. P. 298

- The trees are different.

- The stock prices at maturity are also different.
  - \((Su - D)u, (Su - D)d, (Sd - D)u, (Sd - D)d\) (p. 298).
  - \((S - D/R)u^2, (S - D/R)ud, (S - D/R)d^2\) (ad hoc).

- Note that, as \(d < R < u\),
  \[
  (Su - D)u > (S - D/R)u^2, \\
  (Sd - D)d < (S - D/R)d^2,
  \]

---

Contributed by Mr. Yang, Jui-Chung (D97723002) on March 18, 2009.
The Ad-Hoc Approximation vs. P. 298 (concluded)

- So the ad hoc approximation has a smaller dynamic range.

- This explains why in practice the volatility is usually increased when using the ad hoc approximation.
A General Approach\textsuperscript{a}

- A new tree structure.
- No approximation assumptions are made.
- A mathematical proof that the tree can always be constructed.
- The actual performance is quadratic except in pathological cases (see pp. 689ff).
- Other approaches include adjusting $\sigma$ and approximating the known dividend with a dividend yield.

\textsuperscript{a}Dai (R86526008, D8852600) and Lyuu (2004).
Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.

- The payment of a continuous dividend yield at rate $q$ reduces the growth rate of the stock price by $q$.
  - A stock that grows from $S$ to $S_{\tau}$ with a continuous dividend yield of $q$ would grow from $S$ to $S_{\tau}e^{q\tau}$ without the dividends.

- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.\(^a\)

\(^a\)In pricing European options, we care only about the distribution of $S_{\tau}$. 

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Continuous Dividend Yields (continued)

- The Black-Scholes formulas hold with $S$ replaced by $Se^{-q\tau}$.\(^a\)

\[
C = Se^{-q\tau} N(x) - X e^{-r\tau} N(x - \sigma\sqrt{\tau}),
\]

\[
P = X e^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x),
\]

where \(x \equiv \ln(S/X) + (r - q + \sigma^2/2) \tau\).

- Formulas (29) and (29') remain valid as long as the dividend yield is predictable.

\(^{a}\text{Merton (1973).}\)
Continuous Dividend Yields (continued)

- To run binomial tree algorithms, replace $u$ with $ue^{-q\Delta t}$ and $d$ with $de^{-q\Delta t}$, where $\Delta t \equiv \tau/n$.
  - The reason: The stock price grows at an expected rate of $r - q$ in a risk-neutral economy.

- Other than the changes, binomial tree algorithms stay the same.
  - In particular, $p$ should use the original $u$ and $d$!\(^a\)

\(^a\)Contributed by Ms. Wang, Chuan-Ju (F95922018) on May 2, 2007.
Continuous Dividend Yields (concluded)

• Alternatively, pick the risk-neutral probability as

\[ \frac{e^{(r-q)\Delta t} - d}{u - d}, \]  

(30)

where \( \Delta t \equiv \tau/n. \)

– The reason: The stock price grows at an expected rate of \( r - q \) in a risk-neutral economy.

• The \( u \) and \( d \) remain unchanged.

• Other than the change in Eq. (30), binomial tree algorithms stay the same as if there were no dividends.
Curtailing the Range of Tree Nodes\textsuperscript{a}

- Those nodes can be skipped if they are extremely unlikely to be reached.

- The probability that the price will move more than a certain number of standard deviations from its initial value is negligible.

- Similarly, suppose the stock price at maturity is known.

- The probability that the price will move outside that number of standard deviations in working backward is also negligible.

\textsuperscript{a}Andricopoulos, Widdicks, Duck, Newton (2004).
Curtailing the Range of Tree Nodes (continued)

• In summary, for certain stock prices, the strike price is so low or so high that it could not realistically be reached.

• For these prices the option value is basically deterministic.

• By working only within the said range of stock prices, we can save time without significant loss of accuracy.
Curtailing the Range of Tree Nodes (concluded)

• For time $t$, where $0 < t < T$, the maximum and minimum stock prices $S_{\text{max}}(t)$ and $S_{\text{min}}(t)$ are:

\[
S_{\text{max}}(t) = \min \left( S_0 e^{rt+\eta\sigma\sqrt{t}}, X e^{-r(T-t)+\eta\sigma\sqrt{T-t}} \right),
\]

\[
S_{\text{min}}(t) = \max \left( S_0 e^{rt-\eta\sigma\sqrt{t}}, X e^{-r(T-t)-\eta\sigma\sqrt{T-t}} \right).
\]
Sensitivity Analysis of Options
Cleopatra’s nose, had it been shorter, the whole face of the world would have been changed.

— Blaise Pascal (1623–1662)
Sensitivity Measures ("The Greeks")

- How the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.

- Let \( x \equiv \frac{\ln(S/X) + (r + \sigma^2/2) \tau}{\sigma \sqrt{\tau}} \) (recall p. 280).

- Recall that
  \[
  N'(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} > 0,
  \]
  the density function of standard normal distribution.
Delta

• Defined as
  \[ \Delta \equiv \frac{\partial f}{\partial S}. \]
  – \( f \) is the price of the derivative.
  – \( S \) is the price of the underlying asset.

• The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.
  – Elementary calculus.

• The delta used in the BOPM (p. 226) is the discrete analog.
Delta (concluded)

- The delta of a European call on a non-dividend-paying stock equals
  \[
  \frac{\partial C}{\partial S} = N(x) > 0.
  \]
- The delta of a European put equals
  \[
  \frac{\partial P}{\partial S} = N(x) - 1 < 0.
  \]
- The delta of a long stock is apparently 1.
Solid curves: at-the-money options.
Dashed curves: out-of-the-money calls or in-the-money puts.
Delta Neutrality

• A position with a total delta equal to 0 is delta-neutral.
  – A delta-neutral portfolio is immune to small price changes in the underlying asset.

• Creating one serves for hedging purposes.
  – A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  – Short $\Delta$ shares of stock to hedge a long call.
  – Long $\Delta$ shares of stock to hedge a short call.

• In general, hedge a position in a security with delta $\Delta_1$ by shorting $\Delta_1/\Delta_2$ units of a security with delta $\Delta_2$. 
Theta (Time Decay)

• Defined as the rate of change of a security’s value with respect to time, or $\Theta \equiv -\frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial t}$.

• For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} - rXe^{-r\tau} N(x - \sigma \sqrt{\tau}) < 0.$$  
– The call loses value with the passage of time.

• For a European put,

$$\Theta = -\frac{SN'(x) \sigma}{2\sqrt{\tau}} + rXe^{-r\tau} N(-x + \sigma \sqrt{\tau}).$$  
– Can be negative or positive.
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money options.
Dashed curve: out-of-the-money call or in-the-money put.
Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \equiv \partial^2 \Pi / \partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- In practice, a portfolio with a high gamma needs to be rebalanced more often to maintain delta neutrality.
- Roughly, delta $\sim$ duration, and gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is

$$N'(x)/(S\sigma\sqrt{\tau}) > 0.$$
Vega\textsuperscript{a} (Lambda, Kappa, Sigma)

• Defined as the rate of change of its value with respect to the volatility of the underlying asset

\[ \Lambda \equiv \frac{\partial f}{\partial \sigma}. \]

• Volatility often changes over time.

• A security with a high vega is very sensitive to small changes or estimation error in volatility.

• The vega of a European call or put on a non-dividend-paying stock is \( S \sqrt{\tau} N'(x) > 0. \)
  
  – So higher volatility always increases the option value.

\textsuperscript{a}Vega is not Greek.
Vega (concluded)

• Note that if $S \neq X$, $\tau \to 0$ implies

$$\Lambda \to 0$$

(which answers the question on p. 284 for the Black-Scholes model).

• The Black-Scholes formula (p. 280) implies

$$C \to S,$$

$$P \to X e^{-r\tau},$$

as $\sigma \to \infty$.

• These boundary conditions may be handy for certain numerical methods.
Dotted curve: in-the-money call or out-of-the-money put.
Solid curves: at-the-money option.
Dashed curve: out-of-the-money call or in-the-money put.
Rho

- Defined as the rate of change in its value with respect to interest rates
  \[ \rho \equiv \frac{\partial f}{\partial r}. \]
- The rho of a European call on a non-dividend-paying stock is
  \[ X\tau e^{-r\tau} N(x - \sigma \sqrt{\tau}) > 0. \]
- The rho of a European put on a non-dividend-paying stock is
  \[ -X\tau e^{-r\tau} N(-x + \sigma \sqrt{\tau}) < 0. \]
Dotted curves: in-the-money call or out-of-the-money put.
Solid curves: at-the-money option.
Dashed curves: out-of-the-money call or in-the-money put.
Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,
  \[ \frac{f(S + \Delta S) - f(S - \Delta S)}{2\Delta S} \]
- The computation time roughly doubles that for evaluating the derivative security itself.
An Alternative Numerical Delta

- Use intermediate results of the binomial tree algorithm.
- When the algorithm reaches the end of the first period, $f_u$ and $f_d$ are computed.
- These values correspond to derivative values at stock prices $S_u$ and $S_d$, respectively.
- Delta is approximated by $\frac{f_u - f_d}{S_u - S_d}$.
- Almost zero extra computational effort.

\footnote{Pelsser and Vorst (1994).}
Numerical Gamma

- At the stock price \((S_{uu} + S_{ud})/2\), delta is approximately \((f_{uu} - f_{ud})/(S_{uu} - S_{ud})\).

- At the stock price \((S_{ud} + S_{dd})/2\), delta is approximately \((f_{ud} - f_{dd})/(S_{ud} - S_{dd})\).

- Gamma is the rate of change in deltas between \((S_{uu} + S_{ud})/2\) and \((S_{ud} + S_{dd})/2\), that is,

\[
\frac{f_{uu} - f_{ud}}{S_{uu} - S_{ud}} - \frac{f_{ud} - f_{dd}}{S_{ud} - S_{dd}} \frac{(S_{uu} - S_{dd})/2}{(S_{uu} - S_{dd})/2}.
\]

- Alternative formulas exist (p. 586).
Finite Difference Fails for Numerical Gamma

• Numerical differentiation gives
\[ \frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}. \]

• It does not work (see text).

• But why did the binomial tree version work?
Other Numerical Greeks

• The theta can be computed as
  \[ \frac{f_{ud} - f}{2(\tau/n)} \, . \]

  – In fact, the theta of a European option can be derived from delta and gamma (p. 585).

• For vega and rho, there seems no alternative but to run the binomial tree algorithm twice.\(^a\)

\(^a\)But see pp. 946ff.
Extensions of Options Theory
As I never learnt mathematics, so I have had to think.
— Joan Robinson (1903–1983)
Pricing Corporate Securities\(^a\)

- Interpret the underlying asset as the total value of the firm.
- The option pricing methodology can be applied to pricing corporate securities.
  - The result is called the structural model.
- Assumptions:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

\(^a\)Black and Scholes (1973).
Risky Zero-Coupon Bonds and Stock

• Consider XYZ.com.

• Capital structure:
  – $n$ shares of its own common stock, $S$.
  – Zero-coupon bonds with an aggregate par value of $X$.

• What is the value of the bonds, $B$?

• What is the value of the XYZ.com stock?
Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds’ maturity date, suppose the total value of the firm $V^*$ is less than the bondholders’ claim $X$.
- Then the firm declares bankruptcy, and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain $X$ and the stockholders $V^* - X$.

<table>
<thead>
<tr>
<th></th>
<th>$V^* \leq X$</th>
<th>$V^* &gt; X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonds</td>
<td>$V^*$</td>
<td>$X$</td>
</tr>
<tr>
<td>Stock</td>
<td>0</td>
<td>$V^* - X$</td>
</tr>
</tbody>
</table>
Risky Zero-Coupon Bonds and Stock (continued)

- The stock has the same payoff as a call!
- It is a call on the total value of the firm with a strike price of $X$ and an expiration date equal to the bonds’.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call on the total value of the firm.
- Let $V$ stand for the total value of the firm.
- Let $C$ stand for a call on $V$. 
Risky Zero-Coupon Bonds and Stock (continued)

• Thus

\[ nS = C, \]
\[ B = V - C. \]

• Knowing \( C \) amounts to knowing how the value of the firm is divided between stockholders and bondholders.

• Whatever the value of \( C \), the total value of the stock and bonds at maturity remains \( V^* \).

• The relative size of debt and equity is irrelevant to the firm’s current value \( V \).
Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 11 (p. 280) and the put-call parity,

\[ nS = VN(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}), \]
\[ B = VN(-x) + Xe^{-r\tau}N(x - \sigma\sqrt{\tau}). \]

- Above,

\[ x \equiv \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}. \]

- The continuously compounded yield to maturity of the firm’s bond is

\[ \frac{\ln(X/B)}{\tau}. \]
Risky Zero-Coupon Bonds and Stock (concluded)

- Define the credit spread or default premium as the yield difference between risky and riskless bonds,

\[
\frac{\ln(X/B)}{\tau} - r
\]

\[
= -\frac{1}{\tau} \ln \left( N(-z) + \frac{1}{\omega} N(z - \sigma \sqrt{\tau}) \right).
\]

- \( \omega \equiv X e^{-r \tau} / V. \)

- \( z \equiv (\ln \omega)/(\sigma \sqrt{\tau}) + (1/2) \sigma \sqrt{\tau} = -x + \sigma \sqrt{\tau}. \)

- Note that \( \omega \) is the debt-to-total-value ratio.
A Numerical Example

- XYZ.com’s assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck’s market value per share is $44.5.

- XYZ.com’s securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.

- Each bond promises to pay $1,000 at maturity.

- $n = 1,000$, $V = 44.5 \times n = 44,500$, and $X = 30 \times 1,000 = 30,000$. 
<table>
<thead>
<tr>
<th>Option</th>
<th>Strike</th>
<th>Exp.</th>
<th>Vol.</th>
<th>Last</th>
<th>Vol.</th>
<th>Last</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merck</td>
<td>30</td>
<td>Jul</td>
<td>328</td>
<td>151/4</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>441/2</td>
<td>35</td>
<td>Jul</td>
<td>150</td>
<td>91/2</td>
<td>10</td>
<td>1/16</td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Apr</td>
<td>887</td>
<td>43/4</td>
<td>136</td>
<td>1/16</td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Jul</td>
<td>220</td>
<td>51/2</td>
<td>297</td>
<td>1/4</td>
</tr>
<tr>
<td>441/2</td>
<td>40</td>
<td>Oct</td>
<td>58</td>
<td>6</td>
<td>10</td>
<td>1/2</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Apr</td>
<td>3050</td>
<td>7/8</td>
<td>100</td>
<td>11/8</td>
</tr>
<tr>
<td>441/2</td>
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<td>May</td>
<td>462</td>
<td>13/8</td>
<td>50</td>
<td>13/8</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Jul</td>
<td>883</td>
<td>115/16</td>
<td>147</td>
<td>13/4</td>
</tr>
<tr>
<td>441/2</td>
<td>45</td>
<td>Oct</td>
<td>367</td>
<td>23/4</td>
<td>188</td>
<td>21/16</td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such a call is selling for $15.25$.
- So XYZ.com’s stock is worth $15.25 \times n = 15,250$ dollars.
- The entire bond issue is worth

\[ B = 44,500 - 15,250 = 29,250 \]

dollars.

- Or $975$ per bond.
A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $X$ par value plus $n$ written European puts on Merck at a strike price of $30$.
  - By the put-call parity.
- The difference between $B$ and the price of the default-free bond is the value of these puts.
- The next table shows the total market values of the XYZ.com stock and bonds under various debt amounts $X$. 
<table>
<thead>
<tr>
<th>Promised payment to bondholders</th>
<th>Current market value of bonds</th>
<th>Current market value of stock</th>
<th>Current total value of firm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$B$</td>
<td>$nS$</td>
<td>$V$</td>
</tr>
<tr>
<td>30,000</td>
<td>29,250.0</td>
<td>15,250.0</td>
<td>44,500</td>
</tr>
<tr>
<td>35,000</td>
<td>35,000.0</td>
<td>9,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>40,000</td>
<td>39,000.0</td>
<td>5,500.0</td>
<td>44,500</td>
</tr>
<tr>
<td>45,000</td>
<td>42,562.5</td>
<td>1,937.5</td>
<td>44,500</td>
</tr>
</tbody>
</table>
A Numerical Example (continued)

- Suppose the promised payment to bondholders is $45,000.
- Then the relevant option is the July call with a strike price of $45,000/n = 45$ dollars.
- Since that option is selling for $1\frac{15}{16}$, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1,937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.
A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option’s terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one key difference between options and corporate securities.
  - Parameters such volatility, dividend, and strike price are under partial control of the stockholders.
A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45,000$ dollars.
- The table on p. 348 says the total market value of the bonds should be $42,562.5$.
- The new bondholders pay
  \[ 42,562.5 \times (15/45) = 14,187.5 \]
  dollars.
- The remaining stock is worth $1,937.5$. 
A Numerical Example (continued)

• The stockholders therefore gain

$$14,187.5 + 1,937.5 - 15,250 = 875$$
dollars.

• The original bondholders lose an equal amount,

$$29,250 - \frac{30}{45} \times 42,562.5 = 875.$$
A Numerical Example (continued)

- Suppose the stockholders sell \((1/3) \times n\) Merck shares to fund a $14,833.3 cash dividend.

- They now have $14,833.3 in cash plus a call on \((2/3) \times n\) Merck shares.

- The strike price remains \(X = 30,000\).

- This is equivalent to owning \(2/3\) of a call on \(n\) Merck shares with a total strike price of $45,000.

- \(n\) such calls are worth $1,937.5 (p. 348).

- So the total market value of the XYZ.com stock is \((2/3) \times 1,937.5 = 1,291.67\) dollars.
A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence

\[(2/3) \times n \times 44.5 - 1,291.67 = 28,375\]

dollars.

- Hence the stockholders gain

\[14,833.3 + 1,291.67 - 15,250 \approx 875\]

dollars.

- The bondholders watch their value drop from $29,250 to $28,375, a loss of $875.
Further Topics

• Other Examples:
  – Subordinated debts as bull call spreads.
  – Warrants as calls.
  – Callable bonds as American calls with 2 strike prices.
  – Convertible bonds.

• Securities with a complex liability structure must be solved by trees.\(^a\)

\(^a\)Dai (R86526008, D8852600), Lyuu, and Wang (F95922018) (2010).
Barrier Options$^a$

- Their payoff depends on whether the underlying asset’s price reaches a certain price level $H$.

- A knock-out option is an ordinary European option which ceases to exist if the barrier $H$ is reached by the price of its underlying asset.

- A call knock-out option is sometimes called a down-and-out option if $H < S$.

- A put knock-out option is sometimes called an up-and-out option when $H > S$.

$^a$A former MBA student in finance told me on March 26, 2004, that she did not understand why I covered barrier options until she started working in a bank. She was working for Lehman Brothers in HK as of April, 2006.
Time

Price

$H$

$S$

Barrier hit

Time
Barrier Options (concluded)

• A knock-in option comes into existence if a certain barrier is reached.

• A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.

• An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.

• Formulas exist for all the possible barrier options mentioned above.$^a$

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$^a$Haug (2006).
A Formula for Down-and-In Calls$^a$

- Assume $X \geq H$.
- The value of a European down-and-in call on a stock paying a dividend yield of $q$ is

$$Se^{-q\tau} \left( \frac{H}{S} \right)^{2\lambda} N(x) - Xe^{-r\tau} \left( \frac{H}{S} \right)^{2\lambda-2} N(x - \sigma\sqrt{\tau}),$$

(31)

\[-x \equiv \frac{\ln(H^2/(SX))+(r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}.\]

\[-\lambda \equiv (r - q + \sigma^2/2)/\sigma^2.\]

- A European down-and-out call can be priced via the in-out parity (see text).

$^a$Merton (1973).
A Formula for Up-and-In Puts

• Assume $X \leq H$.

• The value of a European up-and-in put is

\[ X e^{-r\tau} \left( \frac{H}{S} \right)^{2\lambda-2} N(-x + \sigma \sqrt{\tau}) - Se^{-q\tau} \left( \frac{H}{S} \right)^{2\lambda} N(-x). \]

• Again, a European up-and-out put can be priced via the in-out parity.

\(^a\)Merton (1973).
Are American Options Barrier Options?\textsuperscript{a}

- American options are barrier options with the exercise boundary as the barrier and the payoff as the rebate?
- One salient difference is that the exercise boundary must be derived during backward induction.
- But the barrier in a barrier option is given a priori.

\textsuperscript{a}Contributed by Mr. Yang, Jui-Chung (D97723002) on March 25, 2009.
Interesting Observations

• Assume $H < X$.

• Replace $S$ in the pricing formula Eq. (29) on p. 307 for the call with $H^2/S$.

• Equation (31) on p. 359 for the down-and-in call becomes Eq. (29) when $r - q = \sigma^2/2$.

• Equation (31) becomes $S/H$ times Eq. (29) when $r - q = 0$. 
Interesting Observations (concluded)

• Replace $S$ in the pricing formula for the down-and-in call, Eq. (31), with $H^2/S$.

• Equation (31) becomes Eq. (29) when $r - q = \sigma^2/2$.

• Equation (31) becomes $H/S$ times Eq. (29) when $r - q = 0$.\(^a\)

• Why?

\(^a\)Contributed by Mr. Chou, Ming-Hsin (R02723073) on April 24, 2014.
Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Below is for the down-and-out option.
$S = 8$, $X = 6$, $H = 4$, $R = 1.25$, $u = 2$, and $d = 0.5$.
Backward-induction: $C = (0.5 \times C_u + 0.5 \times C_d)/1.25$. 