Binomial Distribution

- Denote the binomial distribution with parameters n and p by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

  – $n! = 1 \times 2 \times \cdots \times n$.

  – Convention: $0! = 1$.

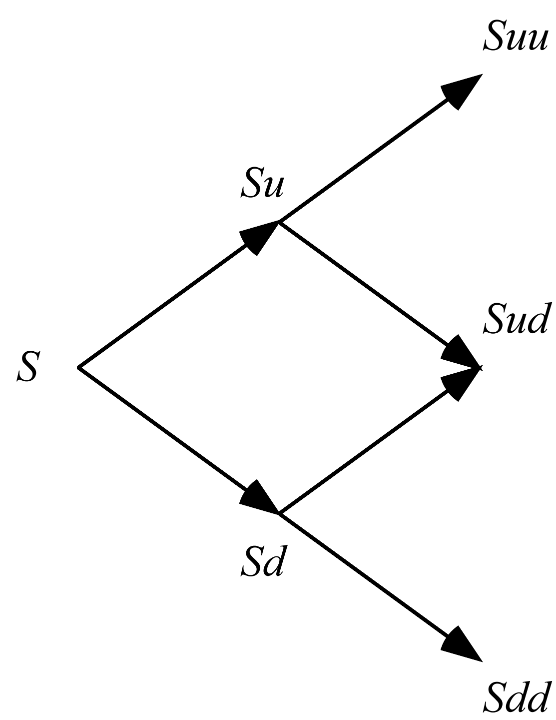
- Suppose you flip a coin n times with p being the probability of getting heads.

- Then $b(j; n, p)$ is the probability of getting j heads.

Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two: Suu, Sud, and Sdd.
  - There are 4 paths.
  - But the tree *combines*.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.[a]

[a]It is Markovian.


Suu
Su
Sud
S
Sd
Sdd

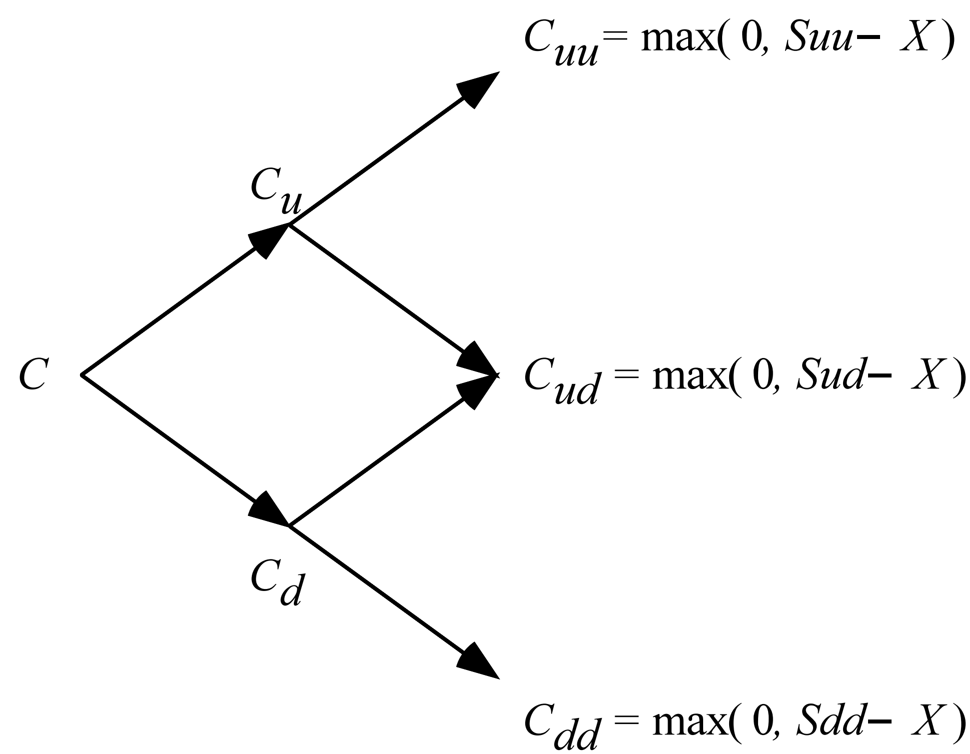
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let C_{uu} be the call's value at time two if the stock price is Suu.
- Thus,

$$C_{uu} = \max(0, Suu - X).$$

- C_{ud} and C_{dd} can be calculated analogously,

$$\begin{aligned} C_{ud} &= \max(0, Sud - X), \\ C_{dd} &= \max(0, Sdd - X). \end{aligned}$$


$C_{uu} = \max(0, Suu - X)$
C_u
C
$C_{ud} = \max(0, Sud - X)$
C_d
$C_{dd} = \max(0, Sdd - X)$

Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time 1 can be obtained by applying the same logic:

$$
\begin{aligned}
C_u &= \frac{pC_{uu} + (1-p)\, C_{ud}}{R}, \quad (26)\\
C_d &= \frac{pC_{ud} + (1-p)\, C_{dd}}{R}.
\end{aligned}
$$

- Deltas can be derived from Eq. (23) on p. 228.
- For example, the delta at C_u is

$$\frac{C_{uu} - C_{ud}}{Suu - Sud}.$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- Compute
$$\frac{pC_u + (1 - p)\, C_d}{R}$$
as the option price.
- The values of delta h and B can be derived from Eqs. (23)–(24) on p. 228.

Early Exercise

- Since the call will not be exercised at time 1 even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)\,C_d}{R} \geq \frac{[\,pu + (1-p)\,d\,]\,S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

  – The call again will not be exercised at present.[a]

- So

$$C = hS + B = \frac{pC_u + (1-p)\,C_d}{R}.$$

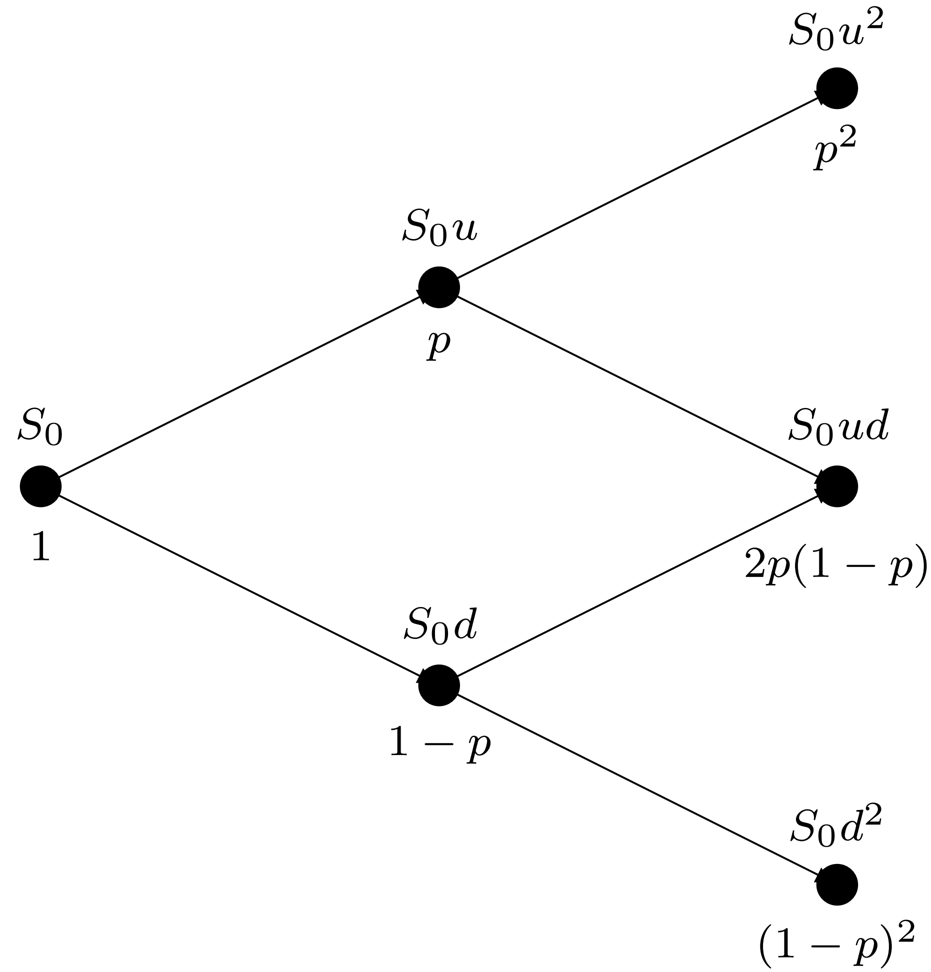
[a]Consistent with Theorem 4 (p. 210).

Backward Induction[a]

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happened at C_u and C_d, too, as demonstrated in Eq. (26) on p. 239.
- This recursive procedure is called backward induction.
- C equals

$$
\begin{aligned}
& [\, p^2 C_{uu} + 2p(1-p)\, C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\
= \; & [\, p^2 \max\left(0, Su^2 - X\right) + 2p(1-p) \max\left(0, Sud - X\right) \\
& +(1-p)^2 \max\left(0, Sd^2 - X\right)\,]/R^2.
\end{aligned}
$$

[a] Ernst Zermelo (1871–1953).


S_0u^2
p^2
S_0u
p
S_0
1
S_0ud
$2p(1-p)$
S_0d
$1-p$
S_0d^2
$(1-p)^2$

Backward Induction (continued)

- In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, Su^j d^{n-j} - X\right)}{R^n}.$$

  - The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.

- Similarly,

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - Su^j d^{n-j}\right)}{R^n}.$$

Backward Induction (concluded)

- Note that
$$p_j \equiv \frac{\binom{n}{j} p^j (1-p)^{n-j}}{R^n}$$
is the state price[a] for the state $Su^j d^{n-j}$, $j = 0, 1, \ldots, n$.

- In general,
$$\text{option price} = \sum_j p_j \times \text{payoff at state } j.$$

---

[a]Recall p. 187. One can obtain the undiscounted state price $\binom{n}{j} p^j (1-p)^{n-j}$—the risk-neutral probability—for the state $Su^j d^{n-j}$ with $(X_M - X_L)^{-1}$ units of the butterfly spread where $X_L = Su^{j-1} d^{n-j+1}$, $X_M = Su^j d^{n-j}$, and $X_H = Su^{j-1+1} d^{n-j-1}$.

Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\hat{r}n} E^{\pi}[\,\mathcal{D}\,].$$

  - E^{π} means the expectation is taken under the risk-neutral probability.
- The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- But it does not depend on predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.[a]
  - Changes in value are due entirely to capital gains.

[a]Except at the beginning, of course, when you have to put up the option value C or P before the replication starts.

Hakansson's Paradox[a]

- If options can be replicated, why are they needed at all?

[a] Hakansson (1979).

Can You Figure Out u, d without Knowing q?[a]

- Yes, you can, under BOPM.
- Let us observe the time series of past stock prices, e.g.,

$$\overbrace{S, Su,}^{u \text{ is available}} \quad Su^2, \underbrace{Su^3, Su^3 d}_{d \text{ is available}}, \ldots$$

- So with sufficiently long history, you will figure out u and d without knowing q.

[a]Contributed by Mr. Hsu, Jia-Shuo (`D97945003`) on March 11, 2009.

The Binomial Option Pricing Formula

- The stock prices at time n are

$$Su^n, Su^{n-1}d, \ldots, Sd^n.$$

- Let a be the minimum number of upward price moves for the call to finish in the money.
- So a is the smallest nonnegative integer j such that

$$Su^j d^{n-j} \geq X,$$

or, equivalently,

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil.$$

The Binomial Option Pricing Formula (concluded)

- Hence,

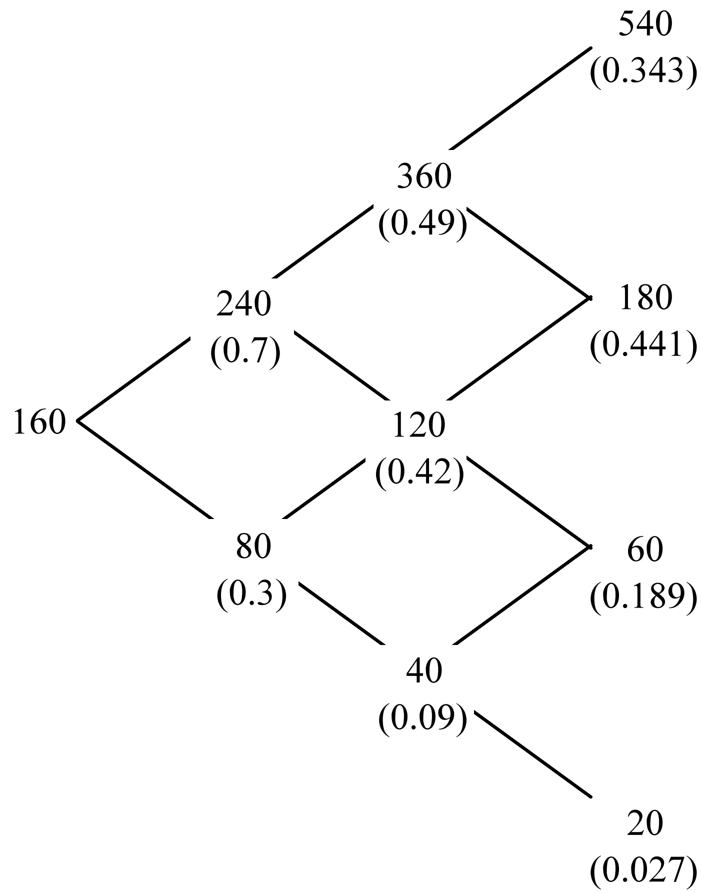
$$
\begin{aligned}
& C \\
= \;& \frac{\sum_{j=a}^{n} \binom{n}{j} p^{j} (1-p)^{n-j} \left(S u^{j} d^{n-j} - X \right)}{R^{n}} \qquad (27) \\
= \;& S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^{j} [\,(1-p)\, d\,]^{n-j}}{R^{n}} \\
& - \frac{X}{R^{n}} \sum_{j=a}^{n} \binom{n}{j} p^{j} (1-p)^{n-j} \\
= \;& S \sum_{j=a}^{n} b\,(j; n, pu/R) - X e^{-\hat{r}n} \sum_{j=a}^{n} b(j; n, p).
\end{aligned}
$$

Numerical Examples

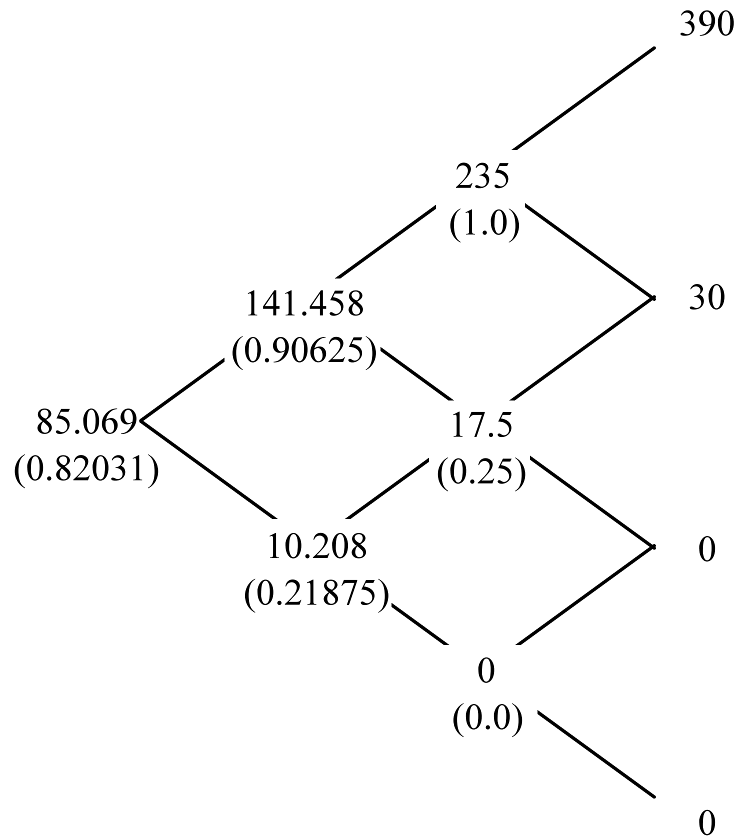
- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period ($R = e^{0.18232} = 1.2$).
  - Hence $p = (R - d)/(u - d) = 0.7$.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Or, the PV of the expected payoff at expiration:

$$\frac{390 \times 0.343 + 30 \times 0.441 + 0 \times 0.189 + 0 \times 0.027}{(1.2)^3} = 85.069.$$

Binomial process for the stock price
(probabilities in parentheses)



Binomial process for the call price
(hedge ratios in parentheses)



Numerical Examples (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90 and invest \$85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains,

$$90 - 85.069 = 4.931 \text{ dollars},$$

  is the arbitrage profit as we will see.

Numerical Examples (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy
$$0.90625 - 0.82031 = 0.08594$$
more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Numerical Examples (continued)

- The trading strategy is self-financing because the portfolio has a value of

$$0.90625 \times 240 - 76.04232 = 141.45768.$$

- It matches the corresponding call value!

Numerical Examples (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares.
- This generates an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to

$$76.04232 \times 1.2 - 78.75 = 12.5$$

dollars.

Numerical Examples (continued)

Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Numerical Examples (concluded)

Time 3 (the case of declining price):

- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of

$$0.25 \times 60 = 15$$

  dollars.
- Use it to repay the debt of $12.5 \times 1.2 = 15$ dollars.

Applications besides Exploiting Arbitrage Opportunities[a]

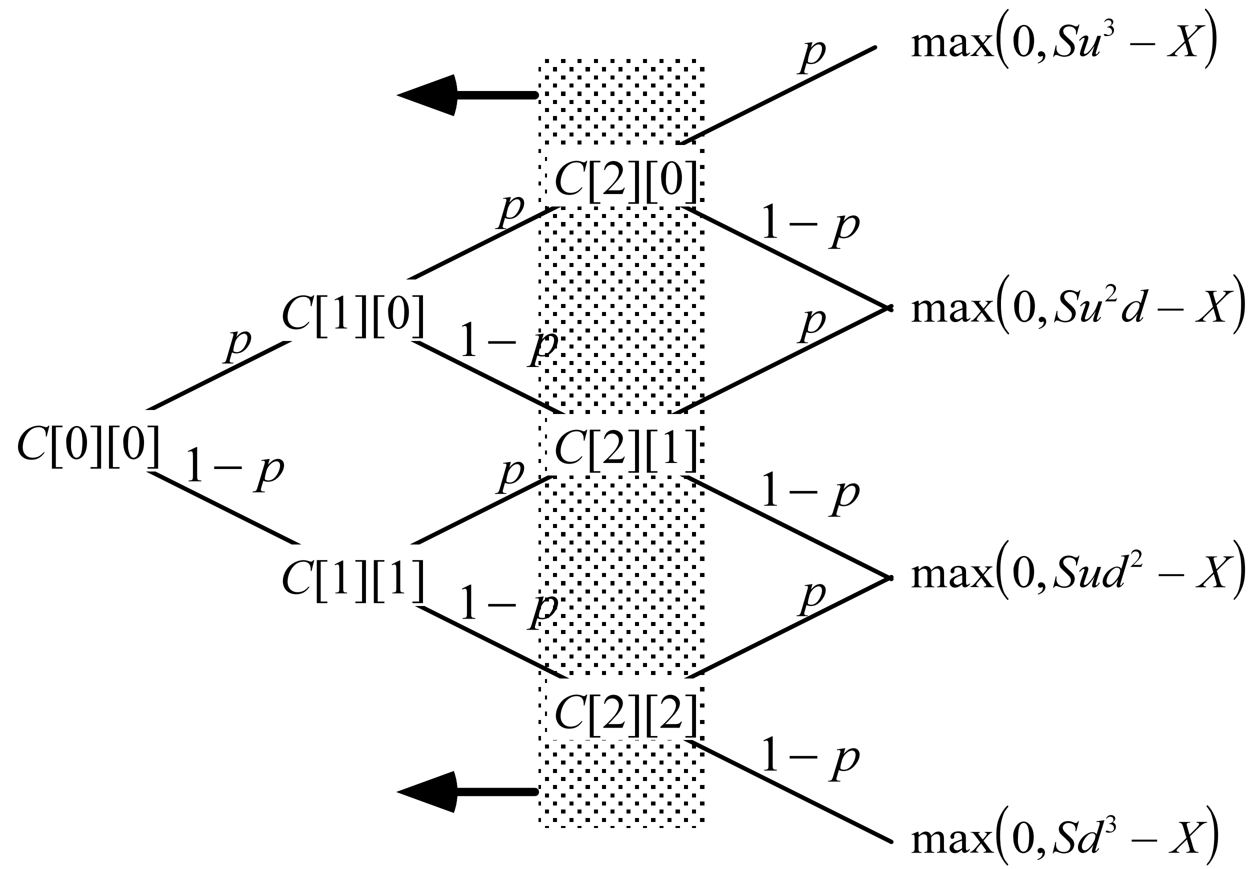
- Replicate an option using stocks and bonds.
  - Set up a portfolio to replicate the call with \$85.069.
- Hedge the options we issued.
  - Set up a portfolio to replicate the call with \$85.069 to counterbalance its values exactly.[b]
- · · ·

[a] Thanks to a lively class discussion on March 16, 2011.
[b] Hedge and replication are mirror images.

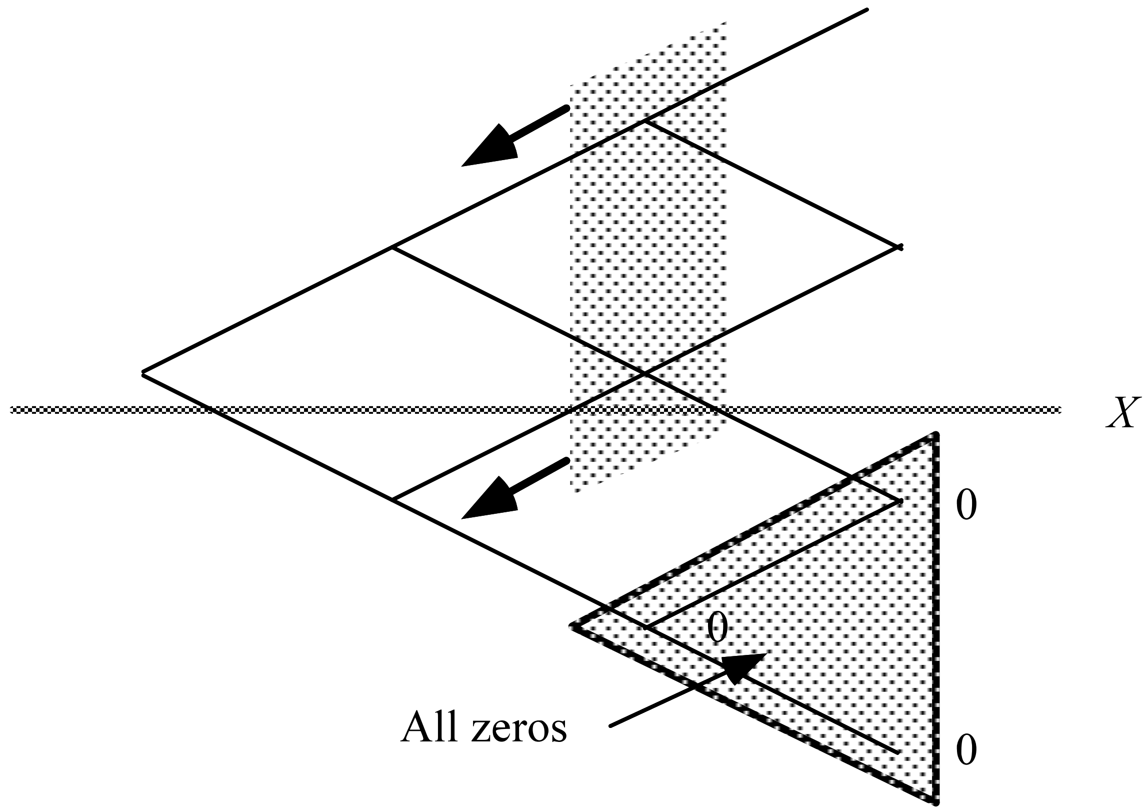
Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$ because there are $\sim n^2/2$ nodes.
- The memory requirement is $O(n^2)$.
  - Can be easily reduced to $O(n)$ by reusing space.[a]
- To price European puts, simply replace the payoff.

[a]But watch out for the proper updating of array entries.


max(0, Su^3 − X)
max(0, Su^2d − X)
max(0, Sud^2 − X)
max(0, Sd^3 − X)
C[0][0]
C[1][0]
C[1][1]
C[2][0]
C[2][1]
C[2][2]
p
1 − p

Further Time Improvement for Calls



Optimal Algorithm

- We can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j; n, p) = \frac{p(n-j+1)}{(1-p)\,j}\, b(j-1; n, p).$$

Optimal Algorithm (continued)

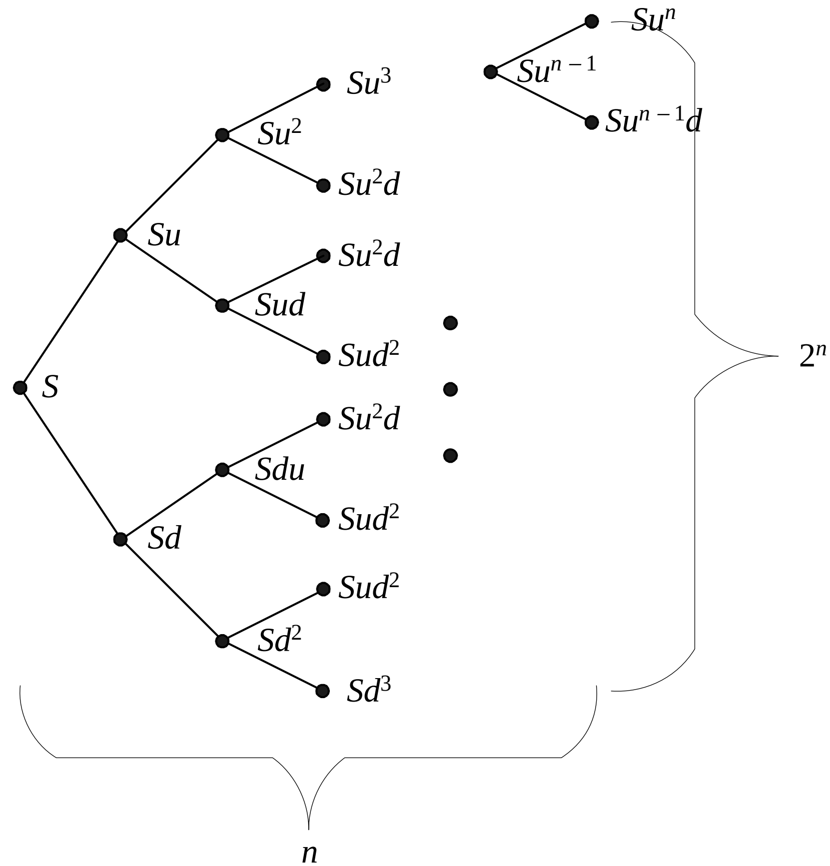
- The following program computes $b(j; n, p)$ in $b[\,j\,]$:
- It runs in $O(n)$ steps.

$b[\,a\,] := \binom{n}{a}\, p^a (1-p)^{n-a}$;
for $j = a+1, a+2, \ldots, n$ **do**
  $b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j)$;
end for

Optimal Algorithm (concluded)

- With the $b(j;n,p)$ available, the risk-neutral valuation formula (27) on p. 251 is trivial to compute.
- But we only need a single variable to store the $b(j;n,p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique *cannot* be applied to American options because of early exercise.
- So binomial tree algorithms for American options usually run in $O(n^2)$ time.

The Bushy Tree



Toward the Black-Scholes Formula

- The binomial model seems to suffer from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As n increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.
- Any proper calibration of the model parameters makes the BOPM converge to the continuous-time model.
- We now skim through the proof.

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as $n \to \infty$.
- First, $\hat{r} = r\tau/n$.
  - The period gross return $R = e^{\hat{r}}$.

Toward the Black-Scholes Formula (continued)

- Let
$$\widehat{\mu} \equiv \frac{1}{n}\, E\left[\ln \frac{S_\tau}{S}\right]$$
denote the expected value of the continuously compounded rate of return per period.

- Let
$$\widehat{\sigma}^2 \equiv \frac{1}{n}\, \mathrm{Var}\left[\ln \frac{S_\tau}{S}\right]$$
denote the variance of that return.

Toward the Black-Scholes Formula (continued)

- Under the BOPM, it is not hard to show that

$$\begin{aligned} \widehat{\mu} &= q\ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q)\ln^2(u/d). \end{aligned}$$

- Assume the stock's true continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
- Call σ the stock's (annualized) volatility.

Toward the Black-Scholes Formula (continued)

- The BOPM converges to the distribution only if

$$
\begin{aligned}
n\widehat{\mu} &= n[\, q \ln(u/d) + \ln d \,] \to \mu\tau, \\
n\widehat{\sigma}^2 &= nq(1-q) \ln^2(u/d) \to \sigma^2\tau.
\end{aligned}
$$

- We need one more condition to have a solution for u, d, q.

Toward the Black-Scholes Formula (continued)

- Impose

$$ud = 1.$$

  - It makes nodes at the same horizontal level of the tree have identical price (review p. 263).
  - Other choices are possible (see text).
  - Exact solutions for u, d, q are also feasible: 3 equations for 3 variables.[a]

[a]Chance (2008).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\,\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}\,. \quad (28)$$

- With Eqs. (28), it can be checked that

$$\begin{aligned} n\widehat{\mu} &= \mu\tau, \\ n\widehat{\sigma}^2 &= \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right]\sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- The choices (28) result in the CRR binomial model.[a]
- A more common choice for the probability is actually

$$q = \frac{R - d}{u - d}.$$

by Eq. (25) on p. 232.

- Their numerical properties are essentially identical.

[a] Cox, Ross, and Rubinstein (1979).

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $d < R < u$ may not hold under Eqs. (28) on p. 274.
  - If this happens, the probabilities lie outside $[\,0, 1\,]$.[a]
- The problem disappears when n satisfies

$$e^{\sigma \sqrt{\tau/n}} > e^{r\tau/n},$$

  i.e., when $n > r^2\tau/\sigma^2$ (check it).
  - So it goes away if n is large enough.
  - Other solutions will be presented later.

[a] Many papers and programs forget to check this condition!

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- The central limit theorem says $\ln(S_\tau/S)$ converges to $N(\mu\tau, \sigma^2\tau)$.[a]
- So $\ln S_\tau$ approaches $N(\mu\tau + \ln S, \sigma^2\tau)$.
- Conclusion: S_τ has a lognormal distribution in the limit.

---

[a]The normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.

Toward the Black-Scholes Formula (continued)

Lemma 10 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r-\sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability

$$p \equiv (e^{r\tau/n} - d)/(u-d).$$

- Let $n \to \infty$.[a]

[a] See Lemma 9.3.3 of the textbook.

Toward the Black-Scholes Formula (continued)

- The expected stock price at expiration in a risk-neutral economy is[a]

$$Se^{r\tau}.$$

- The stock's expected annual rate of return[b] is thus the riskless rate r.

---

[a]By Lemma 10 (p. 278) and Eq. (21) on p. 160.

[b]In the sense of $(1/\tau) \ln E[\, S_\tau / S \,]$ (arithmetic average rate of return) not $(1/\tau) E[\, \ln(S_\tau / S) \,]$ (geometric average rate of return).

Toward the Black-Scholes Formula (concluded)[a]

Theorem 11 (The Black-Scholes Formula)

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ P &= Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SN(-x), \end{aligned}$$

where

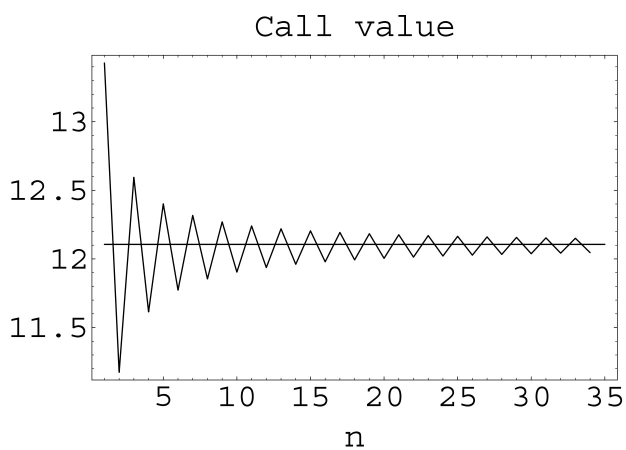
$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

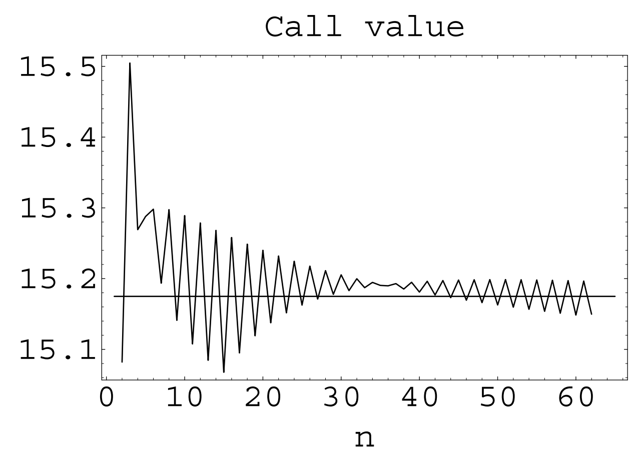
[a] On a United flight from San Francisco to Tokyo on March 7, 2010, a real-estate manager mentioned this formula to me!

BOPM and Black-Scholes Model

- The Black-Scholes formula needs 5 parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take 6 inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$
\begin{aligned}
u &= e^{\sigma\sqrt{\tau/n}}, \\
d &= e^{-\sigma\sqrt{\tau/n}}, \\
\hat{r} &= r\tau/n.
\end{aligned}
$$





- $S = 100$, $X = 100$ (left), and $X = 95$ (right).

BOPM and Black-Scholes Model (concluded)

- The binomial tree algorithms converge reasonably fast.
- The error is $O(1/n)$.[a]
- Oscillations are inherent, however.
- Oscillations can be dealt with by the judicious choices of u and d (see text).

[a]Chang and Palmer (2007).

Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.[a]
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?

---

[a] Implied volatility is hard to compute when τ is small (why?).

Implied Volatility (concluded)

- Implied volatility is

  the wrong number to put in the wrong formula to get the right price of plain-vanilla options.[a]

- Implied volatility is often preferred to historical volatility in practice.
  - Using the historical volatility is like driving a car with your eyes on the rearview mirror?

[a]Rebonato (2004).

Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options.
  - It becomes higher the further the option is in- or out-of-the-money.
- Other patterns have also been observed.

Problems; the Smile (concluded)

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.
- So?

Binomial Tree Algorithms for American Puts

- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs
$$\max(0, X - Su^j d^{n-j})$$
and applies backward induction.
- At each intermediate node, it compares the payoff if exercised and the continuation value.
- It keeps the larger one.

Bermudan Options

- Some American options can be exercised only at discrete time points instead of continuously.
- They are called Bermudan options.
- Their pricing algorithm is identical to that for American options.
- But early exercise is considered for only those nodes when early exercise is permitted.

