Of Spot Rate Curve and Yield Curve

- $y_k$: yield to maturity for the $k$-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.
Shapes

• The spot rate curve often has the same shape as the yield curve.
  – If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).

• But this is only a trend not a mathematical truth.\(^a\)

\(^a\)See a counterexample in the text.
Forward Rates

- The yield curve contains information regarding future interest rates currently “expected” by the market.

- Invest $1 for \( j \) periods to end up with \( [1 + S(j)]^j \) dollars at time \( j \).
  - The maturity strategy.

- Invest $1 in bonds for \( i \) periods and at time \( i \) invest the proceeds in bonds for another \( j - i \) periods where \( j > i \).

- Will have \( [1 + S(i)]^i[1 + S(i, j)]^{j-i} \) dollars at time \( j \).
  - \( S(i, j) \): \( (j - i) \)-period spot rate \( i \) periods from now.
  - The rollover strategy.
Forward Rates (concluded)

• When $S(i, j)$ equals

$$f(i, j) \equiv \left[ \frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1, \quad (15)$$

we will end up with $[1 + S(j)]^j$ dollars again.

• By definition, $f(0, j) = S(j)$.

• $f(i, j)$ is called the (implied) forward rates.
  
  – More precisely, the $(j - i)$-period forward rate $i$ periods from now.
Time Line

\[ f(0, 1) \quad f(1, 2) \quad f(2, 3) \quad f(3, 4) \]

Time 0

\[ S(1) \quad S(2) \quad S(3) \quad S(4) \]
Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i, j)$ and future spot rate $S(i, j)$.
  - This is the subject of the term structure theories.

- We merely looked for the future spot rate that, *if realized*, will equate the two investment strategies.

- $f(i, i + 1)$ are called the instantaneous forward rates or one-period forward rates.
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,
  \[ f(i, j) > S(j) > \cdots > S(i). \]

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,
  \[ f(i, j) < S(j) < \cdots < S(i). \]
(a) forward rate curve
spot rate curve
yield curve

(b) yield curve
spot rate curve
forward rate curve
Forward Rates $\equiv$ Spot Rates $\equiv$ Yield Curve

- The FV of $1$ at time $n$ can be derived in two ways.
- Buy $n$-period zero-coupon bonds and receive
  \[ [1 + S(n)]^n. \]
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The FV is
  \[ [1 + S(1)] [1 + f(1, 2)] \cdots [1 + f(n - 1, n)]. \]
Forward Rates ≡ Spot Rates ≡ Yield Curves
(concluded)

- Since they are identical,

\[
S(n) = \{[1 + S(1)][1 + f(1, 2)] \\
\cdots [1 + f(n - 1, n)]\}^{1/n} - 1.
\]

(16)

- Hence, the forward rates (specifically the one-period forward rates) determine the spot rate curve.

- Other equivalencies can be derived similarly, such as

\[
f(T, T + 1) = \frac{d(T)}{d(T + 1)} - 1.
\]
Locking in the Forward Rate \( f(n, m) \)

- Buy one \( n \)-period zero-coupon bond for \( 1/(1 + S(n))^n \) dollars.

- Sell \( (1 + S(m))^m/(1 + S(n))^n \) \( m \)-period zero-coupon bonds.

- No net initial investment because the cash inflow equals the cash outflow: \( 1/(1 + S(n))^n \).

- At time \( n \) there will be a cash inflow of $1.

- At time \( m \) there will be a cash outflow of \( (1 + S(m))^m/(1 + S(n))^n \) dollars.
Locking in the Forward Rate $f(n, m)$ (concluded)

- This implies the rate $f(n, m)$ between times $n$ and $m$.

$$\frac{(1 + S(m))^m}{(1 + S(n))^n}$$
Forward Contracts

- We had generated the cash flow of a financial instrument called forward contract.

- Agreed upon today, it enables one to
  - Borrow money at time $n$ in the future, and
  - Repay the loan at time $m > n$ with an interest rate equal to the forward rate

$$f(n, m).$$

- Can the spot rate curve be an arbitrary curve?\(^a\)

\(^a\)Contributed by Mr. Dai, Tian-Shyr (R86526008, D88526006) in 1998.
Spot and Forward Rates under Continuous Compounding

- The pricing formula:

\[ P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}. \]

- The market discount function:

\[ d(n) = e^{-nS(n)}. \]

- The spot rate is an arithmetic average of forward rates,

\[ S(n) = \frac{f(0, 1) + f(1, 2) + \cdots + f(n-1, n)}{n}. \]

\[ ^{a}\text{Compare it with Eq. (16) on p. 129.} \]
Spot and Forward Rates under Continuous Compounding (continued)

- The formula for the forward rate:
  \[ f(i, j) = \frac{jS(j) - iS(i)}{j - i}. \]

- The one-period forward rate:
  \[ f(j, j + 1) = -\ln \frac{d(j + 1)}{d(j)}. \]
Spot and Forward Rates under Continuous Compounding (concluded)

• Now,

\[
f(T) \equiv \lim_{\Delta T \to 0} f(T, T + \Delta T)
\]

\[
= S(T) + T \frac{\partial S}{\partial T}.
\]

• So \(f(T) > S(T)\) if and only if \(\partial S/\partial T > 0\) (i.e., a normal spot rate curve).

• If \(S(T) < -T(\partial S/\partial T)\), then \(f(T) < 0\).\(^a\)

\(^a\)Contributed by Mr. Huang, Hsien-Chun (R03922103) on March 11, 2015.
Unbiased Expectations Theory

- Forward rate equals the average future spot rate,
  \[ f(a, b) = E[S(a, b)]. \] (17)

- It does not imply that the forward rate is an accurate predictor for the future spot rate.

- It implies the maturity strategy and the rollover strategy produce the same result at the horizon on the average.
Unbiased Expectations Theory and Spot Rate Curve

• It implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - \( f(j, j + 1) > S(j + 1) \) if and only if \( S(j + 1) > S(j) \) from Eq. (15) on p. 123.
  - So \( E[S(j, j + 1)] > S(j + 1) > \cdots > S(1) \) if and only if \( S(j + 1) > \cdots > S(1) \).

• Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.
More Implications

• The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.

• Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.

• Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.

• That would mean investors are indifferent to risk.
A “Bad” Expectations Theory

- The expected returns on all possible riskless bond strategies are equal for all holding periods.

- So

\[ (1 + S(2))^2 = (1 + S(1)) E[1 + S(1, 2)] \]  \hspace{1cm} (18)

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement,

\[ \frac{1}{E[1 + S(1, 2)]} = \frac{1 + S(1)}{(1 + S(2))^2}. \]
A “Bad” Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond for \((1 + S(2))^{-2}\) dollars and sells it after one period.
  - The expected return\(^a\) is
    
    \[
    E \left[ (1 + S(1, 2))^{-1} \right] / (1 + S(2))^{-2}.
    \]
  
  - Strategy two buys a one-period bond with a return of \(1 + S(1)\).

\(^a\)More precisely, the one-plus return.
A “Bad” Expectations Theory (concluded)

- The theory says the returns are equal:

\[
\frac{1 + S(1)}{(1 + S(2))^2} = E \left[ \frac{1}{1 + S(1,2)} \right].
\]

- Combine this with Eq. (18) on p. 139 to obtain

\[
E \left[ \frac{1}{1 + S(1,2)} \right] = \frac{1}{E[1 + S(1,2)]}.
\]

- But this is impossible save for a certain economy.
  - Jensen’s inequality states that \( E[g(X)] > g(E[X]) \)
    for any nondegenerate random variable \( X \) and
    strictly convex function \( g \) (i.e., \( g''(x) > 0 \)).
  - Use \( g(x) \equiv (1 + x)^{-1} \) to prove our point.
Local Expectations Theory

- The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

\[
E \left[ \frac{(1 + S(1,n))^{-(n-1)}}{(1 + S(n))^{-n}} \right] = 1 + S(1) \quad \text{for all } n > 1.
\]

- This theory is the basis of many interest rate models.
Duration in Practice

• To handle more general types of spot rate curve changes, define a vector \([c_1, c_2, \ldots, c_n]\) that characterizes the perceived type of change.
  – Parallel shift: \([1, 1, \ldots, 1]\).
  – Twist: \([1, 1, \ldots, 1, -1, \ldots, -1]\),
    \(
    [1.8\%, 1.6\%, 1.4\%, 1\%, 0\%, -1\%, -1.4\%, \ldots],
    \) etc.
  – ....

• At least one \(c_i\) should be 1 as the reference point.
Duration in Practice (concluded)

- Let
  \[ P(y) \equiv \sum_{i} C_i / (1 + S(i) + yc_i)^i \]
  be the price associated with the cash flow \( C_1, C_2, \ldots \).

- Define duration as
  \[ -\left. \frac{\partial P(y)/P(0)}{\partial y} \right|_{y=0} \quad \text{or} \quad -\frac{P(\Delta y) - P(-\Delta y)}{P(0)\Delta y}. \]

- Modified duration equals the above when
  \[ [c_1, c_2, \ldots, c_n] = [1, 1, \ldots, 1], \]
  \[ S(1) = S(2) = \cdots = S(n). \]
Some Loose Ends on Dates

• Holidays.
• Weekends.
• Business days ($T + 2$, etc.).
• Shall we treat a year as 1 year whether it has 365 or 366 days?
Fundamental Statistical Concepts
There are three kinds of lies: lies, damn lies, and statistics.
— Misattributed to Benjamin Disraeli (1804–1881)

If 50 million people believe a foolish thing, it’s still a foolish thing.
— George Bernard Shaw (1856–1950)

One death is a tragedy, but a million deaths are a statistic.
— Josef Stalin (1879–1953)
Moments

- The variance of a random variable \( X \) is defined as
  \[
  \text{Var}[X] \equiv E \left[ (X - E[X])^2 \right].
  \]

- The covariance between random variables \( X \) and \( Y \) is
  \[
  \text{Cov}[X,Y] \equiv E \left[ (X - \mu_X)(Y - \mu_Y) \right],
  \]
  where \( \mu_X \) and \( \mu_Y \) are the means of \( X \) and \( Y \), respectively.

- Random variables \( X \) and \( Y \) are uncorrelated if
  \[
  \text{Cov}[X,Y] = 0.
  \]
Correlation

• The standard deviation of $X$ is the square root of the variance,

$$\sigma_X \equiv \sqrt{\text{Var}[X]}.$$  

• The correlation (or correlation coefficient) between $X$ and $Y$ is

$$\rho_{X,Y} \equiv \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y},$$

provided both have nonzero standard deviations.\(^a\)

\(^a\)Paul Wilmott (2009), “the correlations between financial quantities are notoriously unstable.”
Variance of Sum

• Variance of a weighted sum of random variables equals

\[
\text{Var}\left[ \sum_{i=1}^{n} a_i X_i \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \text{Cov}[X_i, X_j].
\]

• It becomes

\[
\sum_{i=1}^{n} a_i^2 \text{Var}[X_i]
\]

when \( X_i \) are uncorrelated.
Conditional Expectation

• “$X \mid I$” denotes $X$ conditional on the information set $I$.

• The information set can be another random variable’s value or the past values of $X$, say.

• The conditional expectation $E[X \mid I]$ is the expected value of $X$ conditional on $I$; it is a random variable.

• The law of iterated conditional expectations:

$$E[X] = E[E[X \mid I]].$$

• If $I_2$ contains at least as much information as $I_1$, then

$$E[X \mid I_1] = E[E[X \mid I_2] \mid I_1].$$ (19)
The Normal Distribution

• A random variable $X$ has the normal distribution with mean $\mu$ and variance $\sigma^2$ if its probability density function is

$$
\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.
$$

• This is expressed by $X \sim N(\mu, \sigma^2)$.

• The standard normal distribution has zero mean, unit variance, and the following distribution function

$$
\text{Prob}[X \leq z] = N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} \, dx.
$$
Moment Generating Function

• The moment generating function of random variable $X$ is

$$\theta_X(t) \equiv E[e^{tX}].$$

• The moment generating function of $X \sim N(\mu, \sigma^2)$ is

$$\theta_X(t) = \exp \left[ \mu t + \frac{\sigma^2 t^2}{2} \right]. \quad (20)$$
The Multivariate Normal Distribution

• If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then
  \[ \sum_i X_i \sim N \left( \sum_i \mu_i, \sum_i \sigma_i^2 \right). \]

• Let $X_i \sim N(\mu_i, \sigma_i^2)$, which may not be independent.

• Suppose
  \[ \sum_{i=1}^{n} t_i X_i \sim N \left( \sum_{i=1}^{n} t_i \mu_i, \sum_{i=1}^{n} \sum_{j=1}^{n} t_i t_j \text{Cov}[X_i, X_j] \right) \]
  for every linear combination $\sum_{i=1}^{n} t_i X_i$.\textsuperscript{a}

• $X_i$ are said to have a multivariate normal distribution.

\textsuperscript{a}Corrected by Mr. Huang, Guo-Hua (R98922107) on March 10, 2010.
Generation of Univariate Normal Distributions

- Let $X$ be uniformly distributed over $(0, 1]$ so that
  \[ \text{Prob}[X \leq x] = x, \quad 0 < x \leq 1. \]

- Repeatedly draw two samples $x_1$ and $x_2$ from $X$ until
  \[ \omega \equiv (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1. \]

- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where\(^a\)
  \[ c \equiv \sqrt{-2(\ln \omega)/\omega}. \]

\(^a\)As they are normally distributed, to prove independence, it suffices to prove that they are uncorrelated, which is easy. Thanks to a lively class discussion on March 5, 2014.
A Dirty Trick and a Right Attitude

• Let $\xi_i$ are independent and uniformly distributed over $(0, 1)$.

• A simple method to generate the standard normal variable is to calculate$^a$

$$\left( \sum_{i=1}^{12} \xi_i \right) - 6.$$

• But why use 12?

• Recall the mean and variance of $\xi_i$ are $1/2$ and $1/12$, respectively.

---

$^a$Jäckel (2002), “this is not a highly accurate approximation and should only be used to establish ballpark estimates.”
• So the general formula is

\[
\frac{\left(\sum_{i=1}^{n} \xi_i\right) - (n/2)}{\sqrt{n/12}}.
\]

• Choosing \(n = 12\) yields a formula without the need of division and square-root operations.\(^a\)

• Always blame your random number generator last.\(^b\)

• Instead, check your programs first.

\(^a\)Contributed by Mr. Chen, Shih-Hang (R02723031) on March 5, 2014.

\(^b\)“The fault, dear Brutus, lies not in the stars but in ourselves that we are underlings.” William Shakespeare (1564–1616), Julius Caesar.
Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation $\rho$ can be generated.

- Let $X_1$ and $X_2$ be independent standard normal variables.

- Set

$$
U \equiv aX_1, \\
V \equiv \rho U + \sqrt{1 - \rho^2} \ aX_2.
$$
Generation of Bivariate Normal Distributions (concluded)

- $U$ and $V$ are the desired random variables with
  \[
  \text{Var}[U] = \text{Var}[V] = a^2, \\
  \text{Cov}[U, V] = \rho a^2.
  \]

- Note that the mapping between $(X_1, X_2)$ and $(U, V)$ is one-to-one.
The Lognormal Distribution

- A random variable $Y$ is said to have a lognormal distribution if $\ln Y$ has a normal distribution.

- Let $X \sim N(\mu, \sigma^2)$ and $Y \equiv e^X$.

- The mean and variance of $Y$ are

$$
\mu_Y = e^{\mu + \sigma^2/2} \quad \text{and} \quad \sigma_Y^2 = e^{2\mu + \sigma^2} \left( e^{\sigma^2} - 1 \right),
$$

(respectively).

- They follow from $E[Y^n] = e^{n\mu + n^2\sigma^2/2}$. 

Option Basics
The shift toward options as the center of gravity of finance [...] — Merton H. Miller (1923–2000)
Calls and Puts

• A call gives its holder the right to buy a number of the underlying asset by paying a strike price.
Calls and Puts (continued)

- A put gives its holder the right to sell a number of the underlying asset for the strike price.

\[
\text{strike price} \quad \downarrow \\
\text{option premium} \quad \downarrow \\
\text{stock}
\]
Calls and Puts (concluded)

- An embedded option has to be traded along with the underlying asset.

- How to price options?
  - It can be traced to Aristotle’s (384 B.C.–322 B.C.) *Politics*, if not earlier.
Exercise

• When a call is exercised, the holder pays the strike price in exchange for the stock.

• When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.

• An option can be exercised prior to the expiration date: early exercise.
American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- An American option is worth at least as much as an otherwise identical European option.