Day Count Conventions: Actual/Actual

- The first “actual” refers to the actual number of days in a month.

- The second refers to the actual number of days in a coupon period.

- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.
Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date \( D_1 \equiv (y_1, m_1, d_1) \) to date \( D_2 \equiv (y_2, m_2, d_2) \) is
  \[
  360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1)
  \]
- But if \( d_1 \) or \( d_2 \) is 31, we need to change it to 30 before applying the above formula.
Day Count Conventions: 30/360 (concluded)

- An equivalent formula without any adjustment is (check it)

\[
360 \times (y_2 - y_1) + 30 \times (m_2 - m_1 - 1) \\
+ \max(30 - d_1, 0) + \min(d_2, 30).
\]

- Many variations regarding 31, Feb 28, and Feb 29.
Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let
  \[
  \omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}.
  \]
  \hspace{1cm} (9)
- The price is now calculated by
  \[
  PV = \frac{C}{(1 + \frac{r}{m})^{\omega}} + \frac{C}{(1 + \frac{r}{m})^{\omega+1}} \cdots
  \]
  \[
  = \sum_{i=0}^{n-1} \frac{C}{(1 + \frac{r}{m})^{\omega+i}} + \frac{F}{(1 + \frac{r}{m})^{\omega+n-1}}.
  \]  \hspace{1cm} (10)
$C(1 - \omega)$

coupon payment date

(1 - \omega)\%  \omega\%

coupon payment date
Accrued Interest (AI)

• The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.

• The buyer pays the invoice price: the quoted price plus the accrued interest.

• The accrued interest equals

\[
C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1 - \omega).
\]

• The yield to maturity is the \( r \) satisfying Eq. (10) when \( PV \) is the invoice price.
Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.
Example ("30/360") (concluded)

• The accrued interest is \((10/2) \times (1 - \frac{60}{180}) = 3.3333\) per $100 of par value.

• The yield to maturity is 3%.

• This can be verified by Eq. (10) on p. 71 with
  
  \(- \omega = 60/180,\)
  
  \(- m = 2,\)
  
  \(- F = 100,\)
  
  \(- C = 5,\)
  
  \(- PV = 111.2891 + 3.3333,\)
  
  \(- r = 0.03.\)
Price Behavior (2) Revisited

• Before: A bond selling at par if the yield to maturity equals the coupon rate.

• But it assumed that the settlement date is on a coupon payment date.

• Now suppose the settlement date for a bond selling at par (i.e., the quoted price is equal to the par value) falls between two coupon payment dates.

• Then its yield to maturity is less than the coupon rate.
  – The short reason: Exponential growth to $C$ is replaced by linear growth, hence “overpaying.”
Bond Price Volatility
“Well, Beethoven, what is this?”
— Attributed to Prince Anton Esterházy
Price Volatility

- Volatility measures how bond prices respond to interest rate changes.
- It is key to the risk management of interest rate-sensitive securities.
Price Volatility (concluded)

- What is the sensitivity of the percentage price change to changes in interest rates?

- Define price volatility by

\[
\frac{\partial P}{\partial y} - \frac{P}{P}.
\]
Price Volatility of Bonds

- The price volatility of a level-coupon bond is

\[
- \frac{(C/y) n - (C/y^2) ((1 + y)^{n+1} - (1 + y)) - nF}{(C/y) ((1 + y)^{n+1} - (1 + y)) + F(1 + y)}.
\]

- \( F \) is the par value.
- \( C' \) is the coupon payment per period.

- For bonds without embedded options,

\[
- \frac{\partial P}{\partial y} \frac{P}{P} > 0.
\]
Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset’s cash flows.
- The weights are the cash flows’ PVs divided by the asset’s price.
- Formally,
  \[ MD \equiv \frac{1}{P} \sum_{i=1}^{n} i \frac{C_i}{(1+y)^i}. \]
- The Macaulay duration, in periods, is equal to
  \[ MD = -(1+y) \frac{\partial P}{\partial y} \frac{1}{P}. \] (11)
MD of Bonds

• The MD of a level-coupon bond is

\[ MD = \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n} \right]. \]  (12)

• It can be simplified to

\[ MD = \frac{c(1+y) \left[ (1+y)^n - 1 \right] + ny(y-c)}{cy \left[ (1+y)^n - 1 \right] + y^2}, \]

where \( c \) is the period coupon rate.

• The MD of a zero-coupon bond equals \( n \), its term to maturity.

• The MD of a level-coupon bond is less than \( n \).
Remarks

- Equations (11) on p. 82 and (12) on p. 83 hold only if the coupon $C$, the par value $F$, and the maturity $n$ are all independent of the yield $y$.
  - That is, if the cash flow is independent of yields.

- To see this point, suppose the market yield declines.

- The MD will be lengthened.

- But for securities whose maturity actually decreases as a result, the price volatility (as originally defined) may decrease.
How Not To Think about MD

• The MD has its origin in measuring the length of time a bond investment is outstanding.

• But it should be seen mainly as measuring price volatility.

• Many, if not most, duration-related terminology cannot be comprehended otherwise.
Conversion

- For the MD to be year-based, modify Eq. (12) on p. 83 to
  \[
  \frac{1}{P} \left[ \sum_{i=1}^{n} \frac{i}{k} \left( \frac{C}{1 + \frac{y}{k}} \right)^i + \frac{n}{k} \frac{F}{(1 + \frac{y}{k})^n} \right],
  \]
  where \( y \) is the annual yield and \( k \) is the compounding frequency per annum.

- Equation (11) on p. 82 also becomes
  \[
  \text{MD} = - \left( 1 + \frac{y}{k} \right) \frac{\partial P}{\partial y} \frac{1}{P}.
  \]

- By definition, \( \text{MD (in years)} = \frac{\text{MD (in periods)}}{k} \). 
Modified Duration

- Modified duration is defined as

\[
\text{modified duration} \equiv - \frac{\partial P}{\partial y} \frac{1}{P} = \frac{\text{MD}}{(1 + y)}. \tag{13}
\]

- By the Taylor expansion,

percent price change \(\approx -\text{modified duration} \times \text{yield change.}\)
Example

• Consider a bond whose modified duration is 11.54 with a yield of 10%.

• If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

\[-11.54 \times 0.001 = -0.01154 = -1.154\% .\]
Modified Duration of a Portfolio

- The modified duration of a portfolio equals

\[ \sum_i \omega_i D_i. \]

- \( D_i \) is the modified duration of the \( i \)th asset.
- \( \omega_i \) is the market value of that asset expressed as a percentage of the market value of the portfolio.
Effective Duration

• Yield changes may alter the cash flow or the cash flow may be so complex that simple formulas are unavailable.

• We need a general numerical formula for volatility.

• The effective duration is defined as

\[
\frac{P_--P_+}{P_0(y_+-y_-)}. 
\]

- \( P_- \) is the price if the yield is decreased by \( \Delta y \).
- \( P_+ \) is the price if the yield is increased by \( \Delta y \).
- \( P_0 \) is the initial price, \( y \) is the initial yield.
- \( \Delta y \) is small.

• See plot on p. 91.
Effective Duration (concluded)

• One can compute the effective duration of just about any financial instrument.

• Duration of a security can be longer than its maturity or negative!

• Neither makes sense under the maturity interpretation.

• An alternative is to use

\[ \frac{P_0 - P_+}{P_0 \Delta y} \]

– More economical but theoretically less accurate.
The Practices

• Duration is usually expressed in percentage terms — call it \( D\% \) — for quick mental calculation.

• The percentage price change expressed in percentage terms is then approximated by

\[
-D\% \times \Delta r
\]

when the yield increases instantaneously by \( \Delta r\% \).

– Price will drop by 20% if \( D\% = 10 \) and \( \Delta r = 2 \) because \( 10 \times 2 = 20 \).

• \( D\% \) equals modified duration as originally defined (prove it!).
Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.

- Define dollar duration as

\[
\text{modified duration} \times \text{price (\% of par)} = -\frac{\partial P}{\partial y}.
\]

- The approximate *dollar* price change per $100 of par value is

\[
\text{price change} \approx -\text{dollar duration} \times \text{yield change}.
\]

- One can hedge a bond with a dollar duration $D$ by bonds with a dollar duration $-D$. 
Convexity

• Convexity is defined as

\[
\text{convexity (in periods)} = \frac{\partial^2 P}{\partial y^2} \frac{1}{P}.
\]

• The convexity of a level-coupon bond is positive (prove it!).

• For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude (see plot next page).

• So between two bonds with the same price and duration, the one with a higher convexity is more valuable.\(^a\)

\(^a\)Do you spot a problem here (Christensen and Sørensen (1994))?
Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

\[
\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}
\]

when there are \( k \) periods per annum.
Use of Convexity

- The approximation \( \Delta P/P \approx - \text{duration} \times \text{yield change} \) works for small yield changes.

- For larger yield changes, use

\[
\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y} \frac{1}{P} \Delta y + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \frac{1}{P} (\Delta y)^2
\]

\[
= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.
\]

- Recall the figure on p. 96.
The Practices

• Convexity is usually expressed in percentage terms — call it \( C\% \) — for quick mental calculation.

• The percentage price change expressed in percentage terms is approximated by

\[
- D\% \times \Delta r + C\% \times (\Delta r)^2 / 2
\]

when the yield increases instantaneously by \( \Delta r\% \).

– Price will drop by 17% if \( D\% = 10, \ C\% = 1.5, \) and \( \Delta r = 2 \) because

\[
-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.
\]

• \( C\% \) equals convexity divided by 100 (prove it!).
Effective Convexity

- The effective convexity is defined as

\[
\frac{P_+ + P_- - 2P_0}{P_0 \left(0.5 \times (y_+ - y_-)\right)^2},
\]

- \(P_-\) is the price if the yield is decreased by \(\Delta y\).
- \(P_+\) is the price if the yield is increased by \(\Delta y\).
- \(P_0\) is the initial price, \(y\) is the initial yield.
- \(\Delta y\) is small.

- Effective convexity is most relevant when a bond’s cash flow is interest rate sensitive.

- Numerically, choosing the right \(\Delta y\) is a delicate matter.
Approximate $d^2 f(x)^2/dx^2$ at $x = 1$, Where $f(x) = x^2$

- The difference of $((1 + \Delta x)^2 + (1 - \Delta x)^2 - 2)/(\Delta x)^2$ and $2$:

![Graph showing error]

- This numerical issue is common in financial engineering but does not admit easy solutions yet (see pp. 761ff).
Interest Rates and Bond Prices: Which Determines Which?\textsuperscript{a}

- If you have one, you have the other.
- So they are just two names given to the same thing: cost of fund.
- Traders most likely work with prices.
- Banks most likely work with interest rates.

\textsuperscript{a}Contributed by Mr. Wang, Cheng (R01741064) on March 5, 2014.
Term Structure of Interest Rates
Why is it that the interest of money is lower, when money is plentiful?
— Samuel Johnson (1709–1784)

If you have money, don’t lend it at interest. Rather, give [it] to someone from whom you won’t get it back.
— Thomas Gospel 95
Term Structure of Interest Rates

• Concerned with how interest rates change with maturity.

• The set of yields to maturity for bonds form the term structure.
  – The bonds must be of equal quality.
  – They differ solely in their terms to maturity.

• The term structure is fundamental to the valuation of fixed-income securities.
Yield (%) vs. Year
Term Structure of Interest Rates (concluded)

- Term structure often refers exclusively to the yields of zero-coupon bonds.

- A yield curve plots the yields to maturity of coupon bonds against maturity.

- A par yield curve is constructed from bonds trading near par.
Four Typical Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.
Spot Rates

• The \( i \)-period spot rate \( S(i) \) is the yield to maturity of an \( i \)-period zero-coupon bond.

• The PV of one dollar \( i \) periods from now is by definition

\[
[1 + S(i)]^{-i}.
\]

  It is the price of an \( i \)-period zero-coupon bond.

• The one-period spot rate is called the short rate.

• Spot rate curve: Plot of spot rates against maturity:

\[
S(1), S(2), \ldots, S(n).
\]
Problems with the PV Formula

• In the bond price formula (3) on p. 32,

\[ \sum_{i=1}^{n} \frac{C}{(1 + y)^i} + \frac{F}{(1 + y)^n}, \]

every cash flow is discounted at the same yield \( y \).

• Consider two riskless bonds with different yields to maturity because of their different cash flow streams:

\[ \sum_{i=1}^{n_1} \frac{C}{(1 + y_1)^i} + \frac{F}{(1 + y_1)^{n_1}}, \]
\[ \sum_{i=1}^{n_2} \frac{C}{(1 + y_2)^i} + \frac{F}{(1 + y_2)^{n_2}}. \]
Problems with the PV Formula (concluded)

- The yield-to-maturity methodology discounts their *contemporaneous* cash flows with different rates.
- But shouldn’t they be discounted at the same rate?
Spot Rate Discount Methodology

- A cash flow \( C_1, C_2, \ldots, C_n \) is equivalent to a package of zero-coupon bonds with the \( i \)th bond paying \( C_i \) dollars at time \( i \).
Spot Rate Discount Methodology (concluded)

- So a level-coupon bond has the price

\[ P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}. \]  \hspace{1cm} (14)

- This pricing method incorporates information from the term structure.

- It discounts each cash flow at the corresponding spot rate.
Discount Factors

• In general, any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i).$$

- Above, $d(i) \equiv [1 + S(i)]^{-i}$, $i = 1, 2, \ldots, n$, are called discount factors.
- $d(i)$ is the PV of one dollar $i$ periods from now.
- This formula, now just a definition, will be justified on p. 195.

• The discount factors are often interpolated to form a continuous function called the discount function.
Extracting Spot Rates from Yield Curve

• Start with the short rate \( S(1) \).
  
  – Note that short-term Treasuries are zero-coupon bonds.

• Compute \( S(2) \) from the two-period coupon bond price \( P \) by solving

\[
P = \frac{C}{1 + S(1)} + \frac{C + 100}{[1 + S(2)]^2}.
\]
Extracting Spot Rates from Yield Curve (concluded)

- Inductively, we are given the market price $P$ of the $n$-period coupon bond and

$$S(1), S(2), \ldots , S(n - 1).$$

- Then $S(n)$ can be computed from Eq. (14) on p. 113, repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{[1 + S(i)]^i} + \frac{F}{[1 + S(n)]^n}.$$

- The running time can be made to be $O(n)$ (see text).
- The procedure is called bootstrapping.
Some Problems

• Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).

• Some maturities might be missing from the data points (the incompleteness problem).

• Treasuries might not be of the same quality.

• Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.\textsuperscript{a}

\textsuperscript{a}Any economic justifications?
Yield Spread

- Consider a *risky* bond with the cash flow \( C_1, C_2, \ldots, C_n \) and selling for \( P \).

- Yield spread is the difference between the IRR of the risky bond and that of a riskless bond with comparable maturity.
Static Spread

- Were the risky bond riskless, it would fetch

\[ P^* = \sum_{t=1}^{n} \frac{C_t}{[1 + S(t)]^t}. \]

- But as risk must be compensated, in reality \( P < P^* \).

- The static spread is the amount \( s \) by which the spot rate curve has to shift in parallel to price the risky bond:

\[ P = \sum_{t=1}^{n} \frac{C_t}{[1 + s + S(t)]^t}. \]

- Unlike the yield spread, the static spread incorporates information from the term structure.