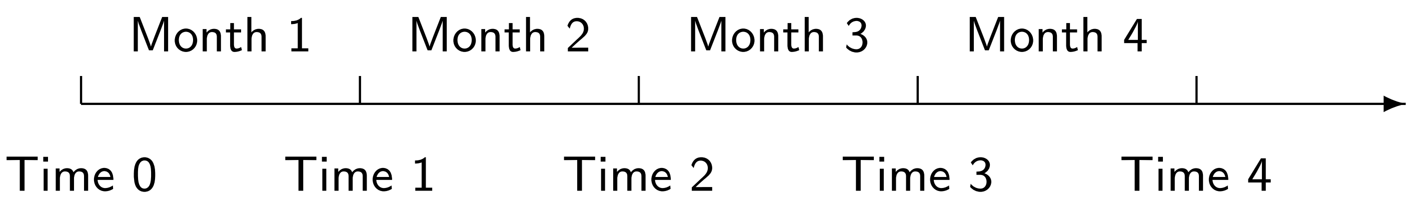
Analysis of Mortgage-Backed Securities

Oh, well, if you cannot measure,
measure anyhow.
— Frank H. Knight (1885–1972)

Uniqueness of MBS

- Compared with other fixed-income securities, the MBS is unique in two respects.
- Its cash flow consists of principal and interest (P&I).
- The cash flow may vary because of prepayments in the underlying mortgages.

Time Line

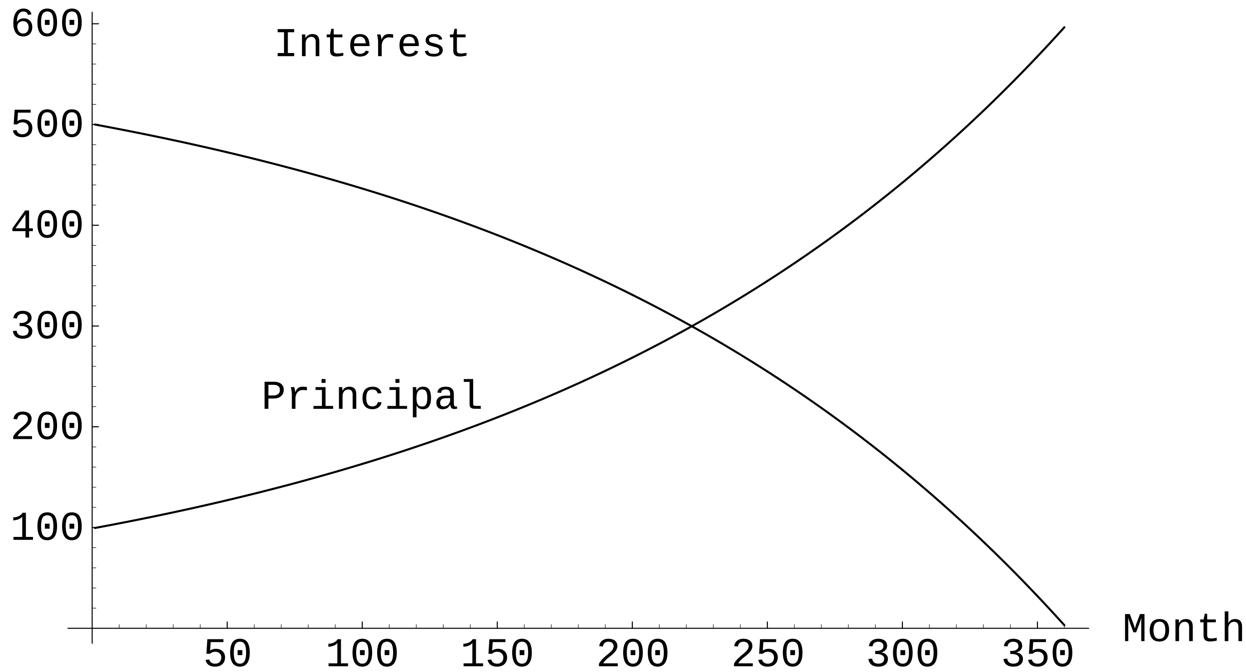


- Mortgage payments are paid in arrears.
- A payment for month i occurs at time i, that is, end of month i.
- The end of a month will be identified with the beginning of the coming month.

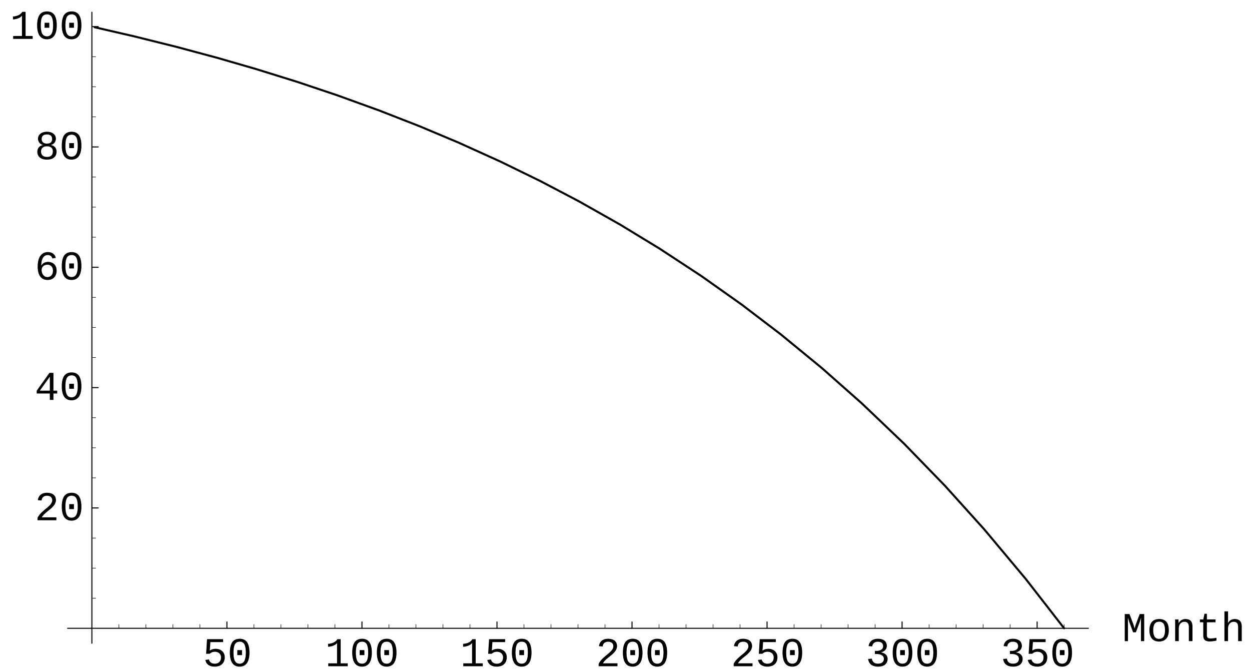
Cash Flow Analysis

- A traditional mortgage has a fixed term, a fixed interest rate, and a fixed monthly payment.
- Page 1095 illustrates the scheduled P&I for a 30-year, 6% mortgage with an initial balance of \$100,000.
- Page 1096 depicts how the remaining principal balance decreases over time.

Scheduled Principal and Interest Payments



Scheduled Remaining Principal Balances



Cash Flow Analysis (continued)

- In the early years, the P&I consists mostly of interest.
- Then it gradually shifts toward principal payment with the passage of time.
- However, the total P&I payment remains the same each month, hence the term *level* pay.
- In the absence of prepayments and servicing fees, identical characteristics hold for the pool's P&I payments.

Cash Flow Analysis (continued)

- From Eq. (6) on p. 46 the remaining principal balance after the kth payment is

$$C \, \frac{1 - (1 + r/m)^{-n+k}}{r/m}. \tag{132}$$

  - C is the scheduled P&I payment of an n-month mortgage making m payments per year.
  - r is the annual mortgage rate.

- For mortgages, $m = 12$.

Cash Flow Analysis (continued)

- The scheduled remaining principal balance after k payments can be expressed as a portion of the original principal balance:

$$\begin{aligned} \text{Bal}_k &\equiv 1 - \frac{(1+r/m)^k - 1}{(1+r/m)^n - 1} \\ &= \frac{(1+r/m)^n - (1+r/m)^k}{(1+r/m)^n - 1}. \end{aligned} \tag{133}$$

- This equation can be verified by dividing Eq. (132) (p. 1098) by the same equation with $k = 0$.

Cash Flow Analysis (continued)

- The remaining principal balance after k payments is

$$\mathrm{RB}_k \equiv \mathcal{O} \times \mathrm{Bal}_k,$$

  where $\mathcal{O}$ will denote the original principal balance.

- The term factor denotes the portion of the remaining principal balance to its original principal balance.
- So Bal_k is the monthly factor when there are no prepayments.
- It is also known as the amortization factor.

Cash Flow Analysis (concluded)

- When the idea of factor is applied to a mortgage pool, it is called the paydown factor on the pool or simply the pool factor.

An Example

- The remaining balance of a 15-year mortgage with a 9% mortgage rate after 54 months is

$$
\begin{aligned}
& \mathcal{O} \times \frac{(1+(0.09/12))^{180} - (1+(0.09/12))^{54}}{(1+(0.09/12))^{180} - 1} \\
= \quad & \mathcal{O} \times 0.824866.
\end{aligned}
$$

- In other words, roughly 82.49% of the original loan amount remains after 54 months.

P&I Analysis

- By the amortization principle, the tth interest payment equals

$$I_t \equiv \mathrm{RB}_{t-1} \times \frac{r}{m} = \mathcal{O} \times \frac{r}{m} \times \frac{(1+r/m)^n - (1+r/m)^{t-1}}{(1+r/m)^n - 1}.$$

- The principal part of the tth monthly payment is

$$\begin{aligned} P_t &\equiv \mathrm{RB}_{t-1} - \mathrm{RB}_t \\ &= \mathcal{O} \times \frac{(r/m)(1+r/m)^{t-1}}{(1+r/m)^n - 1}. \end{aligned} \tag{134}$$

P&I Analysis (concluded)

- The scheduled P&I payment at month t, or $P_t + I_t$, is

$$
\begin{aligned}
& (\mathrm{RB}_{t-1} - \mathrm{RB}_t) + \mathrm{RB}_{t-1} \times \frac{r}{m} \\
= \quad & \mathcal{O} \times \left[\frac{(r/m)(1+r/m)^n}{(1+r/m)^n - 1} \right], \qquad (135)
\end{aligned}
$$

  indeed a level pay independent of t.

- The term within the brackets, called the payment factor or annuity factor, is the monthly payment for each dollar of mortgage.

An Example

- The mortgage on pp. 40ff has a monthly payment of

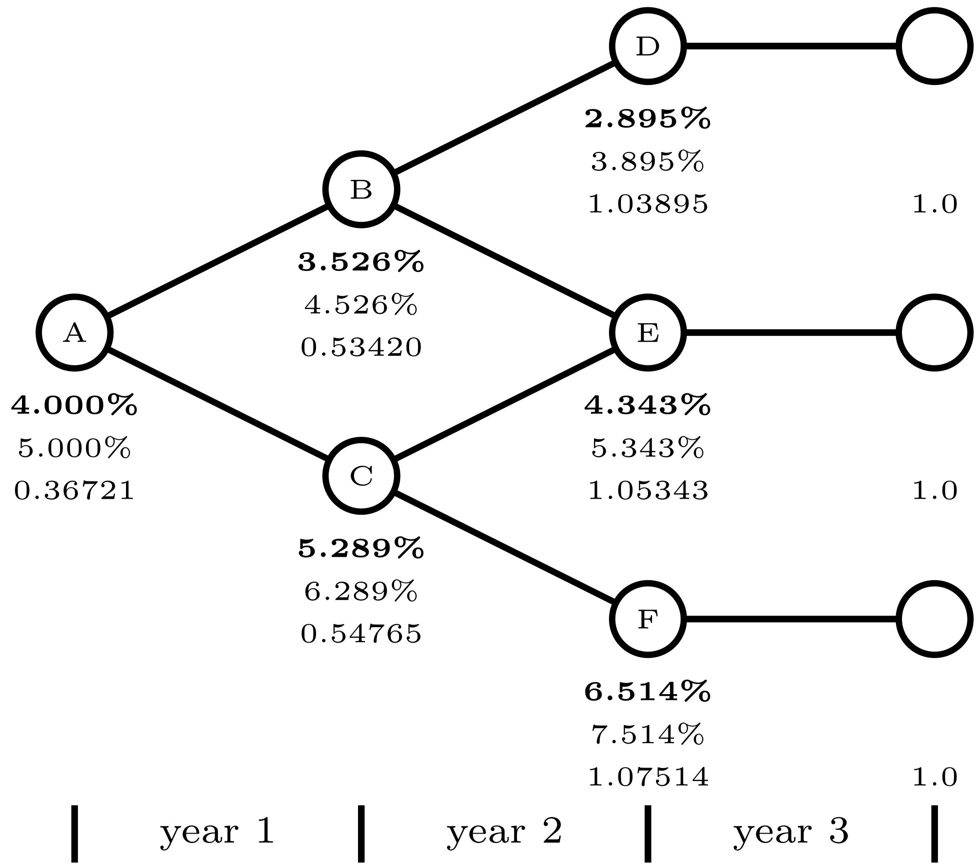
$$250000 \times \frac{(0.08/12) \times (1 + (0.08/12))^{180}}{(1 + (0.08/12))^{180} - 1} = 2389.13$$

by Eq. (135) on p. 1104.

- This number agrees with the number derived earlier.

Pricing Adjustable-Rate Mortgages

- We turn to ARM pricing as an interesting application of derivatives pricing and the analysis above.
- Consider a 3-year ARM with an interest rate that is 1% above the 1-year T-bill rate at the beginning of the year.
- This 1% is called the margin.
- Assume this ARM carries annual, not monthly, payments.
- The T-bill rates follow the binomial process, in boldface, on p. 1107, and the risk-neutral probability is 0.5.



Stacked at each node are the T-bill rate, the mortgage rate, and the payment factor for a mortgage initiated at that node and ending at year 3 (based on the mortgage rate at the same node). The short rates are from p. 861.

Pricing Adjustable-Rate Mortgages (continued)

- How much is the ARM worth to the issuer?
- Each new coupon rate at the reset date determines the level mortgage payment for the months until the next reset date as if the ARM were a fixed-rate loan with the new coupon rate and a maturity equal to that of the ARM.
- For example, for the interest rate tree on p. 1107, the scenario $\text{A} \rightarrow \text{B} \rightarrow \text{E}$ will leave our three-year ARM with a remaining principal at the end of the second year different from that under the scenario $\text{A} \rightarrow \text{C} \rightarrow \text{E}$.

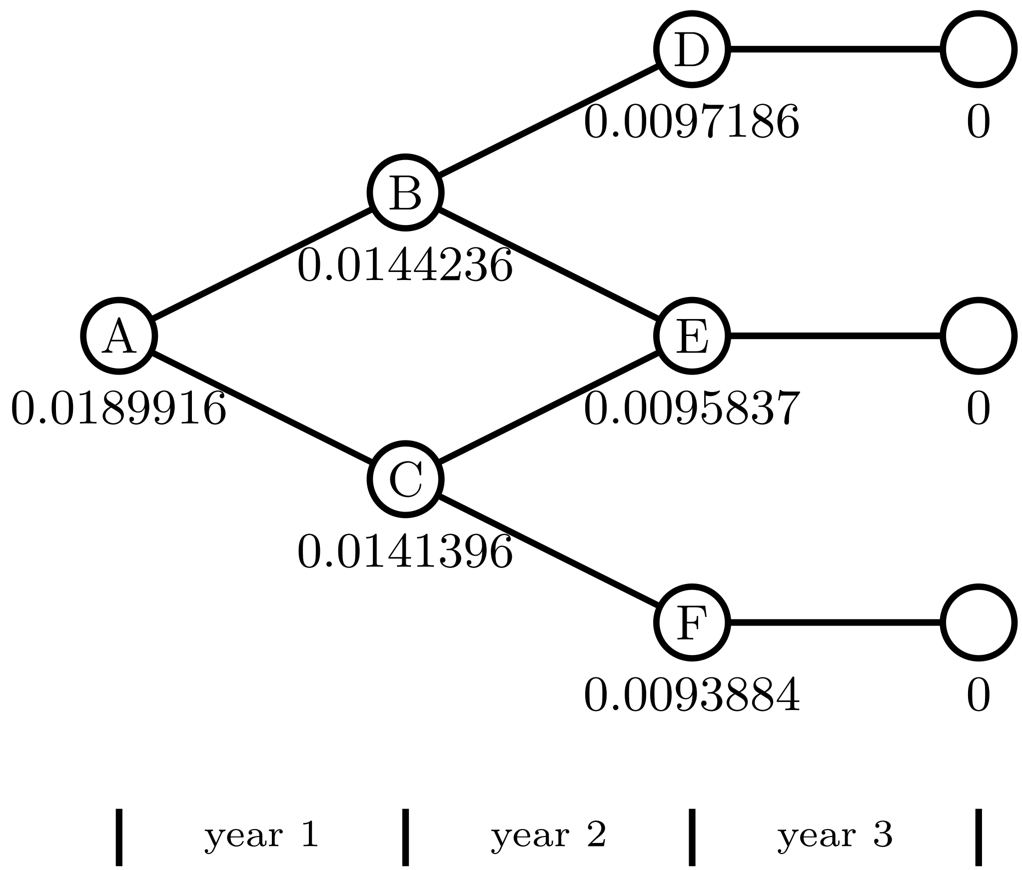
Pricing Adjustable-Rate Mortgages (continued)

- This path dependency calls for care in algorithmic design to avoid exponential complexity.
- Attach to each node on the binomial tree the annual payment per \$1 of principal for a mortgage initiated at that node and ending at year 3.
  - In other words, the payment factor.
- At node B, for example, the annual payment factor can be calculated by Eq. (135) on p. 1104 with $r = 0.04526$, $m = 1$, and $n = 2$ as

$$\frac{0.04526 \times (1.04526)^2}{(1.04526)^2 - 1} = 0.53420.$$

Pricing Adjustable-Rate Mortgages (continued)

- The payment factors for other nodes on p. 1107 are calculated in the same manner.
- We now apply backward induction to price the ARM (see p. 1111).
- At each node on the tree, the net value of an ARM of value \$1 initiated at that node and ending at the end of the third year is calculated.
- For example, the value is zero at terminal nodes since the ARM is immediately repaid.


D
0.0097186
0
B
0.0144236
A
0.0189916
E
0.0095837
0
C
0.0141396
F
0.0093884
0
year 1
year 2
year 3

Pricing Adjustable-Rate Mortgages (continued)

- At node D, the value is

$$\frac{1.03895}{1.02895} - 1 = 0.0097186,$$

which is simply the net present value of the payment 1.03895 next year.

  - Recall that the issuer makes a loan of \$1 at D.

- The values at nodes E and F can be computed similarly.

Pricing Adjustable-Rate Mortgages (continued)

- At node B, we first figure out the remaining principal balance after the payment one year hence as

$$1 - (0.53420 - 0.04526) = 0.51106,$$

because \$0.04526 of the payment of \$0.53426 constitutes the interest.

- The issuer will receive \$0.01 above the T-bill rate next year, and the value of the ARM is either \$0.0097186 or \$0.0095837 per \$1, each with probability 0.5.

Pricing Adjustable-Rate Mortgages (continued)

- The ARM's value at node B thus equals

$$\frac{0.51106 \times (0.0097186 + 0.0095837)/2 + 0.01}{1.03526} = 0.0144236.$$

- The values at nodes C and A can be calculated similarly as

$$\begin{aligned} & \frac{(1-(0.54765-0.06289)) \times (0.0095837+0.0093884)/2+0.01}{1.05289} \\ = \quad & 0.0141396 \\ & \\ & \frac{(1-(0.36721-0.05)) \times (0.0144236+0.0141396)/2+0.01}{1.04} \\ = \quad & 0.0189916, \end{aligned}$$

respectively.

Pricing Adjustable-Rate Mortgages (concluded)

- The value of the ARM to the issuer is hence \$0.0189916 per \$1 of loan amount.
- The above idea of scaling has wide applicability in pricing certain classes of path-dependent securities.

More on ARMs

- ARMs are indexed to publicly available indices such as:
  - LIBOR
  - The constant maturity Treasury rate (CMT)
  - The Cost of Funds Index (COFI).
- COFI is based on an average cost of funds.
- So it moves relatively sluggishly compared with LIBOR.
- Since 1990, the need for securitization gradually shift in LIBOR's favor.[a]

[a] See Morgenson (2012). The LIBOR rate-fixing scandal broke in June 2012.

More on ARMs (continued)

- If the ARM coupon reflects fully and instantaneously current market rates, then the ARM security will be priced close to par and refinancings rarely occur.
- In reality, adjustments are imperfect in many ways.
- At the reset date, a margin is added to the benchmark index to determine the new coupon.

More on ARMs (concluded)

- ARMs often have periodic rate caps that limit the amount by which the coupon rate may increase or decrease at the reset date.
- They also have lifetime caps and floors.
- To attract borrowers, mortgage lenders usually offer a below-market initial rate (the "teaser" rate).
- The reset interval, the time period between adjustments in the ARM coupon rate, is often annual, which is not frequent enough.
- But these terms are easy to incorporate into the pricing algorithm.

Expressing Prepayment Speeds

- The cash flow of a mortgage derivative is determined from that of the mortgage pool.
- The single most important factor complicating this endeavor is the unpredictability of prepayments.
- Recall that prepayment represents the principal payment made in excess of the scheduled principal amortization.

Expressing Prepayment Speeds (concluded)

- Compare the amortization factor Bal_t of the pool with the reported factor to determine if prepayments have occurred.
- The amount by which the reported factor exceeds the amortization factor is the prepayment amount.

Single Monthly Mortality

- A SMM of ω means $\omega\%$ of the scheduled remaining balance at the end of the month will prepay (recall p. 1074).
- In other words, the SMM is the percentage of the remaining balance that prepays for the month.
- Suppose the remaining principal balance of an MBS at the beginning of a month is \$50,000, the SMM is 0.5%, and the scheduled principal payment is \$70.
- Then the prepayment for the month is

$$0.005 \times (50{,}000 - 70) \approx 250$$

dollars.

Single Monthly Mortality (concluded)

- If the same monthly prepayment speed s is maintained since the issuance of the pool, the remaining principal balance at month i will be $\text{RB}_i \times (1 - s/100)^i$.
- It goes without saying that prepayment speeds must lie between 0% and 100%.

An Example

- Take the mortgage on p. 1102.
- Its amortization factor at the 54th month is 0.824866.
- If the actual factor is 0.8, then the (implied) SMM for the initial period of 54 months is

$$100 \times \left[1 - \left(\frac{0.8}{0.824866} \right)^{1/54} \right] = 0.0566677.$$

- In other words, roughly 0.057% of the remaining principal is prepaid per month.

Conditional Prepayment Rate

- The conditional prepayment rate (CPR) is the annualized equivalent of a SMM,

$$\mathrm{CPR} = 100 \times \left[1 - \left(1 - \frac{\mathrm{SMM}}{100} \right)^{12} \right].$$

- Conversely,

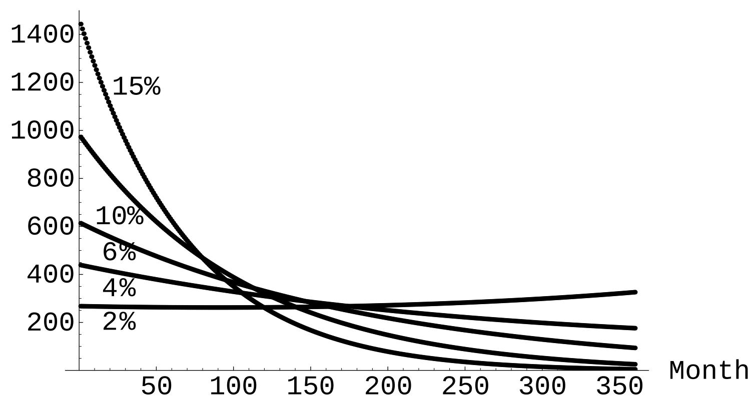
$$\mathrm{SMM} = 100 \times \left[1 - \left(1 - \frac{\mathrm{CPR}}{100} \right)^{1/12} \right].$$

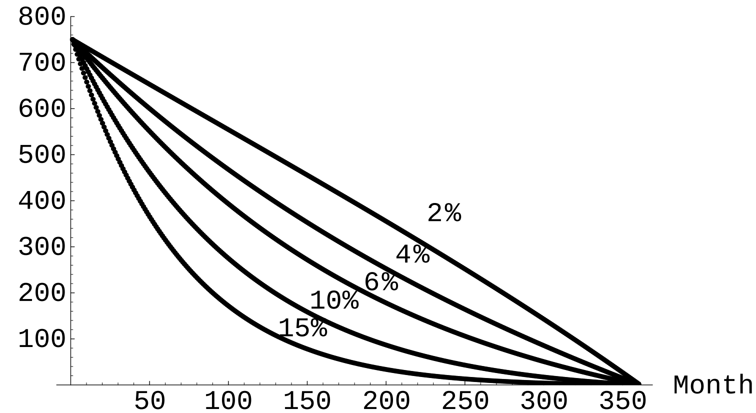
Conditional Prepayment Rate (concluded)

- For example, the SMM of 0.0566677 on p. 1123 is equivalent to a CPR of

$$100 \times \left[1 - \left(1 - \left(\frac{0.0566677}{100} \right)^{12} \right) \right] = 0.677897.$$

- Roughly 0.68% of the remaining principal is prepaid annually.
- The figures on 1126 plot the principal and interest cash flows under various prepayment speeds.
- Observe that with accelerated prepayments, the principal cash flow is shifted forward in time.





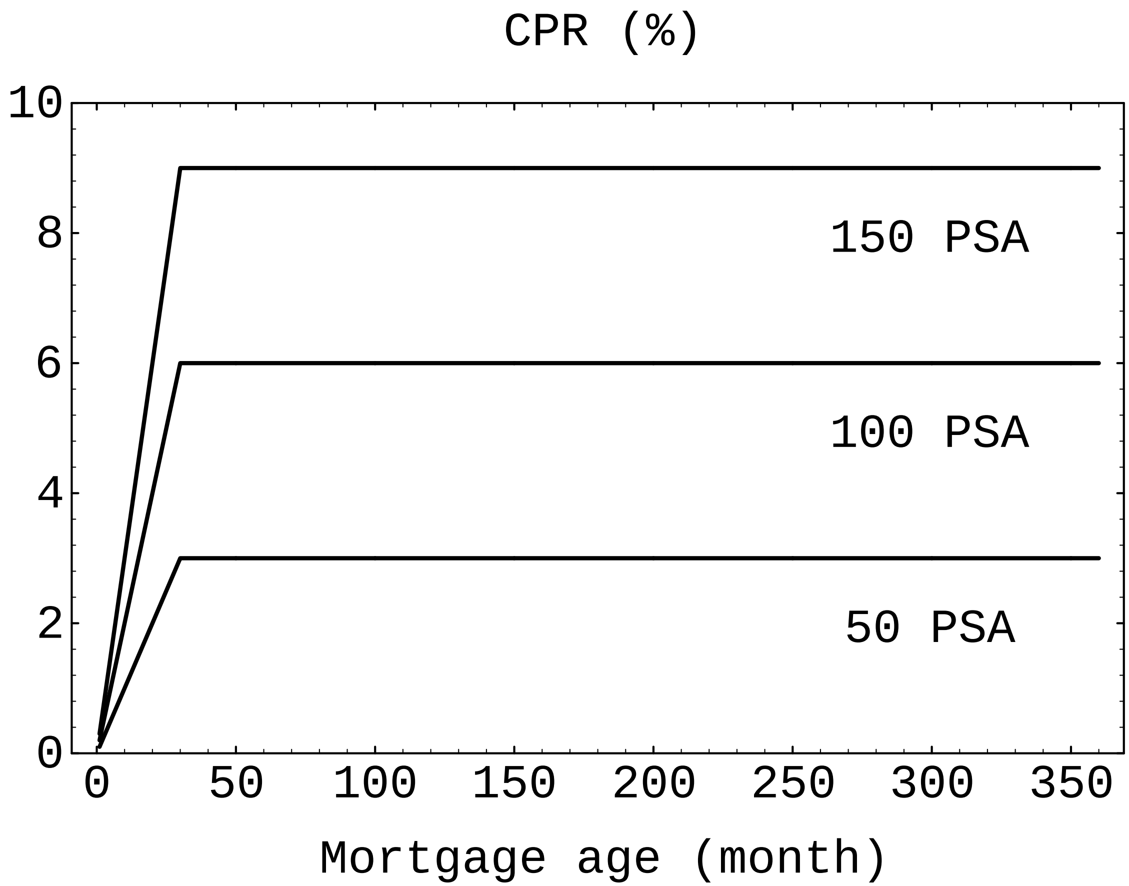
Principal (left) and interest (right) cash flows at various CPRs. The 6% mortgage has 30 years to maturity and an original loan amount of \$100,000.

PSA

- In 1985 the Public Securities Association (PSA) standardized a prepayment model.
- The PSA standard is expressed as a monthly series of CPRs.
  - It reflects the increase in CPR that occurs as the pool seasons.
- At the time the PSA proposed its standard, a seasoned 30-year GNMA's typical prepayment speed was $\sim 6\%$ CPR.

PSA (continued)

- The PSA standard postulates the following prepayment speeds:
  - The CPR is 0.2% for the first month.
  - It increases thereafter by 0.2% per month until it reaches 6% per year for the 30th month.
  - It then stays at 6% for the remaining years.
- The PSA benchmark is also referred to as 100 PSA.
- Other speeds are expressed as some percentage of PSA.
  - 50 PSA means one-half the PSA CPRs.
  - 150 PSA means one-and-a-half the PSA CPRs.


CPR (%)
150 PSA
100 PSA
50 PSA
0
2
4
6
8
10
0
50
100
150
200
250
300
350
Mortgage age (month)

PSA (concluded)

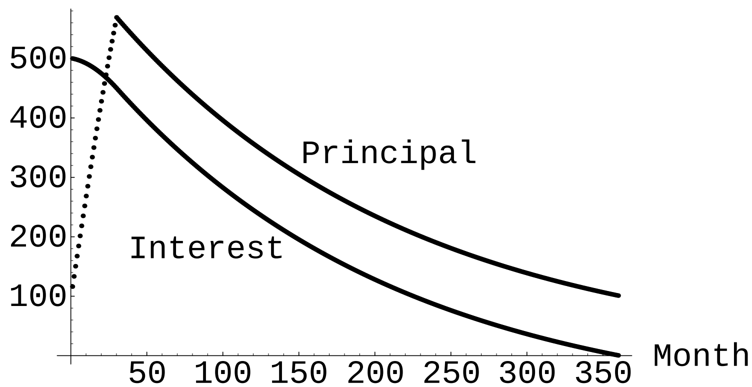
- Mathematically,

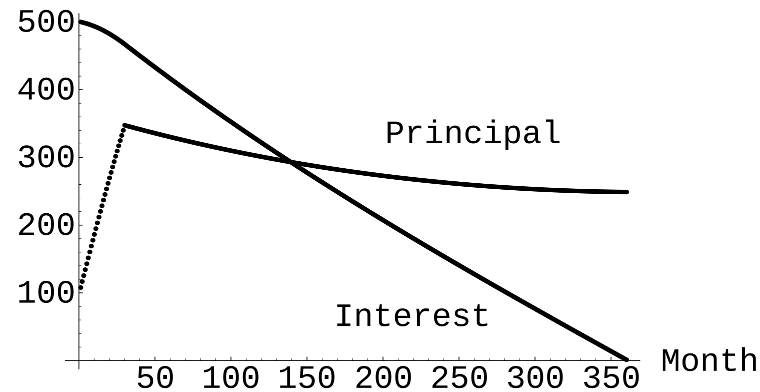
$$\text{CPR} = \begin{cases} 6\% \times \frac{\text{PSA}}{100} & \text{if the pool age exceeds 30 months} \\ \\ 0.2\% \times m \times \frac{\text{PSA}}{100} & \text{if the pool age } m \leq 30 \text{ months} \end{cases}$$

- Conversely,

$$\text{PSA} = \begin{cases} 100 \times \frac{\text{CPR}}{6} & \text{if the pool age exceeds 30 months} \\ \\ 100 \times \frac{\text{CPR}}{0.2 \times m} & \text{if the pool age } m \leq 30 \text{ months} \end{cases}$$

Cash Flows at 50 and 100 PSAs





The 6% mortgage has 30 years to maturity and an original loan amount of \$100,000. The 100 PSA scenario is on the left, and the 50 PSA is on the right.

Prepayment Vector

- The PSA tries to capture how prepayments vary with age.
- But it should be viewed as a market convention rather than a model.
- A vector of PSAs generated by a prepayment model should be used to describe the monthly prepayment speed through time.
- The monthly cash flows can be derived thereof.

Prepayment Vector (continued)

- Similarly, the CPR should be seen purely as a measure of speed rather than a model.
- If one treats a single CPR number as the true prepayment speed, that number will be called the constant prepayment rate.
- This simple model crashes with the empirical fact that pools with new production loans typically prepay at a slower rate than seasoned pools.
- A vector of CPRs should be preferred.

Prepayment Vector (concluded)

- A CPR/SMM vector is easier to work with than a PSA vector because of the lack of dependence on the pool age.
- But they are all equivalent as a CPR vector can always be converted into an equivalent PSA vector and vice versa.

Cash Flow Generation

- Each cash flow is composed of the principal payment, the interest payment, and the principal prepayment.
- Let B_k denote the actual remaining principal balance at month k.
- The pool's actual remaining principal balance at time $i-1$ is B_{i-1}.

Cash Flow Generation (continued)

- The principal and interest payments at time i are

$$\overline{P_i} \equiv B_{i-1}\left(\frac{\text{Bal}_{i-1}-\text{Bal}_i}{\text{Bal}_{i-1}}\right) \tag{136}$$

$$= B_{i-1}\,\frac{r/m}{(1+r/m)^{n-i+1}-1} \tag{137}$$

$$\overline{I_i} \equiv B_{i-1}\,\frac{r-\alpha}{m} \tag{138}$$

  – α is the servicing spread (or servicing fee rate), which consists of the servicing fee for the servicer as well as the guarantee fee.

Cash Flow Generation (continued)

- The prepayment at time i is

$$\mathrm{PP}_i = B_{i-1}\,\frac{\mathrm{Bal}_i}{\mathrm{Bal}_{i-1}} \times \mathrm{SMM}_i.$$

  – SMM_i is the prepayment speed for month i.

- If the total principal payment from the pool is $\overline{P_i} + \mathrm{PP}_i$, the remaining principal balance is

$$\begin{aligned}
B_i &= B_{i-1} - \overline{P_i} - \mathrm{PP}_i \\
&= B_{i-1}\left[1 - \left(\frac{\mathrm{Bal}_{i-1} - \mathrm{Bal}_i}{\mathrm{Bal}_{i-1}}\right) - \frac{\mathrm{Bal}_i}{\mathrm{Bal}_{i-1}} \times \mathrm{SMM}_i\right] \\
&= \frac{B_{i-1} \times \mathrm{Bal}_i \times (1 - \mathrm{SMM}_i)}{\mathrm{Bal}_{i-1}}. \qquad (139)
\end{aligned}$$

Cash Flow Generation (continued)

- Equation (139) can be applied iteratively to yield[a]

$$B_i = \mathrm{RB}_i \times \prod_{j=1}^{i} (1 - \mathrm{SMM}_j). \tag{140}$$

- Define

$$b_i \equiv \prod_{j=1}^{i} (1 - \mathrm{SMM}_j).$$

[a] RB_i is defined on p. 1100.

Cash Flow Generation (continued)

- Then the scheduled P&I is[a]

$$\overline{P_i} = b_{i-1} P_i \quad \text{and} \quad \overline{I_i} = b_{i-1} I'_i. \tag{141}$$

  - $I'_i \equiv \mathrm{RB}_{i-1} \times (r - \alpha)/m$ is the scheduled interest payment.

- The scheduled cash flow and the b_i determined by the prepayment vector are all that are needed to calculate the projected actual cash flows.

[a] P_i and I_i are defined on p. 1103.

Cash Flow Generation (concluded)

- If the servicing fees do not exist (that is, $\alpha = 0$), the projected monthly payment *before* prepayment at month i becomes

$$\overline{P_i} + \overline{I_i} = b_{i-1}(P_i + I_i) = b_{i-1}C. \tag{142}$$

  – C is the scheduled monthly payment on the original principal.

- See Figure 29.10 in the text for a linear-time algorithm for generating the mortgage pool's cash flow.

Cash Flows of Sequential-Pay CMOs

- Take a 3-tranche sequential-pay CMO backed by \$3,000,000 of mortgages with a 12% coupon and 6 months to maturity.
- The 3 tranches are called A, B, and Z.
- All three tranches carry the same coupon rate of 12%.

Cash Flows of Sequential-Pay CMOs (continued)

- The Z tranche consists of Z bonds.
  - A Z bond receives no payments until all previous tranches are retired.
  - Although a Z bond carries an explicit coupon rate, the owed interest is accrued and added to the principal balance of that tranche.
  - The Z bond thus protects earlier tranches from extension risk
- When a Z bond starts receiving cash payments, it becomes a pass-through instrument.

Cash Flows of Sequential-Pay CMOs (continued)

- The Z tranche's coupon cash flows are initially used to pay down the tranches preceding it.
- Its existence (as in the ABZ structure here) accelerates the principal repayments of the sequential-pay bonds.
- Assume the ensuing monthly interest rates are 1%, 0.9%, 1.1%, 1.2%, 1.1%, 1.0%.
- Assume that the SMMs are 5%, 6%, 5%, 4%, 5%, 6%.
- We want to calculate the cash flow and the then fair price of each tranche.

Cash Flows of Sequential-Pay CMOs (continued)

- Compute the pool's cash flow by invoking the algorithm in Figure 29.10 in the text.
  - $n = 6$, $r = 0.01$, and $\text{SMM} = [\,0.05, 0.06, 0.05, 0.04, 0.05, 0.06\,]$.
- Individual tranches' cash flows and remaining principals thereof can be derived by allocating the pool's principal and interest cash flows based on the CMO structure.
- See the next table for the breakdown.

Month		1	2	3	4	5	6
Interest rate		1.0%	0.9%	1.1%	1.2%	1.1%	1.0%
SMM		5.0%	6.0%	5.0%	4.0%	5.0%	6.0%
Remaining principal (B_i)							
	3,000,000	2,386,737	1,803,711	1,291,516	830,675	396,533	0
A	1,000,000	376,737	0	0	0	0	0
B	1,000,000	1,000,000	783,611	261,215	0	0	0
Z	1,000,000	1,010,000	1,020,100	1,030,301	830,675	396,533	0
Interest ($\overline{I_i}$)		30,000	23,867	18,037	12,915	8,307	3,965
A		20,000	3,767	0	0	0	0
B		10,000	20,100	18,037	2,612	0	0
Z		0	0	0	10,303	8,307	3,965
Principal		613,263	583,026	512,195	460,841	434,142	396,534
A		613,263	376,737	0	0	0	0
B		0	206,289	512,195	261,215	0	0
Z		0	0	0	199,626	434,142	396,534

Cash Flows of Sequential-Pay CMOs (concluded)

- Note that the Z tranche's principal is growing at 1% per month until all previous tranches are retired.
- Before that time, the interest due the Z tranche is used to retire A's and B's principals.
- For example, the $10,000 interest due tranche Z at month one is directed to tranche A instead.
  - It reduces A's remaining principal from $386,737 by $10,000 to $376,737.
  - But it increases Z's from $1,000,000 to $1,010,000.
- At month four, the interest amount that goes into tranche Z, $10,303, is exactly what is required of Z's remaining principal of $1,030,301.

Pricing Sequential-Pay CMOs

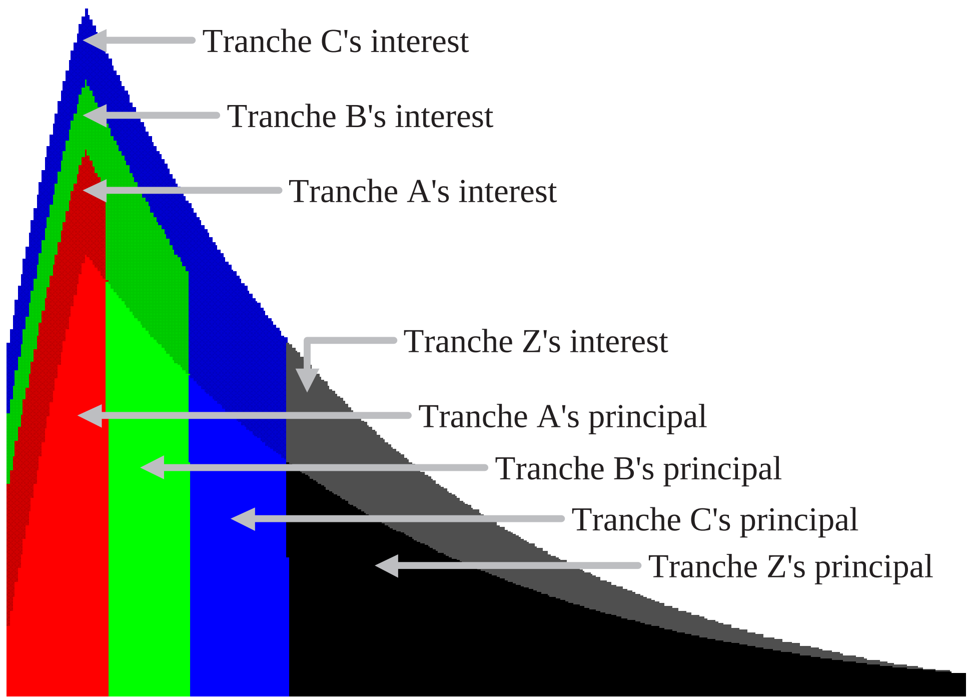
- We now price the tranches:

$$
\begin{aligned}
\text{tranche A} &= \frac{20000 + 613263}{1.01} + \frac{3767 + 376737}{1.01 \times 1.009} = 1000369, \\
\text{tranche B} &= \frac{10000 + 0}{1.01} + \frac{20100 + 206289}{1.01 \times 1.009} + \frac{18037 + 512195}{1.01 \times 1.009 \times 1.011} \\
&\quad + \frac{2612 + 261215}{1.01 \times 1.009 \times 1.011 \times 1.012} \\
&= 999719, \\
\text{tranche Z} &= \frac{10303 + 199626}{1.01 \times 1.009 \times 1.011 \times 1.012} \\
&\quad + \frac{8307 + 434142}{1.01 \times 1.009 \times 1.011 \times 1.012 \times 1.011} \\
&\quad + \frac{3965 + 396534}{1.01 \times 1.009 \times 1.011 \times 1.012 \times 1.011 \times 1.01} \\
&= 997238.
\end{aligned}
$$

Pricing Sequential-Pay CMOs (concluded)

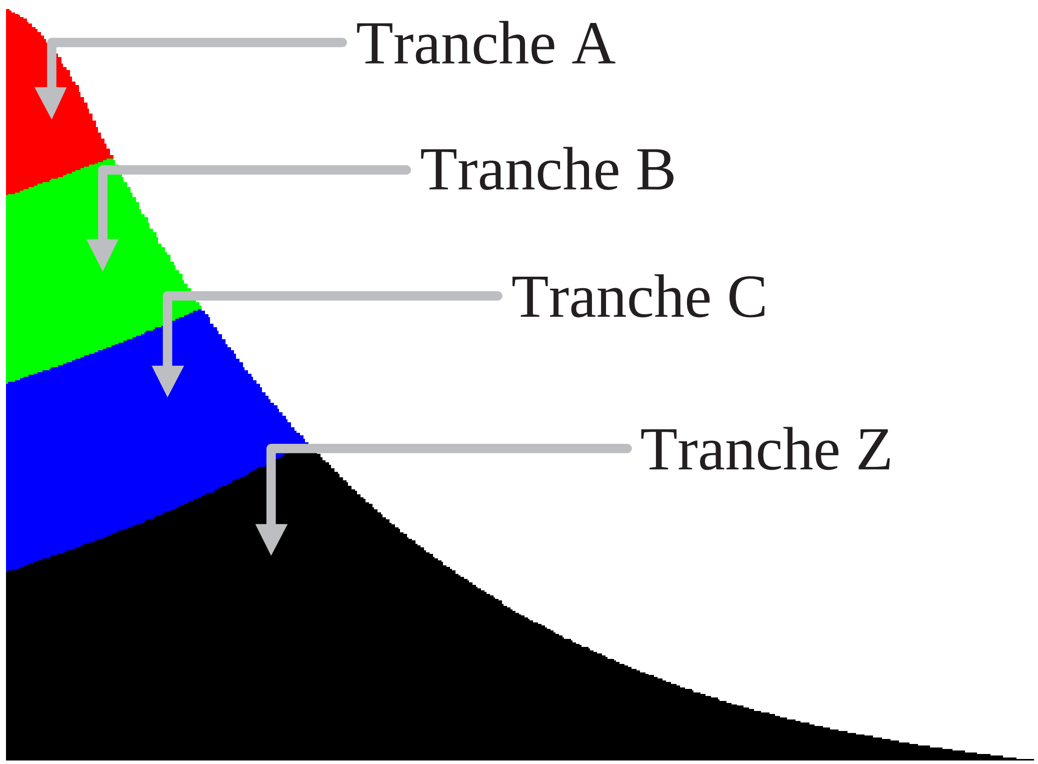
- This CMO has a total theoretical value of \$2,997,326.
- It is slightly less than its par value of \$3,000,000.
- See the algorithm in Figure 29.12 in the text for the cash flow generator.

A 4-Tranche Example: Cash Flows



The mortgage rate is 6%, the actual prepayment speed is 150 PSA, and each tranche has an identical original principal amount.

A 4-Tranche Example: Remaining Principals



Pricing Sequential-Pay CMOs: Methodology

- Suppose we have the interest rate path and the prepayment vector for that interest rate path.
- Then a CMO's cash flow can be calculated and the CMO priced.
- Unfortunately, the remaining principal of a CMO under prepayments is path dependent.
  - For example, a period of high rates before dropping to the current level is not likely to result in the same remaining principal as a period of low rates before rising to the current level.

Pricing Sequential-Pay CMOs: Methodology (concluded)

- If we try to price a 30-year CMO on a binomial interest rate model, there will be $2^{360} \approx 2.35 \times 10^{108}$ paths!
- Hence Monte Carlo simulation is the method of choice.
- First, one interest rate path is generated.
- Based on that path, the prepayment model is applied to generate the pool's principal, prepayment, and interest cash flows.
- Now, the cash flows of individual tranches can be generated and their present values derived.
- Repeat the above procedure over many interest rate scenarios and average the present values.

MBS Valuation Methodologies

1. Static cash flow yield.
2. Option modeling.
3. Option-adjusted spread (OAS).

Cash Flow Yield

- To price an MBS, one starts with its cash flow: The periodic P&I under a static prepayment assumption as given by a prepayment vector.
- The invoice price is now

$$\sum_{i=1}^{n} C_i/(1+r)^{\omega-1+i}.$$

  - C_i is the cash flow at time i.
  - n is the weighted average maturity (WAM).
  - r is the discount rate.
  - ω is the fraction of period from settlement until the first P&I payment date.

Cash Flow Yield (continued)

- The r that equates the above with the market price is called the (static) cash flow yield.
- The static cash flow yield methodology compares the cash flow yield on an MBS with that on "comparable" bonds.
- The implied PSA is the single PSA speed producing the same cash flow yield.[a]

[a] Fabozzi (1991).

Cash Flow Yield (concluded)

- This simple methodology has obvious weaknesses (some generic).
  - It is static.
  - The projected cash flow may not be reinvested at the cash flow yield.[a]
  - The MBS may not be held until the final payout date.
  - The actual prepayment behavior is likely to deviate from the assumptions.

[a]This deficiency can be remedied somewhat by adopting the static spread methodology on p. 117.

The Option Pricing Methodology

- Virtually all mortgage loans give the homeowner the right to prepay the mortgage at any time.
- The totality of these rights to prepay constitutes the embedded call option of the pass-through.
- In contrast, the MBS investor is short the embedded call.
- Therefore,

$$\begin{aligned} & \text{pass-through price} \\ = \; & \text{noncallable pass-through price} - \text{call option price.} \end{aligned}$$

The Option Pricing Methodology (continued)

- The option pricing methodology prices the call option by an option pricing model.
- It then estimates the market price of the noncallable pass-through by

$$\begin{aligned} & \text{noncallable pass-through price} \\ = \; & \text{pass-through price} + \text{call option price}. \end{aligned}$$

- The above price is finally used to compute the yield on this theoretical bond which does not prepay.
- This yield is called the option-adjusted yield.

The Option Pricing Methodology (continued)

- The option pricing methodology suffers from several difficulties (some generic).
  - The Black-Scholes model is not satisfactory for pricing fixed-income securities.[a]
  - There may not exist a benchmark to compare the option-adjusted yield with to obtain the yield spread.
  - This methodology does not incorporate the shape of the yield curve.

[a]See Section 24.7 of the textbook.

The Option Pricing Methodology (concluded)

- (continued)
  - Prepayment options are often "irrationally" exercised.
  - A partial exercise is possible as the homeowner can prepay a portion of the loan.
    - There is not one option but many, one per homeowner.
  - Valuation of the call option becomes very complicated for CMO bonds.

The OAS Methodology[a]

- The OAS methodology has four major components.
- The interest rate model is the first component.
- The second component is the prepayment model.[b]
  - Deterministic models that are accurate on average seem good enough for pass-throughs, IOs, and POs.[c]
- The OAS methodology is meant to identify investments with the best potential for excess returns.

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[a] The idea can be traced to Waldman and Modzelewski (1985), if not earlier.

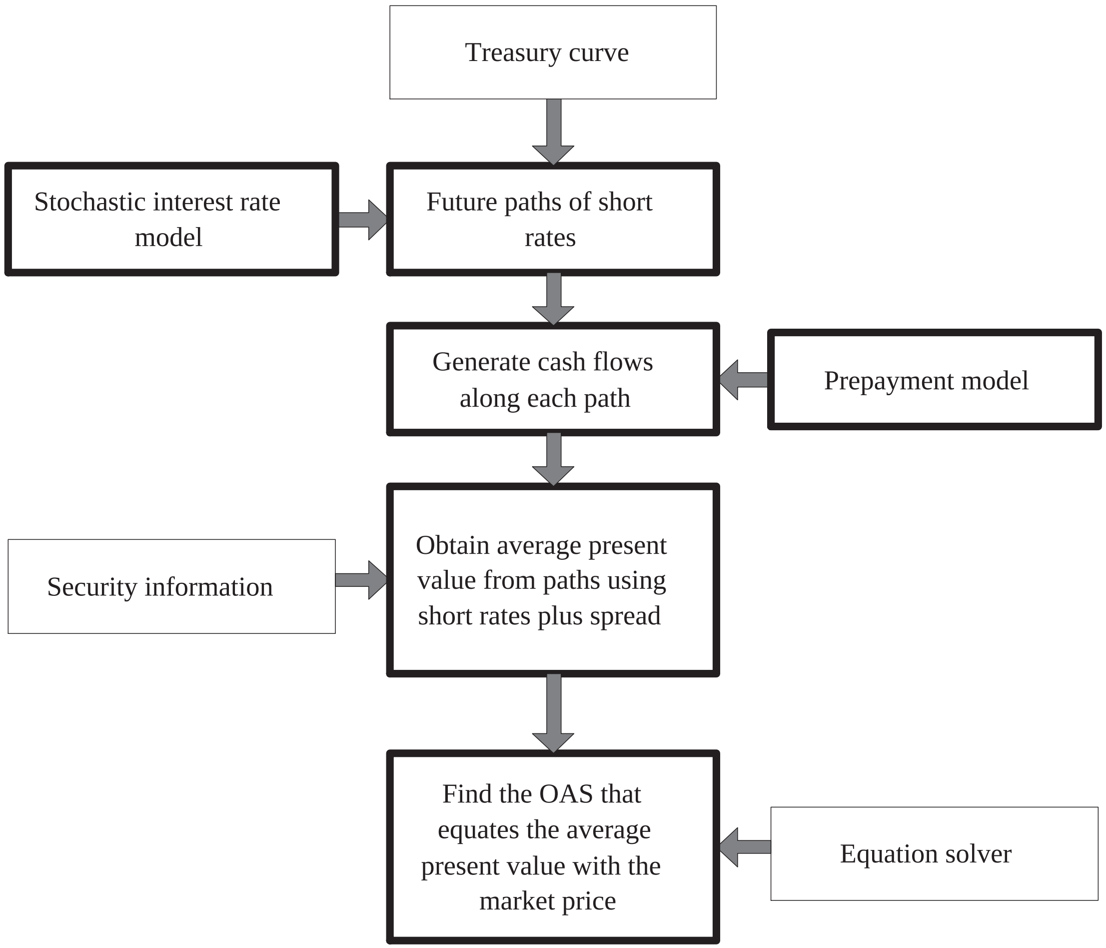
[b] May be the single most important component.

[c] Hayre (1997) and Hayre and Rajan (1995).

The OAS Methodology (continued)

- The cash flow generator is the third component.
  - It calculates the current coupon rates for the interest rate paths given by the interest rate model.
  - It then generates the P&I cash flows for the pool as well as allocating them for individual securities based on the prepayment model and security information.
  - Note that the pool cash flow drives many securities.
- Finally, the equation solver calculates the OAS.

The OAS Methodology (continued)



The OAS Methodology (continued)

- The general valuation formula for uncertain cash flows can be written as

$$\text{PV} = \lim_{N \to \infty} \frac{1}{N} \sum_{N \text{ paths } r^*} \sum_n \frac{C_n^*}{(1 + r_1^*)(1 + r_2^*) \cdots (1 + r_n^*)}. \tag{143}$$

  - r^* denotes a risk-neutral interest rate path for which r_i^* is the ith one-period rate.
  - C_n^* is the cash flow at time n under this scenario.

The OAS Methodology (continued)

- The Monte Carlo calculation of OAS is closely related to Eq. (143) on p. 1164.
- The interest rate model randomly produces a set of risk-neutral rate paths.
- The cash flow is then generated for each path.
- Finally, we solve for the spread s that makes the average discounted cash flow equal the market price,

$$
\begin{aligned}
& P \\
= \quad & \lim_{N \to \infty} \frac{1}{N} \sum_{N \text{ paths } r^*} \sum_n \frac{C_n^*}{(1 + r_1^* + s)(1 + r_2^* + s) \cdots (1 + r_n^* + s)}.
\end{aligned}
$$

- This spread s is the OAS.

The OAS Methodology (continued)

- A common alternative averages the cash flows first and then calculates the OAS as the spread that equates this average cash flow with the market price.
- Although this approach is more efficient, it will generally give a different spread.

The OAS Methodology (continued)

- OAS calculation is very time consuming.
- The majority of the cost lies in generating the cash flows.
- This is because CMOs can become arbitrarily complex in their rules for allocating the cash flows.
- Such complexity requires special care in software design.

The OAS Methodology (concluded)

- Although extremely popular, the OAS methodology suffers from several difficulties.
  - It may be difficult to interpret the OAS number.
  - The spread must be a constant.
  - It assumes the cash flow can be reinvested with the same OAS.

Collateralized Mortgage Obligations

Capital can be understood only as motion,
not as a thing at rest.
— Karl Marx (1818–1883)

CMOs

- The complexity of a CMO arises from layering different types of payment rules on a prioritized basis.
- In the first-generation CMOs, the sequential-pay CMOs, each class of bond would be retired sequentially.
- A sequential-pay CMO with a large number of tranches will have very narrow cash flow windows for the tranches.
- To further reduce prepayment risk, tranches with a principal repayment schedule were introduced.
- They are called scheduled bonds.

CMOs (continued)

- For example, bonds that guarantee the repayment schedule when the actual prepayment speed lies within a specified range are known as planned amortization class bonds (PACs).[a]
- PACs offer protection against both contraction and extension risks.
- But some investors may desire only protection from one of these risks.
- For them, a bond class known as the targeted amortization class (TAC) was created.

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[a] Introduced in August 1986.

CMOs (continued)

- Scheduled bonds expose certain CMO classes to less prepayment risk.
- However, this can occur only if the redirection in the prepayment risk is absorbed as much as possible by other classes known as the support bonds.
- Pro rata bonds provide another means of layering.
- Principal cash flows to these bonds are divided proportionally, but the bonds can have different interest payment rules.

CMOs (continued)

- Suppose the weighted average coupon (WAC) of the collateral is 10%, tranche B1 receives 40% of the principal, and tranche B2 receives 60% of the principal.
- Given this pro rata structure, many choices of interest payment rules are possible for B1 and B2 as long as the interest payments are nonnegative and the WAC does not exceed 10%.
- The coupon rates can even be floating.

CMOs (continued)

- One possibility is for B1 to have a coupon of 5% and for B2 to have a coupon of 13.33%.
- This works because

$$\frac{40}{100} \times 5\% + \frac{60}{100} \times 13.33\% = 10\%.$$

- Bonds with pass-through coupons that are higher and lower than the collateral coupon have thus been created.
- Bonds like B1 are called synthetic discount securities.
- Bonds like B2 are called synthetic premium securities.

CMOs (continued)

- An extreme case is for B1 to receive 99% of the principal and have a 5% coupon and for B2 to receive only 1% of the principal and have a 505% coupon.[a]
- IOs have either a nominal principal or a notional principal.
  - A nominal principal represents actual principal that will be paid.
  - It is called "nominal" because it is extremely small, resulting in an extremely high coupon rate.

[a]First-generation IOs issued by Fannie Mae took the form of B2 in July 1986.

CMOs (concluded)

- A case in point is the B2 class with a 505% coupon above.
- A notional principal, in contrast, is the amount on which interest is calculated.
- An IO holder owns none of the notional principal.
- Once the notional principal amount declines to zero, no further payments are made on the IO.

Floating-Rate Tranches[a]

- A form of pro rata bonds are floaters and inverse floaters whose combined coupon does not exceed the collateral coupon.
- A floater is a class whose coupon rate varies directly with the change in the reference rate.
- An inverse floater is a class whose coupon rate changes in the direction opposite to the change in the reference rate.
- When the coupon on the inverse floater changes by x times the amount of the change in the reference rate, this multiple x is called its slope.

[a] Created in September 1986.

Floating-Rate Tranches (continued)

- Because the interest comes from fixed-rate mortgages, floaters must have a coupon cap.
- Similarly, inverse floaters must have a coupon floor.
- Suppose the floater has a principal of $P_{\rm f}$ and the inverse floater has a principal of $P_{\rm i}$.
- Define

$$\begin{aligned} \omega_{\rm f} &\equiv P_{\rm f}/(P_{\rm f}+P_{\rm i}), \\ \omega_{\rm i} &\equiv P_{\rm i}/(P_{\rm f}+P_{\rm i}). \end{aligned}$$

Floating-Rate Tranches (concluded)

- The coupon rates of the floater, $c_{\rm f}$, and the inverse floater, $c_{\rm i}$, must satisfy $\omega_{\rm f} \times c_{\rm f} + \omega_{\rm i} \times c_{\rm i} = \text{WAC}$, or

$$c_{\rm i} = \frac{\text{WAC} - \omega_{\rm f} \times c_{\rm f}}{\omega_{\rm i}}.$$

- The slope is clearly $\omega_{\rm f}/\omega_{\rm i}$.
- To make sure that the inverse floater will not encounter a negative coupon, the cap on the floater must be less than $\text{WAC}/\omega_{\rm f}$.
- In fact, caps and floors are related via

$$\textsf{floor} = \frac{\text{WAC} - \omega_{\rm f} \times \textsf{cap}}{\omega_{\rm i}}.$$

An Example

- Take a CMO deal that includes a floater with a principal of \$64 million and an inverse floater with a principal of \$16 million.
- The coupon rate for the floating-rate class is $\text{LIBOR} + 0.65$.
- The coupon rate for the inverse floater is $42.4 - 4 \times \text{LIBOR}$.
- The slope is thus four.

An Example (concluded)

- The WAC of the two classes is

$$\frac{64}{80} \times \textsf{floater coupon rate} + \frac{16}{80} \times \textsf{inverse floater coupon rate} = 9\%$$

regardless of the level of the LIBOR.

- Consequently, the coupon rate on the underlying collateral, 9%, can support the aggregate interest payments that must be made to these two classes.

- If we set a floor of 0% for the inverse floater, the cap on the floater is 11.25%.

Superfloaters

- A variant of the floating-rate CMO is the superfloater introduced in 1987.
- In a conventional floating-rate class, the coupon rate moves up or down on a one-to-one basis with the reference rate.
- A superfloater's coupon rate, in comparison, changes by some multiple of the change in the reference rate.
  - It magnifies any changes in the value of the reference rate.
- Superfloater tranches are bearish because their value generally appreciates with rising interest rates.

An Example

- Suppose the initial LIBOR is 7% and the coupon rate for a superfloater is set by

$$(\text{initial LIBOR} - 40 \text{ basis points}) + 2 \times (\text{change in LIBOR}).$$

- The following table shows how the superfloater changes its coupon rate as LIBOR changes.
  - The coupon rates for a conventional floater of LIBOR plus 50 basis points are also listed for comparison.

LIBOR change (basis points)	−300	−200	−100	0	+100	+200	+300
Superfloater	0.6	2.6	4.6	6.6	8.6	10.6	12.6
Conventional floater	4.5	5.5	6.5	7.5	8.5	9.5	10.5

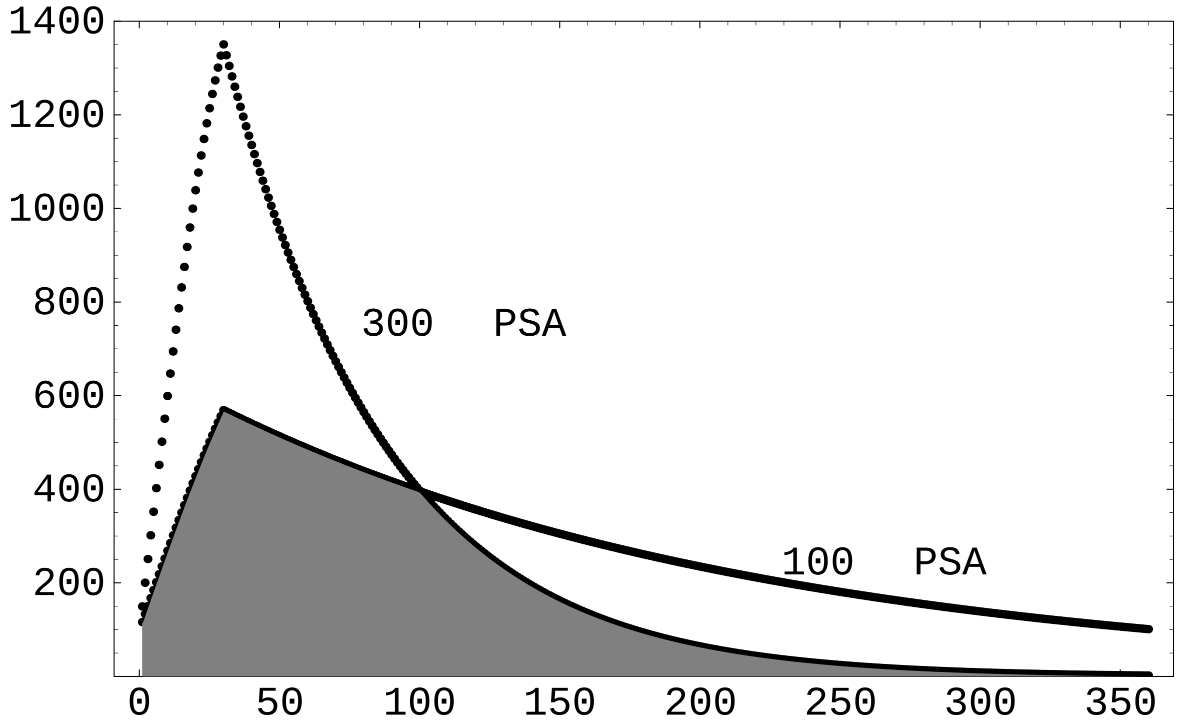
An Example (concluded)

- A superfloater provides a much higher yield than a conventional floater when interest rates rise.
- It provides a much lower yield when interest rates fall or remain stable.
- This can be verified by looking at the above table via spreads in basis points to the LIBOR in the next table.

LIBOR change (basis points)	−300	−200	−100	0	+100	+200	+300
Superfloater	−340	−240	−140	−40	60	160	260
Conventional floater	50	50	50	50	50	50	50

PAC Bonds

- PAC bonds are created by calculating the cash flows from the collateral by use of two prepayment speeds: a fast one and a slow one.
- Consider a PAC band of 100 PSA (the lower collar) to 300 PSA (the upper collar).
- The plot on p. 1187 shows the principal payments at the two collars.
- The principal payments under the higher-speed scenario are higher in the earlier years but lower in later years.



The underlying mortgages are 30-year ones with a total original loan amount of \$100,000,000 (the numbers on the y axis are in thousands) and a coupon rate of 6%.

PAC Bonds (continued)

- The shaded area represents the principal payment schedule that is "guaranteed" for every possible prepayment speed between 100% and 300% PSAs.
- It is calculated by taking the minimum of the principal paydowns at the lower and upper collars.
- This schedule is called the PAC schedule.
- See Figure 30.2 in the text for a linear-time cash flow generator.

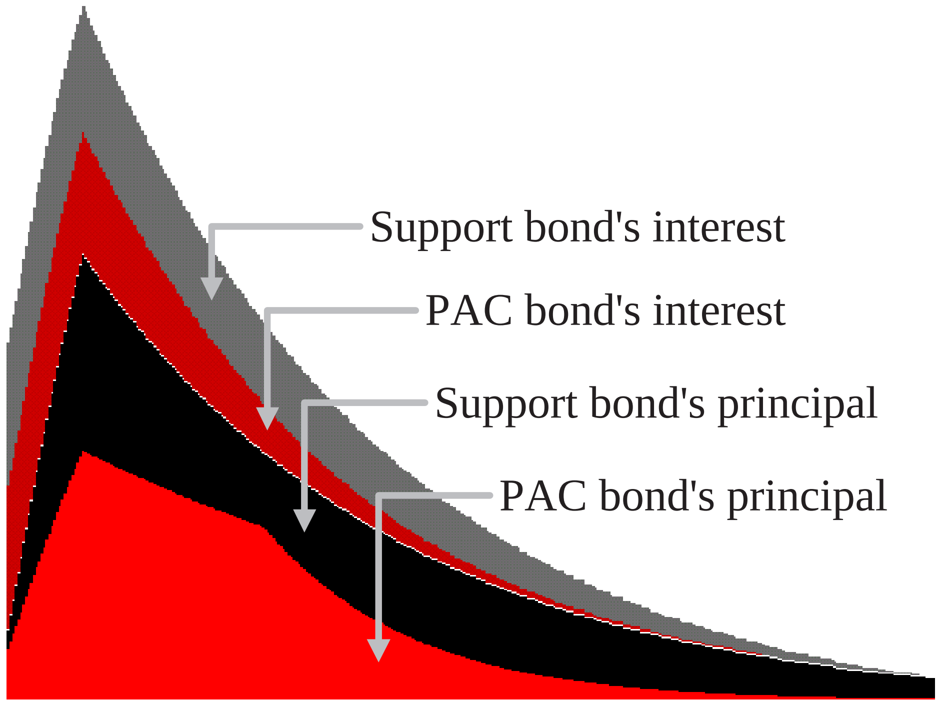
PAC Bonds (continued)

- Adherence to the amortization schedule of the PAC takes priority over those of all other bonds.
- The cash flow of a PAC bond is therefore known as long as its support bonds are not fully paid off.
- Whether this happens depends to a large extent on the CMO structure, such as priority and the relative sizes of PAC and non-PAC classes.
- For example, a relatively small PAC is harder to break than a larger PAC, other things being equal.

PAC Bonds (continued)

- If the actual prepayment speed is 150 PSA, the principal payment pattern of the PAC bond adheres to the PAC schedule.
- The cash flows of the support bond "flow around" the PAC bond (see the plot on p. 1191).
- The cash flows are neither sequential nor pro rata.
- In fact, the support bond pays down *simultaneously* with the PAC bond.
- Because more than one class of bonds may be receiving principal payments at the same time, structures with PAC bonds are simultaneous-pay CMOs.

Cash Flow of a PAC Bond at 150 PSA



The mortgage rate is 6%, the PAC band is 100 PSA to 300 PSA, and the actual prepayment speed is 150 PSA.

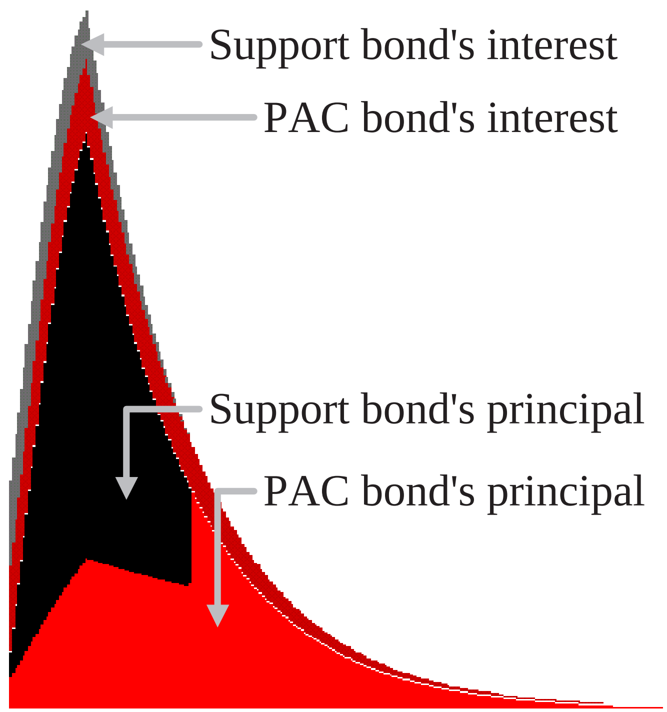
PAC Bonds (continued)

- At the lower prepayment speed of 100 PSA, far less principal cash flow is available in the early years of the CMO.
- As all the principal cash flows go to the PAC bond in the early years, the principal payments on the support bond are deferred and the support bond extends.

PAC Bonds (continued)

- If prepayments move outside the PAC band, the PAC schedule may not be met.
- At 400 PSA, for example, the cash flows to the support bond are accelerated.
- After the support bond is fully paid off, all remaining principal payments go to the PAC bond; its life is shortened.
- See the plot on p. 1194.

Cash Flow of a PAC Bond at 400 PSA



The mortgage rate is 6%, the PAC band is 100 PSA to 300 PSA, and the actual prepayment speed is 400 PSA.

PAC Bonds (concluded)

- The support bond thus absorbs part of the contraction risk.
- Similarly, should the actual prepayment speed fall below the lower collar, then in subsequent periods the PAC bond has priority on the principal payments.
- This reduces the extension risk, which is again absorbed by the support bond.

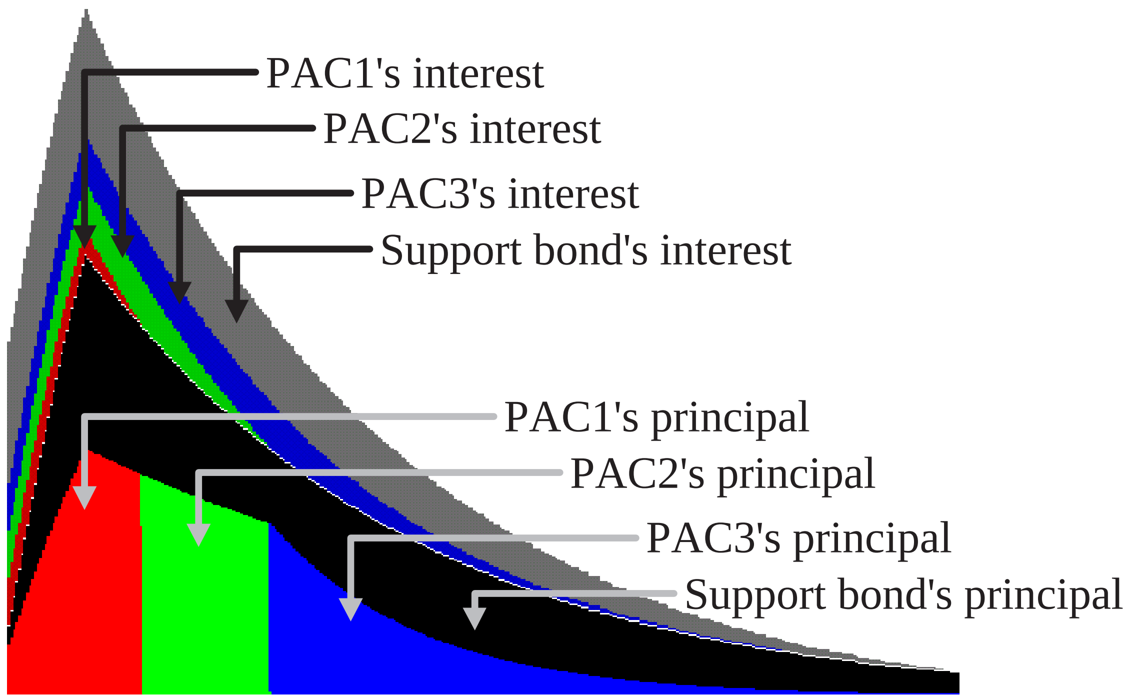
PAC Drift

- The PAC band guarantees that if prepayments occur at any single constant speed within the band *and* stay there, the PAC schedule will be met.
- However, the PAC schedule may not be met even if prepayments on the collateral always vary within the band over time.
- This is because the band that guarantees the original PAC schedule can expand and contract, depending on actual prepayments.
- This phenomenon is known as PAC drift.

Sequential PACs

- PACs can be divided sequentially to provide narrower paydown structures.
- These sequential PACs narrow the range of years over which principal payments occur.
- See the plot on p. 1198.
- Although these bonds are all structured with the same band, the actual range of speeds over which their schedules will be met may differ.

Cash Flow of a Sequential PAC Bond



The mortgage rate is 6%, the PAC band is 100 PSA to 300 PSA, and the actual prepayment speed is 150 PSA. The three PAC bonds have identical original principal amounts.

Sequential PACs (concluded)

- We can take a CMO bond and further structure it.
- For example, the sequential PACs can be split by use of a pro rata structure to create high and low coupon PACs.
- We can also replace the B tranche in a four-tranche ABCZ sequential CMO with a PAC class that amortizes starting in year four, say.
- But note that tranche C may start to receive prepayments that are in excess of the schedule of the PAC bond.
- It may even be retired earlier than tranche B.

Finis